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FEKETE-SZEGÖ INEQUALITIES FOR $Q-$ STARLIKE AND $Q-$ CONVEX FUNCTIONS

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Abstract. Let $\mathcal{S}_q^*(\phi)$ and $\mathcal{C}_q(\phi)$ denote the classes of normalized functions $f(z) = z + a_2z^2 + a_3z^3 + ...$, which are defined in the open unit disk $\mathbb{D}$ and satisfying $zD_qf(z)/f(z) \prec \phi(z)$ and $D_q(zD_qf(z))/D_qf(z) \prec \phi(z)$, where ϕ is the function with real part, respectively. In this paper, we investigate new results of Fekete-Szegö inequalities for the classes $\mathcal{S}_q^*(\phi)$ and $\mathcal{C}_q(\phi)$.

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1. Introduction

Let $\mathcal{A}$ be the class of functions f, defined by $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, that are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ and Ω be the family of functions w, which are analytic in $\mathbb{D}$ and satisfying the conditions $w(0) = 0$, $|w(z)| < 1$ for all $z \in \mathbb{D}$. If f_1 and f_2 are analytic functions in $\mathbb{D}$, then we say that f_1 is subordinate to f_2, written as $f_1 \prec f_2$ if there exists a Schwarz function $w \in \Omega$ such that $f_1(z) = f_2(w(z)), z \in \mathbb{D}$. We also note that if f_2 univalent in $\mathbb{D}$, then $f_1 \prec f_2$ if and only if $f_1(0) = f_2(0)$, $f_1(\mathbb{D}) \subset f_2(\mathbb{D})$ implies $f_1(\mathbb{D}_r) \subset f_2(\mathbb{D}_r)$, where $\mathbb{D}_r = \{z : |z| < r, 0 < r < 1\}$ (see [8]).

Denote by $\mathcal{P}$ the family of functions p of the form $p(z) = 1+c_1z+c_2z^2+c_3z^3+\cdots$, analytic in $\mathbb{D}$ such that p is in $\mathcal{P}$ if and only if

$$p(z) \prec \frac{1+z}{1-z} \Leftrightarrow p(z) = \frac{1+w(z)}{1-w(z)} \tag{1}$$

for some function $w \in \Omega$ and for all $z \in \mathbb{D}$. It is well known that a function f in $\mathcal{A}$ is called starlike ($f \in \mathcal{S}^*$) and convex ($f \in \mathcal{C}$) if there exists a function p in $\mathcal{P}$ such that p may be expressed, respectively, by the following relations:

$$p(z) = z\frac{f'(z)}{f(z)} \quad \text{and} \quad p(z) = 1 + z\frac{f''(z)}{f'(z)}$$

for all $z \in \mathbb{D}$. For definitions and properties of these classes, one may refer to [1] and [8].

Ma and Minda [15] unified various subclasses of starlike and convex functions for which either one of the quantities $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinate to a more general superordinate function. The classes $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ of Ma-Minda starlike and Ma-Minda convex functions, are respectively characterized by $zf'(z)/f(z) \prec \phi(z)$ and $1 + zf''(z)/f'(z) \prec \phi(z)$, where function ϕ with positive real part in $\mathbb{D}$, $\phi(0) = 1, \phi'(0) > 0$. The coefficient $|a_3 - a_2^2|$ on the normalized analytic functions f in $\mathbb{D}$ plays an important role in functions theory. The problem of maximizing the absolute value of this coefficient is called Fekete-Szegö [4] problem. Many authors have considered the Fekete-Szegö problem for various subclasses of $\mathcal{A}$, the upper bound for $|a_3 - a_2^2|$ was investigated by many different authors (see [6, 14]).

We denote by $\mathcal{P}$ a class of analytic function in $\mathbb{D}$ with $p(0) = 1$ and $Rep(z) > 0$. Here we assume that $\phi \in \mathcal{P}$ satisfying $\phi(0) = 1, \phi'(0) > 0$ and $\phi(\mathbb{D})$ is symmetric with respect to the real axis. Also, ϕ has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ..., (B_1 > 0). \tag{2}$$

In 1909 and 1910 Jackson [10, 11, 12] initiated a study of $q-$ difference operator D_q defined by

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} \quad \text{for} \quad B\backslash\{0\}, \tag{3}$$

where B is a subset of complex plane $\mathbb{C}$, called $q-$ geometric set if $qz \in B$, whenever $z \in B$. Note that if a subset B of $\mathbb{C}$ is $q-$ geometric, then it contains all geometric sequences $\{zq^n\}_0^\infty$, $zq \in B$. Obviously, $D_q f(z) \to f'(z)$ as $q \to 1^-$. The $q-$ difference operator (3) is also called Jackson $q-$ difference operator. Note that such an operator plays an important role in the theory of hypergeometric series and quantum physics (see for instance [3, 5, 7, 13]).

