Acta Universitatis Apulensis No. 54/2018
ISSN: 1582-5329 pp. 71-88
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2018.54.06

η-RICCI SOLITONS ON 3-DIMENSIONAL $N(K)$-CONTACT METRIC MANIFOLDS

D. Kar, P. Majhi, U.C. De

Abstract. The object of the present paper is to study η-Ricci solitons on 3-dimensional $N(k)$-contact metric manifolds. First we consider ϕ-concircularly semisymmetric η-Ricci soliton on 3-dimensional $N(k)$-contact metric manifolds. Beside these, we also study h-concircularly semisymmetric η-Ricci soliton on 3-dimensional $N(k)$-contact metric manifolds. Moreover we study concircularly semisymmetric η-Ricci soliton on 3-dimensional $N(k)$-contact metric manifolds. Finally, we construct an example of a 3-dimensional $N(k)$-contact metric manifold which admits η-Ricci solitons.

2010 *Mathematics Subject Classification*: $53C15$, $53C25$.

Keywords: keywords, phrases. Ricci soliton, η-Ricci soliton, concircularly ϕ-semisymmetric, h-concircularly semisymmetric, concircularly semisymmetric.

1. Introduction

In 1982, R. S. Hamilton [17] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evaluation equation for matrices on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial_t} g_{ij} = -2R_{ij}. \tag{1}$$

Ricci solitons are special solutions of the Ricci flow equation (1) of the form

$$g_{ij} = \sigma(t){\psi_t}^* g_{ij}$$

with the initial condition $g_{ij}(0) = g_{ij}$, where ψ_t are deffeomorphisms of M and $\sigma(t)$ is the scaling function.
A Ricci soliton is a natural generalization of Einstein metric. We recall the notion of Ricci soliton according to [6]. On the manifold M Ricci soliton is a tuple (g, V, λ)

with g, a Riemannian metric, V a vector field, called potential vector field, λ a real scalar and S is the Ricci tensor such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{2}$$

where $\pounds$ is the Lie derivative and X, Y are arbitrary vector fields on M. Metrics satisfying (2) are interesting and useful in physics and are often reffered as quasi-Einstein ([8],[9]). Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t}g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise blow-up limits for the Ricci flow on compact manifolds. The initial contribution in the direction is due to Friedan [16] Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The fact that equation (2) is a special case of Einstein field equation.
The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. Ricci soliton have been studied by several authors such as ([11], [12], [14],[15],[17],[18]) and many others.
As a generalization of Ricci soliton, the notion of η-Ricci soliton was introduced by Cho and Kimura [10]. This notion has also been studied in [6] for Hopf hupersurfaces in complex space forms. An η-Ricci soliton is a tuple (g, V, λ, μ), where V is a vector field on M, λ, μ are real scalars and g is a Riemannian (or pseudo-Riemannian) metric satisfying the equation

$$\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{3}$$

where S is the Ricci tensor associated to g. In this connection we mention the works of Blaga ([4],[5]) and Prakasha et. al. [23] on η-Ricci solitons. In particular, if $\mu = 0$, then the notion of η-Ricci soliton (g, V, λ, μ) reduces to the notion of Ricci soliton (g, V, λ). If $\mu \neq 0$, then the η-Ricci soliton is named proper η-Ricci soliton.

A transformation of a $(2n + 1)$-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation ([20],[28]). A concircular transformation is always a conformal transformation [20]. Here, geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus, the geometry of concircular transformations, i.e., the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism (see also [1]). An interesting invariant of a concircular transformation is the concircular curvature tensor $\bar{Z}$. It is defined by ([27],[28]).

$$\bar{Z}(X, Y)W = R(X, Y)W - \frac{r}{2n(2n + 1)}[g(Y, W)X - g(X, W)Y], \tag{4}$$

where $X, Y, W \in TM$ and r is the scalar curvature. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus, concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

A Riemannian manifold is called locally symmetric [7] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M, g). A Riemannian manifold $(M, g), n \geq 3$, is called semisymmetric if

$$R.R = 0$$

holds, where R denotes the curvature tensor of the manifold. It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset. Semisymmetric Riemannian manifolds were first studied by E. Cartan, A. Lichnerowich, R. S. Couty and N. S. Sinjukov. A fundamental study on Riemannian semisymmetric manifolds was made by Z. I. Szabó [24] and O. Kowalski [19].

In a recent paper Yildiz et al. [26] studied ϕ-Weyl semisymmetric and h-Weyl semisymmetric (k, μ)-contact manifolds. A (k, μ)-contact manifold is said to be ϕ-Weyl semisymmetric if $C.\phi = 0$ and h-Weyl semisymmetric if $C.h = 0$, where C is the Weyl conformal curvature tensor.

Motivated by the above studies in the present paper we study ϕ-concircularly semisymmetric and h-concircularly semisymmetric η-Ricci solitons on 3-dimensional $N(k)$-contact metric manifolds.

The present paper is organized as follows: After preliminaries in section 3, we consider η-Ricci soliton on 3-dimensional $N(k)$-contact metric manifolds. In the next two sections we study ϕ-concircularly semisymmetric and h-concircularly semisymmetric η-Ricci soliton on 3-dimensional $N(k)$-contact metric manifolds. Section 6 deals with the study of concircularly semisymmetric η-Ricci soliton on 3-dimensional $N(k)$-contact metric manifolds. Finally, we construct an example of a 3-dimensional $N(k)$-contact metric manifold admitting η-Ricci soliton.

2. Preliminaries

A contact manifold is by definition an odd dimensional smooth manifold M^{2n+1} equipped with a global 1-form satisfying $\eta \wedge (d\eta)^n \neq 0$ everywhere. It is well-known that there exists a unique vector field ξ, the characteristic vector field for which $\eta(\xi) = 1$ and $i_\xi d\eta = 0$. Further, one can find an associated Riemannian metric g and a vector field ϕ of type (1,1) such that

$$\eta(X) = g(X, \xi), d\eta(X, Y) = g(X, \phi Y), \phi^2 X = -X + \eta(X)\xi \tag{5}$$

where X and Y are vector fields on M. From (5) it follows that

$$\phi\xi = 0, \eta \circ \phi = 0, g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \quad (6)$$

The manifold M^{2n+1} together with the structure tensor (η, ξ, ϕ, g) is called a contact metric manifold ([1], [2]).

Given the contact metric manifold (M, η, ξ, ϕ, g), we define a symmetric (1,1)-tensor field h as $h = \frac{1}{2}L_\xi\phi$, where $L_\xi\phi$ denotes Lie differentiation in the direction of ξ. We have the following identities ([1], [2]):

$$h\xi = 0, h\phi + \phi h = 0, \quad (7)$$

$$\nabla_X\xi = -\phi X - \phi hX, \quad (8)$$

$$\nabla_\xi\phi = 0, \quad (9)$$

$$R(\xi, X)\xi - \phi R(\xi, \phi X)\xi = 2(h^2 + \phi^2)X, \quad (10)$$

$$(\nabla_\xi h)X = \phi X - h^2\phi X + \phi R(\xi, X)\xi, \quad (11)$$

$$S(\xi, \xi) = 2n - trh^2, \quad (12)$$

$$R(X, Y)\xi = -(\nabla_X\phi)Y + (\nabla_Y\phi)X - (\nabla_X\phi h)Y + (\nabla_Y\phi h)X. \quad (13)$$

Here, ∇ is the Levi-Civita connection and R the Riemannian curvature tensor defined by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

for all vector fields X, Y, Z on M.

If the characteristic vector field ξ is a Killing vector field, the contact metric manifold (M, η, ξ, ϕ, g) is called K-contact manifold. This is the case if and only if $h = 0$. Finally, if the Riemann curvature tensor satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

or, equivalently, if

$$(\nabla_X\phi)Y = g(X, Y)\xi - \eta(Y)X$$

holds, then the manifold is Sasakian. We note that a Sasakian manifold is always K-contact, but the converse only holds in dimension three.