Also, note that $D_q f(0) \to f'(0)$ as $q \to 1^-$ and $D_q^2 f(z) = D_q(D_q f(z))$. In fact, $q-$ calculus is ordinary classical calculus without the notion of limits. Recent interest in $q-$ calculus is because of its applications in various branches of mathematics and physics. For definitions and properties of $q-$ difference operator or $q-$ calculus, one may refer to [3, 5, 7, 13]. In particular, we recall the following properties:

Since

$$D_q z^n = \frac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1},$$

therefore we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} \frac{1-q^n}{1-q} a_n z^{n-1}, \tag{4}$$

where $[n]_q = \frac{1-q^n}{1-q}$. Clearly, as $q \to 1^-$, $[n]_q \to n$.

The class of $q-$ starlike functions was first introduced by Ismail et. al. [9] in 1990 as below:

Definition 1. *A function $f \in \mathcal{A}$ is said to be in the $\mathcal{S}_q^*$ such that*

$$\mathcal{S}_q^* = \left\{ f \in \mathcal{A} : Re\left(\frac{zD_q f(z)}{f(z)} \right) > 0, q \in (0,1), z \in \mathbb{D} \right\}.$$

When $q \to 1^-$ in the limiting sense, then the class $\mathcal{S}_q^*$ reduces to the traditional class $\mathcal{S}^*$.

Also, the class of $q-$ convex functions was introduced by Ahuja et. al. [2] as follows:

Definition 2. *A function $f \in \mathcal{A}$ is said to be in the $\mathcal{C}_q$ such that*

$$\mathcal{C}_q = \left\{ f \in \mathcal{A} : Re\left(\frac{D_q(zD_q f(z)}{D_q f(z)} \right) > 0, q \in (0,1), z \in \mathbb{D} \right\}.$$

When $q \to 1^-$ in the limiting sense, then the class $\mathcal{C}_q$ reduces to the traditional class $\mathcal{C}$.

Using above definitions and principle of subordination, we now introduce the following classes:

$$\mathcal{S}_q^*(\phi) = \left\{ f \in \mathcal{A} : z\frac{D_q f(z)}{f(z)} \prec \phi(z), \phi \in \mathcal{P} \right\}, \tag{5}$$

$$\mathcal{C}_q(\phi) = \left\{ f \in \mathcal{A} : \frac{D_q(zD_q f(z))}{D_q f(z)} \prec \phi(z), \phi \in \mathcal{P} \right\}. \tag{6}$$

The aim of this paper is to give Fekete-Szegö inequalities for the classes $\mathcal{S}_q^*(\phi)$ and $\mathcal{C}_q(\phi)$.

2. Main Results

We first investigate Fekete-Szegö inequalities for the class $\mathcal{S}_q^*(\phi)$. For our main theorems, we need the following result:

Lemma 1. *[16] Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + ...$, then $|c_n| \leq 2$ for $n \geq 1$. If $|c_1| = 2$, then $p(z) \equiv p_1(z) = \frac{1+\gamma_1 z}{1-\gamma_2 z}$ with $\gamma_1 = \frac{c_1}{2}$. Conversely, if $p(z) \equiv p_1(z)$ for some $|\gamma_1| = 1$, then $c_1 = 2\gamma_1$ and $|c_1| = 2$. Furthermore, we have*

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}.$$

If $|c_1| < 2$ *and* $\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}$*, then* $p(z) \equiv p_2(z)$ *where*

$$p_2(z) = \frac{1 + z\frac{\gamma_2 z + \gamma_1}{1 + \overline{\gamma_1}\gamma_2 z}}{1 - z\frac{\gamma_2 z + \gamma_1}{1 + \overline{\gamma_1}\gamma_2 z}}$$

and $\gamma_1 = \frac{c_1}{2}, \gamma_2 = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$*. Conversely, if* $p(z) \equiv p_2(z)$ *for some* $|\gamma_1| = 1$ *and* $|\gamma_2| = 1$*, then* $\gamma_1 = \frac{c_1}{2}, \gamma_2 = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$ *and* $|c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2}$.

Theorem 2. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$*, where the coefficients* B_n *are real with* $B_1 \neq 0$ *. If* f *belongs to the class* $\mathcal{S}_q^*(\phi)$*, then*

$$|a_2| \leq \frac{|B_1|}{[2]_q - 1}, \tag{7}$$

$$|a_3| \leq \frac{|B_1|}{[3]_q - 1} max\left\{1, \left|\frac{B_1}{[2]_q - 1} + \frac{B_2}{B_1}\right|\right\}, \tag{8}$$

$$\left|a_3 - \frac{([2]_q - 1)^2(\frac{B_1}{[2]_q - 1} + \frac{B_2}{B_1} - 1)}{B_1([3]_q - 1)} a_2^2\right| \leq \frac{|B_1|}{[3]_q - 1}. \tag{9}$$

These results are sharp.

Proof. If $f \in \mathcal{S}_q^*(\phi)$, then there is Schwarz function w, analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{zD_q f(z)}{f(z)} = \phi(w(z)). \tag{10}$$

Define the function p by

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \ldots \quad . \tag{11}$$

We can note that $p(0) = 1$ and p is a function with positive real part. Therefore

$$\begin{aligned}\phi(w(z)) &= \phi\left(\frac{p(z) - 1}{p(z) + 1}\right) \\ &= \phi\left(\frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)z^3 + \ldots\right]\right) \\ &= 1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right]z^2 + \ldots \end{aligned} \tag{12}$$

Also, computations shows that

$$\frac{zD_qf(z)}{f(z)} = 1 + ([2]_q - 1)a_2 z + \big(([3]_q - 1)a_3 - ([2]_q - 1)a_2^2\big)z^2 + ... \quad . \tag{13}$$

From equations in (12) and (13), we obtain

$$([2]_q - 1)a_2 = \frac{B_1 c_1}{2} \tag{14}$$

and

$$([3]_q - 1)a_3 - ([2]_q - 1)a_2^2 = \frac{B_1 c_2}{2} - \frac{B_1 c_1^2}{4} + \frac{B_2 c_1^2}{4}. \tag{15}$$

Taking into account Lemma 1, we obtain

$$|a_2| = \left|\frac{B_1 c_1}{2([2]_q - 1)}\right| \leq \frac{|B_1|}{[2]_q - 1}$$

and

$$\begin{aligned} |a_3| &= \left|\frac{B_1}{2([3]_q - 1)}\left[c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}\left(\frac{B_1}{[2]_q - 1} + \frac{B_2}{B_1}\right)\right]\right| \\ &\leq \frac{|B_1|}{2([3]_q - 1)}\left[2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left|\frac{B_1}{[2]_q - 1} + \frac{B_2}{B_1}\right|\right] \\ &\leq \frac{|B_1|}{2([3]_q - 1)}\left[2 + \frac{|c_1|^2}{2}\left(\left|\frac{B_1}{[2]_q - 1} + \frac{B_2}{B_1}\right| - 1\right)\right] \\ &\leq \frac{|B_1|}{[3]_q - 1} max\left\{1, \left|\frac{B_1}{[2]_q - 1} + \frac{B_2}{B_1}\right|\right\}. \end{aligned}$$

Furthermore, using (14) and (15) we get

$$\left|a_3 - \frac{([2]_q - 1)^2(\frac{B_1}{[2]_q - 1} + \frac{B_2}{B_1} - 1)}{B_1([3]_q - 1)}a_2^2\right| = \frac{|B_1 c_2|}{2([3]_q - 1)} \leq \frac{|B_1|}{[3]_q - 1}.$$

An examination of the proof shows that equality in (7) is attained, when $c_1 = 2$. Equivalently, we have $p(z) = p_1(z) = (1+z)/(1-z)$. Therefore, the extremal function in $\mathcal{S}_q^*(\phi)$ is given by

$$z\frac{D_qf(z)}{f(z)} = \phi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right). \tag{16}$$

In equality (8), for the first case, equality holds if $c_1 = 0, c_2 = 2$. Equivalently, we have $p(z) = p_2(z) = (1 + z^2)/(1 - z^2)$. Therefore, the extremal function in $\mathcal{S}_q^*(\phi)$ is given by

$$z\frac{D_qf(z)}{f(z)} = \phi\left(\frac{p_2(z) - 1}{p_2(z) + 1}\right). \tag{17}$$

In (8), for the second case, the equality holds if $c_1 = 2, c_2 = 2$. Therefore, the extremal function in $\mathcal{S}_q^*(\phi)$ is given by (16). Obtained extremal function for (7) is also valid for (9).