The k-nullity distribution $N(k)$ of a Riemannian manifolds is defined by [25]

$$N(k) : p \to N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\},$$

k being a real number. If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold as $N(k)$-contact metric manifold [25]. If $k = 1$, then the manifold is Sasakian and if $k = 0$, then the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [1].

However, for a $N(k)$-contact metric manifold M of dimension $(2n+1)$, we have ([1], [2])

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{14}$$

where $h = \frac{1}{2}\pounds_\xi \phi$,

$$h^2 = (k-1)\phi^2, \tag{15}$$

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{16}$$

$$\begin{aligned} S(X,Y) &= 2(n-1)g(X,Y) + 2(n-1)g(hX,Y) \\ &\quad [2nk - 2(n-1)]\eta(X)\eta(Y), \qquad n \geq 1. \end{aligned} \tag{17}$$

$$S(Y,\xi) = 2nk\eta(X), \tag{18}$$

$$(\nabla_X \eta)(Y) = g(X + hX, \phi Y), \tag{19}$$

$$(\nabla_X h)(Y) = \{(1-k)g(X,\phi Y) + g(X, h\phi Y)\}\xi + \eta(Y)[h(\phi X + \phi hX)], \tag{20}$$

for any vector fields X, Y, Z, where R is the Riemannian curvature tensor and S is the Ricci tensor. $N(k)$-contact metric manifolds have been studied by several authors such as ([13], [21], [22]) and many others.

The curvature tensor of a 3-dimensional Riemannian manifold is given by

$$\begin{aligned} R(X,Y)Z &= [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad -\frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{21}$$

where S and r are the Ricci tensor and scalar curvature respectively and Q is the Ricci operator defined by $g(QX, Y) = S(X, Y)$.

In [3] Blair et al. proved that in a three dimensional contact metric manifold with ξ belonging to the k-nullity distribution, the following conditions hold:

$$QX = (\frac{r}{2} - k)X + (3k - \frac{r}{2})\eta(X)\xi, \tag{22}$$

$$S(X, Y) = (\frac{r}{2} - k)g(X, Y) + (3k - \frac{r}{2})\eta(X)\eta(Y), \tag{23}$$

$$\nabla_X \xi = -(1 + \alpha)\phi X, \tag{24}$$

where $\alpha = \pm\sqrt{1 - k}$.

Lemma 1. *[3] Let M^3 be a contact metric manifold with contact metric structure (ϕ, ξ, η, g). Then the following conditions are equivalent:*
i) M^3 is η-Einstein
ii) $Q\phi = \phi Q$
iii) ξ belongs to the k-nullity distribution.

Lemma 2. *[3] Let M^3 be a contact metric manifold on which $Q\phi = \phi Q$. Then M^3 is either Sasakian, flat or of constant ξ-sectional curvature $k < 1$ and constant ϕ-sectional curvature $-k$.*

3. η-Ricci solitons on 3-dimensional $N(k)$ contact metric manifolds

In this section we consider η-Ricci soliton on 3-dimensional $N(k)$-contact metric manifolds. Then

$$\begin{aligned}
(\pounds_\xi g)(X, Y) &= \pounds_\xi g(X, Y) - g(\pounds_\xi X, Y) - g(X, \pounds_\xi Y) \\
&= \xi g(X, Y) - g([\xi, X], Y) - g(X, [\xi, Y]) - g(X, [\xi, Y]) \\
&= \nabla_\xi g(X, Y) - g(\nabla_\xi X, Y) + g(\nabla_X \xi, Y) - g(X, \nabla_\xi Y) \\
&\quad + g(X, \nabla_Y, \xi) \\
&= (\nabla_\xi g)(X, Y) + g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \\
&= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \\
&= g(-\phi X - \phi h X, Y) + g(X, -\phi Y - \phi h Y) \\
&= -g(\phi X, Y) - g(\phi h X, Y) - g(X, \phi Y) - g(X, \phi h Y).
\end{aligned} \tag{25}$$