In fact, Theorem 2 gives a special case of Fekete-Szegö problem for real

$$\mu = \frac{([2]_q - 1)^2(\frac{B_1}{[2]_q - 1} + \frac{B_2}{B_1} - 1)}{B_1([3]_q - 1)},$$

which obtain the naturally and simple estimate. Thus the proof is completed.

We now consider $|a_3 - \mu a_2^2|$ for complex μ.

Theorem 3. *Let μ be a nonzero complex number and let $f \in \mathcal{S}_q^*(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{|B_1|}{[3]_q - 1} max\left\{1, \left|\frac{B_2}{B_1} + \frac{B_1}{[2]_q - 1}\left(1 - \frac{([3]_q - 1)}{([2]_q - 1)}\mu\right)\right|\right\}. \tag{18}$$

This result is sharp.

Proof. Applying (14) and (15), we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{B_1}{2([3]_q - 1)}\left[c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}\left(\frac{B_2}{B_1} + \frac{B_1}{[2]_q - 1}\right)\right] - \mu\frac{B_1^2 c_1^2}{4([2]_q - 1)^2} \\ &= \frac{B_1}{2([3]_q - 1)}\left[c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}\left(\frac{B_2}{B_1} + \frac{B_1}{[2]_q - 1}\left(1 - \frac{([3]_q - 1)}{([2]_q - 1)}\mu\right)\right)\right]. \end{aligned}$$

In view of Lemma 1,

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{|B_1|}{2([3]_q - 1)}\left[2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left(\left|\frac{B_2}{B_1} + \frac{B_1}{[2]_q - 1}\left(1 - \frac{[3]_q - 1}{[2]_q - 1}\mu\right)\right|\right)\right] \\ &= \frac{|B_1|}{2([3]_q - 1)}\left[2 + \frac{|c_1|^2}{2}\left(\left|\frac{B_2}{B_1} + \frac{B_1}{[2]_q - 1}\left(1 - \frac{[3]_q - 1}{[2]_q -)}\mu\right)\right| - 1\right)\right] \\ &\leq \frac{|B_1|}{[3]_q - 1}\max\left\{1, \left|\frac{B_2}{B_1} + \frac{B_1}{[2]_q - 1}\left(1 - \frac{[3]_q - 1}{[2]_q - 1}\mu\right)\right|\right\}. \end{aligned}$$

Equality is attained for the first case on choosing $c_1 = 0, c_2 = 2$ in (17) and for the second case on choosing $c_1 = 2, c_2 = 2$ in (16). Thus the proof is completed.

Corollary 4. *Taking $q \to 1^-$ in Theorem 3, we obtain*

$$|a_3 - \mu a_2^2| \leq \frac{|B_1|}{2} max\left\{1, \left|\frac{B_2}{B_1} + B_1\left(1 - 2\mu\right)\right|\right\}.$$

This result is sharp.

We now investigate Fekete-Szegö inequalities for the class $\mathcal{C}_q(\phi)$:

Theorem 5. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$, where the coefficients B_n are real with $B_1 \neq 0$. If f belongs to the class $\mathcal{C}_q(\phi)$, then*

$$|a_2| \leq \frac{|B_1|}{[2]_q([2]_q - 1)}, \tag{19}$$

$$|a_3| \leq \frac{|B_1|}{[3]_q([3]_q - 1)} max\left\{1, \left|\frac{B_1}{[2]_q - 1} + \frac{B_2}{B_1}\right|\right\}, \tag{20}$$

$$\left|a_3 - \frac{[2]_q^2([2]_q - 1)^2(\frac{B_1}{[2]_q - 1} + \frac{B_2}{B_1} - 1)}{B_1[3]_q([3]_q - 1)} a_2^2\right| \leq \frac{|B_1|}{[3]_q([3]_q - 1)}. \tag{21}$$

These results are sharp.

Proof. If $f \in \mathcal{C}_q(\phi)$, then there is Schwarz function w, analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{D_q(zD_q f(z))}{D_q f(z)} = \phi(w(z)). \tag{22}$$

Computations shows that

$$\frac{D_q(zD_q f(z))}{D_q f(z)} = 1 + [2]_q([2]_q - 1)a_2 z + \big([3]_q([3]_q - 1)a_3 - [2]_q^2([2]_q - 1)a_2^2\big)z^2 + ... \tag{23}$$

From equations in (12) and (23), we obtain

$$[2]_q([2]_q - 1)a_2 = \frac{B_1 c_1}{2} \tag{24}$$

and

$$[3]_q([3]_q - 1)a_3 - [2]_q^2([2]_q - 1)a_2^2 = \frac{B_1 c_2}{2} - \frac{B_1 c_1^2}{4} + \frac{B_2 c_1^2}{4}. \tag{25}$$

Taking into account Lemma 1, we obtain

$$|a_2| = \left|\frac{B_1 c_1}{2[2]_q([2]_q - 1)}\right| \leq \frac{|B_1|}{[2]_q([2]_q - 1)}$$

and

$$\begin{aligned}
|a_3| &= \left|\frac{B_1}{2[3]_q([3]_q-1)}\left[c_2-\frac{c_1^2}{2}+\frac{c_1^2}{2}\left(\frac{B_1}{[2]_q-1}+\frac{B_2}{B_1}\right)\right]\right| \\
&\le \frac{|B_1|}{2[3]_q([3]_q-1)}\left[|c_2-\frac{c_1^2}{2}|+\frac{|c_1|^2}{2}\left|\frac{B_1}{[2]_q-1}+\frac{B_2}{B_1}\right|\right] \\
&\le \frac{|B_1|}{2[3]_q([3]_q-1)}\left[2-\frac{|c_1|^2}{2}+\frac{|c_1|^2}{2}\left|\frac{B_1}{[2]_q-1}+\frac{B_2}{B_1}\right|\right] \\
&= \frac{|B_1|}{2[3]_q([3]_q-1)}\left[2+\frac{|c_1|^2}{2}\left(\left|\frac{B_1}{[2]_q-1}+\frac{B_2}{B_1}\right|-1\right)\right] \\
&\le \frac{|B_1|}{[3]_q([3]_q-1)}\max\left\{1,\left|\frac{B_1}{[2]_q-1}+\frac{B_2}{B_1}\right|\right\}.
\end{aligned}$$

Also, in view of (24) and (25), we obtain

$$\left|a_3-\frac{[2]_q^2([2]_q-1)^2(\frac{B_1}{[2]_q-1}+\frac{B_2}{B_1}-1)}{B_1[3]_q([3]_q-1)}a_2^2\right| = \frac{|B_1c_2|}{2[3]_q([3]_q-1)} \le \frac{|B_1|}{[3]_q([3]_q-1)}.$$

Equality in (19) holds if

$$\frac{D_q(zD_qf(z))}{D_qf(z)} = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) \tag{26}$$

and in (20) holds if

$$\frac{D_q(zD_qf(z))}{D_qf(z)} = \phi\left(\frac{p_2(z)-1}{p_2(z)+1}\right), \tag{27}$$

where p_1, p_2 are given in Lemma 1.

In Theorem 5, a special case of Fekete-Szegö problem for real

$$\mu = \frac{[2]_q^2([2]_q-1)^2(\frac{B_1}{[2]_q-1}+\frac{B_2}{B_1}-1)}{B_1[3]_q([3]_q-1)}$$

occurred very naturally and simple estimate was obtained. Thus the proof is completed.

Now, we consider $|a_3-\mu a_2^2|$ for complex μ.

Theorem 6. *Let μ be a nonzero complex number and let $f \in \mathcal{C}_q(\phi)$, then*

$$|a_3-\mu a_2^2| \le \frac{|B_1|}{[3]_q([3]_q-1)}max\left\{1,\left|\frac{B_2}{B_1}+\frac{B_1}{[2]_q-1}\left(1-\frac{[3]_q([3]_q-1)}{[2]_q^2([2]_q-1)}\mu\right)\right|\right\}. \tag{28}$$

This result is sharp.