Then for η-Ricci soliton,

$$\pounds_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{26}$$

from which we get

$$\begin{aligned} 2S(X,Y) &= -(\pounds_\xi g)(X,Y) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y) \\ &= g(\phi X,Y) + g(\phi hX,Y) + g(X,\phi Y) + g(X,\phi hY) \\ &\quad -2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y) \\ &= 2g(\phi hX,Y) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y), \end{aligned} \tag{27}$$

from which it follows that

$$\begin{aligned} S(X,Y) &= g(\phi hX,Y) - \lambda g(X,Y) - \mu\eta(X)\eta(Y) \\ &= -g(hX,\phi Y) - \lambda g(X,Y) - \mu\eta(X)\eta(Y). \end{aligned} \tag{28}$$

Again from (28), we have

$$QX = \phi hX - \lambda X - \mu\eta(X)\xi. \tag{29}$$

In view of (28) we can state the following:

Theorem 3. *The Ricci tensor in 3-dimensional $N(k)$-contact metric manifolds admitting η-Ricci soliton is given by (28).*

4. ϕ-concircularly semisymmetric η-Ricci soliton on 3-dimensional $N(k)$-contact metric manifolds

Let us suppose that the manifold be ϕ-concircularly semisymmetric. Then we have

$$\bar{Z}.\phi = 0, \tag{30}$$

where $\bar{Z}$ is the concircular curvature tensor given by

$$\bar{Z}(U,V)W = R(U,V)W - \frac{r}{6}[g(V,W)U - g(U,W)V]. \tag{31}$$

From (30), it follows that

$$\bar{Z}(U,V)\phi W - \phi(\bar{Z}(U,V)W) = 0. \tag{32}$$

Using (31) in (32) we get

$$R(U,V)\phi W - \phi(R(U,V)W) - \frac{r}{6}[g(V,\phi W)U - g(U,\phi W)V] + \frac{r}{6}[g(V,W)\phi U - g(U,W)\phi V] = 0. \tag{33}$$

Now, from (21) we obtain

$$\begin{aligned} R(U,V)\phi W &= g(hV,W)U - g(hU,W)V - [\phi hU - (2\lambda + \frac{r}{2})U \\ &\quad -\mu\eta(U)\xi]g(V,\phi W) + [-\phi hV + (2\lambda + \frac{r}{2})V \\ &\quad +\mu\eta(V)\xi]g(U,\phi W). \end{aligned} \tag{34}$$

and

$$\begin{aligned} \phi(R(U,V)W) &= -g(hV,\phi W)\phi U + g(hU,\phi W)\phi V + [-hU \\ &\quad -(2\lambda + \frac{r}{2})\phi U]g(V,W) - [hV + (2\lambda + \frac{r}{2})\phi V]g(U,W) \\ &\quad -\mu\eta(V)\eta(W)\phi U + \mu\eta(U)\eta(W)\phi V. \end{aligned} \tag{35}$$

Using (34) and (35) in (33) yields

$$\begin{aligned} & g(hV,W)U - g(hU,W)V - [\phi hU - (2\lambda + \frac{r}{2})U - \mu\eta(U)\xi]g(V,\phi W) \\ & +[-\phi hV + (2\lambda + \frac{r}{2})V + \mu\eta(V)\xi]g(U,\phi W) + g(hV,\phi W)\phi U - g(hU,\phi W)\phi V \\ & -[-hU - (2\lambda + \frac{r}{2})\phi U]g(V,W) + [hV + (2\lambda + \frac{r}{2})\phi V]g(U,W) \\ & +\mu\eta(V)\eta(W)\phi U - \mu\eta(U)\eta(W)\phi V - \frac{r}{6}[g(V,\phi W)U - g(U,\phi W)V] \\ & +\frac{r}{6}[g(V,W)\phi U - g(U,W)\phi V] = 0. \end{aligned} \tag{36}$$

Taking inner product of (36) we obtain

$$\begin{aligned} & g(hV,W)g(U,X) - g(hU,W)g(V,X) - [g(\phi hU,X) - (2\lambda + \frac{r}{2})g(U,X) \\ & -\mu\eta(U)\eta(X)]g(V,\phi W) + [-g(\phi hV,X) + (2\lambda + \frac{r}{2})g(V,X) \\ & +\mu\eta(V)\eta(X)]g(U,\phi W) + g(hV,\phi W)g(\phi U,X) - g(hU,\phi W)g(\phi V,X) \\ & -[-g(hU,X) - (2\lambda + \frac{r}{2})(\phi U,X)]g(V,W) + [g(hV,X) \\ & +(2\lambda + \frac{r}{2})(\phi V,X)]g(U,W) + \mu\eta(V)\eta(W)g(\phi U,X) \\ & -\mu\eta(U)\eta(W)g(\phi V,X) - \frac{r}{6}[g(V,\phi W)g(U,X) - g(U,\phi W)g(V,X)] \\ & +\frac{r}{6}[g(V,W)(\phi U,X) - g(U,W)(\phi V,X)] = 0. \end{aligned} \tag{37}$$

Contracting over V, W in (37) we get

$$\begin{aligned}
&g(U,X)\mathrm{trh} - \mathrm{g(hU,X)} - [\mathrm{g}(\phi\mathrm{hU,X}) - (2\lambda + \frac{\mathrm{r}}{2})\mathrm{g(U,X)} - \mu\eta(\mathrm{U})\eta(\mathrm{X})]\mathrm{tr}\phi \\
&-g(h\phi X, \phi U) - (2\lambda + \frac{r}{2})g(\phi U, X) + g(he_i, \phi e_i)g(\phi U, X) \\
&-g(\phi hU, \phi X) - 3[-g(hU,X) - (2\lambda + \frac{r}{2})g(\phi U, X)] + g(hX, U) \\
&-(2\lambda + \frac{r}{2})g(\phi X, U) - \frac{r}{6}[\mathrm{tr}\phi \mathrm{g(U,X)} + \mathrm{g}(\phi\mathrm{U,X})] \\
&+\frac{r}{6}[3g(\phi U, X) + g(\phi X, U)] + \mu g(\phi U, X) + \mu\eta(U)g(\phi X, \xi) = 0, \qquad (38)
\end{aligned}$$

from which it follows that

$$3g(hU,X) + (6\lambda + \frac{5r}{3} + \mu)g(\phi U, X) = 0. \qquad (39)$$

Substituting $U = \phi U$ in (39) yields

$$-3S(U,X) - (9\lambda + \frac{5r}{3} + \mu)g(U,X) + (6\lambda + \frac{5r}{3} - 2\mu)\eta(U)\eta(X) = 0. \qquad (40)$$

In view of (40) we obtain

$$S(X,U) = ag(X,U) + b\eta(X)\eta(U), \qquad (41)$$

where

$$a = -\frac{1}{3}(9\lambda + \frac{5r}{3} + \mu), \qquad b = \frac{1}{3}(6\lambda + \frac{5r}{3} - 2\mu). \qquad (42)$$

Hence we conclude the following:

Theorem 4. *A ϕ-concircularly semisymmetric η-Ricci soliton on a 3-dimensional $N(k)$-contact metric manifold is η-Einstein.*

Comparing (23) with (41), we get

$$ag(X,Y) + b\eta(X)\eta(Y) = (\frac{r}{2} - k)g(X,Y) + (3k - \frac{r}{2})\eta(X)\eta(Y). \qquad (43)$$

Contracting over X, Y in the above equation we have

$$3a + b = r. \qquad (44)$$

Putting $X = Y = \xi$ in (43) we get

$$a + b = 2k. \qquad (45)$$

Using (42) in (44), (45) respectively we obtain

$$21\lambda - 5\mu = -13r. \tag{46}$$

and

$$\lambda + \mu = -2k. \tag{47}$$

Solving the equations (46) and (47) we infer

$$\lambda = -\frac{13r + 10k}{26}, \qquad \mu = \frac{13r - 42k}{26}. \tag{48}$$

Thus we can state the following:

Theorem 5. *A ϕ-concircularly semisymmetric η-Ricci soliton on a 3-dimensional $N(k)$-contact metric manifold is of the type $(g, \xi, -\frac{13r+10k}{26}, \frac{13r-42k}{26})$.*

5. h-concircularly semisymmetric η-Ricci soliton on 3-dimensional $N(k)$-contact metric manifolds

This section deals with h-concircularly semisymmetric η-Ricci soliton on 3-dimensional $N(k)$-contact metric manifold. Then we have

$$\bar{Z}.h = 0, \tag{49}$$

From (49), it follows that

$$\bar{Z}(U, V)hW - h(\bar{Z}(U, V)W) = 0. \tag{50}$$

Making use of (31) in (50) we have

$$\begin{aligned} &R(U, V)hW - h(R(U, V)W) - \frac{r}{6}[g(V, hW)U - g(U, hW)V] \\ &+\frac{r}{6}[g(V, W)hU - g(U, W)hV] = 0. \end{aligned} \tag{51}$$

From (29) we have

$$QU = -h\phi U - \lambda U - \mu\eta(U)\xi. \tag{52}$$

Then from (52) it follows that

$$h(QU) = (k - 1)\phi U - \lambda hU. \tag{53}$$

Now, from (21) we obtain

$$\begin{aligned} h(R(U,V)W) &= -g(hV,\phi W)hU + g(hU,\phi W)hV + [(k-1)\phi U \\ &\quad -(2\lambda+\frac{r}{2})hU]g(V,W) - [(k-1)\phi V - (2\lambda+\frac{r}{2})hV]g(U,W) \\ &\quad -\mu\eta(V)\eta(W)hU + \mu\eta(U)\eta(W)hV. \end{aligned} \tag{54}$$

and

$$\begin{aligned} R(U,V)hW &= (1-k)g(V,\phi W)U + (k-1)g(U,\phi W)V + [\phi hU - (2\lambda+\frac{r}{2})U \\ &\quad -\mu\eta(U)\xi]g(V,hW) - [\phi hU - (2\lambda+\frac{r}{2})U - \mu\eta(U)\xi]g(V,hW). \end{aligned} \tag{55}$$

Using (54) and (55) in (51) we get

$$\begin{aligned} &(1-k)g(V,\phi W)U + (k-1)g(U,\phi W)V + [\phi hU - (2\lambda+\frac{r}{2})U \\ &-\mu\eta(U)\xi]g(V,hW) - [\phi hV - (2\lambda+\frac{r}{2})V - \mu\eta(V)\xi]g(U,hW) \\ &+g(hV,\phi W)hU - g(hU,\phi W)hV - [(k-1)\phi U - (2\lambda+\frac{r}{2})hU]g(V,W) \\ &+[(k-1)\phi V - (2\lambda+\frac{r}{2})hV]g(U,W) + \mu\eta(V)\eta(W)hU - \mu\eta(U)\eta(W)hV \\ &-\frac{r}{6}[g(V,hW)U - g(U,hW)V] + \frac{r}{6}[g(V,W)hU - g(U,W)hV] = 0. \end{aligned} \tag{56}$$

Taking inner product of (56) we infer

$$\begin{aligned} &(k-1)g(\phi V,W)g(U,X) - (k-1)g(\phi U,W)g(V,X) + [g(\phi hU,X) \\ &-(2\lambda+\frac{r}{2})g(U,X) - \mu\eta(U)\eta(X)]g(V,hW) - [g(\phi hV,X) \\ &-(2\lambda+\frac{r}{2})g(V,X) - \mu\eta(V)\eta(X)]g(hU,W) + g(hV,\phi W)g(hU,X) \\ &+g(\phi hU,W)g(hV,X) - [(k-1)g(\phi U,X) - (2\lambda+\frac{r}{2})h(hU,X)]g(V,W) \\ &+[(k-1)g(\phi V,X) - (2\lambda+\frac{r}{2})h(hV,X)]g(U,W) + \mu\eta(V)\eta(Wg(hU,X) \\ &-\mu\eta(U)\eta(W)g(V,hX) - \frac{r}{6}[g(V,hW)g(U,X) - g(U,hW)g(V,X)] \\ &+\frac{r}{6}[g(V,W)g(hU,X) - g(U,W)g(hV,X]) = 0. \end{aligned} \tag{57}$$