Proof. Applying (24) and (25), we have

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{B_1}{2[3]_q([3]_q-1)}\left[c_2-\frac{c_1^2}{2}+\frac{c_1^2}{2}\left(\frac{B_2}{B_1}+\frac{B_1}{[2]_q-1}\right)\right]-\mu\frac{B_1^2c_1^2}{4[2]_q^2([2]_q-1)^2} \\ &= \frac{B_1}{2[3]_q([3]_q-1)}\left[c_2-\frac{c_1^2}{2}+\frac{c_1^2}{2}\left(\frac{B_2}{B_1}+\frac{B_1}{[2]_q-1}\left(1-\frac{[3]_q([3]_q-1)}{[2]_q^2([2]_q-1)}\mu\right)\right)\right]. \end{aligned}$$

In view of Lemma 1,

$$\begin{aligned} |a_3-\mu a_2^2| &\leq \frac{|B_1|}{2[3]_q([3]_q-1)}\left[2-\frac{|c_1|^2}{2}+\frac{|c_1|^2}{2}\left(\left|\frac{B_2}{B_1}+\frac{B_1}{[2]_q-1}\left(1-\frac{[3]_q([3]_q-1)}{[2]_q^2([2]_q-1)}\mu\right)\right|\right)\right] \\ &= \frac{|B_1|}{2[3]_q([3]_q-1)}\left[2+\frac{|c_1|^2}{2}\left(\left|\frac{B_2}{B_1}+\frac{B_1}{[2]_q-1}\left(1-\frac{[3]_q([3]_q-1)}{[2]_q^2([2]_q-1)}\mu\right)\right|-1\right)\right] \\ &\leq \frac{|B_1|}{[3]_q([3]_q-1)}\max\left\{1,\left|\frac{B_2}{B_1}+\frac{B_1}{[2]_q-1}\left(1-\frac{[3]_q([3]_q-1)}{[2]_q^2([2]_q-1)}\mu\right)\right|\right\}. \end{aligned}$$

This result is sharp for the functions given in (26) and (27). This completes the proof.

Corollary 7. *Taking $q \to 1^-$ in Theorem 6, we obtain*

$$|a_3-\mu a_2^2| \leq \frac{|B_1|}{6}max\left\{1,\left|\frac{B_2}{B_1}+B_1\left(1-\frac{3}{2}\mu\right)\right|\right\}.$$

This result is sharp.

References

[1] O. P. Ahuja, *The Bieberbach conjecture and its impact on the developments in geometric function theory*, Math. Chronicle, 15 (1986), 1-28.

[2] O. P. Ahuja, A. Çetinkaya, Y. Polatoğlu, *Bieberbach-de Branges and Fekete-Szegö inequalities for certain families of q− convex and q− close-to-convex functions*, J. Comput. Anal. Appl. (Accepted, 2017).

[3] G. E. Andrews, *Applications of basic hypergeometric functions*, SIAM Rev. 16 (1974), 441-484.

[4] M. Fekete, G. Szegö, *Eine bemerkung über ungerade Schlichte Funktionen*, J. Lond. Math. Soc. 8 (1933), 85-89.

[5] N. J. Fine, *Basic hypergeometric series and applications*, Math. Surveys Monogr. 1988.

[6] B. A Frasin, M. Darus, *On the Fekete-Szegö problem*, Int. J. Math. Math. Sci. 24 (2000), 577-581.

[7] G. Gasper, M. Rahman, *Basic hypergeometric series*, Cambridge University Press, 2004.

[8] A. W. Goodman, *Univalent functions, Volume I and II*, Polygonal Pub. House, 1983.

[9] M. E. H. Ismail, E. Merkes, D. Steyr, *A generalization of starlike functions*, Complex Variables Theory Appl. 14 (1990), 77-84.

[10] F. H. Jackson, *On $q-$ functions and a certain difference operator*, Trans. Royal Soc. Edinburgh, 46 (1909), 253-281.

[11] F. H. Jackson, *On $q-$ difference integrals*, Quart. J. Pure Appl. Math. 41 (1910), 193-203.

[12] F. H. Jackson, *$q-$ difference equations*, Amer. J. Math. 32 (1910), 305-314.

[13] V. Kac, P. Cheung, *Quantum calculus*, Springer, 2001.

[14] S. Kanas, H. E. Darwish, *Fekete-Szegö problem for starlike and convex functions of complex order*, Appl. Math. Lett. 23 (2010), 777-782.

[15] W. Ma, D. Minda, *A unified treatment of some special classes of univalent functions*, in: Proceedings of the Conference on Complex Analysis, Tianjin, 1992, pp. 157-169, Conf. Proc. Lecture Notes Anal. 1. Int. Press, Cambridge, MA, 1994.

[16] C. Pommerenke, *Univalent Functions*, Studia Mathematica Mathematische Lehrbucher, Vandenhoeck and Ruprecht, 1975.

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