Contracting over V,W in (57) yields

$$2(k-1)g(\phi X,U) + (6\lambda+2r+\mu)g(hU,X) = 0. \tag{58}$$

Substituting $X = \phi X$ in (58) we obtain

$$(6\lambda + 2r + \mu)S(U,X) = -[2(k-1) + \lambda(6\lambda + 2r + \mu)]g(U,X) + [2(k-1) - \mu(6\lambda + 2r + \mu)]\eta(U)\eta(X). \tag{59}$$

From (59) it follows that

$$S(U,X) = ag(U,X) + b\eta(U)\eta(X), \tag{60}$$

where

$$a = \lambda - \frac{2(k-1)}{6\lambda + 2r + \mu}, \qquad b = \frac{2(k-1)}{6\lambda + 2r + \mu} - \mu. \tag{61}$$

Thus we can state the following:

Theorem 6. *A h-concircularly semisymmetric η-Ricci soliton on a 3-dimensional $N(k)$-contact metric manifold is η-Einstein.*

6. Concircularlly semisymmetric η-Ricci soliton on 3-dimensional $N(k)$-contact metric manifolds

This section is devoted to study of conformally semisymmetric η-Ricci soliton on 3-dimensional $N(k)$-contact metric manifolds. Then

$$R.\bar{Z} = 0. \tag{62}$$

This implies

$$R(X,Y)\bar{Z}(U,V)W - \bar{Z}(R(X,Y)U,V)W - \bar{Z}(U,R(X,Y)V)W - \bar{Z}(U,V)R(X,Y)W = 0. \tag{63}$$

From the equation (63) we get

$$\begin{aligned} R(X,Y)R(U,V)W - \frac{r}{6}[g(V,W)R(X,Y)U - g(U,W)R(X,Y)V] \\ -R(R(X,Y)U,V)W + \frac{r}{6}[g(V,W)R(X,Y)U - g(R(X,Y)U,W)V] \\ -R(U,R(X,Y)V)W + \frac{r}{6}[g(R(X,Y)V,W)U - g(U,W)R(X,Y)V] \\ -R(U,V)R(X,Y)W + \frac{r}{6}[g(V,R(X,Y)W)U - g(U,R(X,Y)W)V] = 0. \end{aligned} \tag{64}$$

Putting $X = U = \xi$ in (64) we obtain

$$\begin{aligned} R(\xi,Y)R(\xi,V)W - R(R(\xi,Y)\xi,V)W - R(\xi,R(\xi,Y)V)W \\ -R(\xi,V)R(\xi,Y)W + \frac{r}{6}[g(R(\xi,Y)V,W)\xi + g(R(\xi,Y)W,V)\xi \\ -g(R(\xi,Y)\xi,W)V - \eta(R(\xi,Y)W)V] = 0. \end{aligned} \tag{65}$$

Putting $X = \xi$ in (21) yields

$$\begin{aligned} & R(\xi,V)W = S(V,W)\xi + (\lambda+\mu)\eta(W)V - (\lambda+\mu)g(V,W)\xi - \eta(W)QV \\ & -\frac{r}{2}[g(V,W)\xi - \eta(W)V]. \end{aligned} \tag{66}$$

Now

$$\begin{aligned} R(\xi,Y)R(\xi,V)W = S(V,W)R(\xi,Y)\xi + (\lambda+\mu)\eta(W)R(\xi,Y)V \\ -(\lambda+\mu)g(V,W)R(\xi,Y)\xi - \eta(W)R(\xi,Y)QV \\ -\frac{r}{2}[g(V,W)R(\xi,Y)\xi - \eta(W)R(\xi,Y)V]. \end{aligned} \tag{67}$$

Using (6.6) in (6.4) we have

$$\begin{aligned} & k(\lambda+\mu)Y - \eta(W)R(\xi,Y)QV + S(R(\xi,Y)\xi,W)V + kg(V,W)QY \\ & -S((R(\xi,Y)V,W)\xi + \eta(W)Q(R(\xi,Y)V) - S(R(\xi,Y)W,V)\xi \\ & -(\lambda+\mu)\eta(R(\xi,Y)W)V + \frac{r}{2}[-k\eta(Y)\eta(W)V \\ & +kg(Y,W)V - \eta(R(\xi,Y)W)V] = 0. \end{aligned} \tag{68}$$

Now

$$\begin{aligned} S(R(\xi,Y)\xi,W)V &= k\eta(Y)S(W,\xi)V - kS(X,W)V \\ &= -k(\lambda+\mu)\eta(W)\eta(Y)V - kS(X,W)V. \end{aligned} \tag{69}$$

Using (69) in (68) yields

$$\begin{aligned} & k(\lambda+\mu)g(V,W)Y - \eta(W)R(\xi,Y)QV - k[\lambda+\mu-\frac{r}{2}]\eta(Y)\eta(W)V \\ & +kg(V,W)QY + \eta(W)Q(R(\xi,Y)V) - (\lambda+\mu+\frac{r}{2})\eta(R(\xi,Y)W)V \\ & +\frac{kr}{2}g(Y,W)V - kS(Y,W)V - S(R(\xi,Y)V,W)\xi \\ & -S(R(\xi,Y)W,V) = 0. \end{aligned} \tag{70}$$

Substituting $V = \xi$ in (70) we obtain

$$2k(\lambda+\mu)\eta(W)Y - [(k+1)(\lambda+\mu) - \frac{r}{2}]\eta(Y)\eta(W)\xi$$
$$-\frac{r}{2}\eta(R(\xi,Y)W)\xi + \frac{kr}{2}g(Y,W)\xi = 0. \tag{71}$$

Now,

$$\eta(R(\xi,Y)W) = S(Y,W) + 2(\lambda+\mu)\eta(Y)\eta(W) - (\lambda+\mu)g(Y,W)$$
$$-\frac{r}{2}[g(Y,W) - \eta(Y)\eta(W)]. \tag{72}$$

With the help of (72), from (71) we get

$$2k(\lambda+\mu)\eta(W)Y - [(\lambda+\mu)(r+k+1) - \frac{r}{2} + \frac{r^2}{4}]\eta(Y)\eta(W)\xi$$
$$+[(\lambda+\mu)\frac{r}{2} + \frac{r^2}{4} + \frac{kr}{2}]g(Y,W)\xi - \frac{r}{2}S(Y,W)\xi = 0. \tag{73}$$

Contracting W in (73) we have

$$r = \frac{4k(k-1)}{k+1}. \tag{74}$$

Taking inner product of (73) with respect to ξ and then using (74) yields

$$S(Y,W) = ag(Y,W) + b\eta(Y)\eta(W), \tag{75}$$

where

$$a = \frac{k(k-3)}{k+1}, \qquad b = \frac{k^2-k+2}{k-1}. \tag{76}$$

Thus we can state the following:

Theorem 7. *A concircularly semisymmetric η-Ricci soliton on a 3-dimensional $N(k)$-contact metric manifold is η-Einstein.*

Hence from Lemma 2 and Theorems 4, 6, 7 we have the following:

Proposition 1. *A ϕ-concircularlly semisymmetric, h-concircularlly semisymmetric and concircularlly semisymmetric η-Ricci soliton on 3-dimensional N(k)-contact metric manifold satisfies $Q\phi = \phi Q$.*

Moreover from Lemma 2 and Theorems 4, 6, 7 we are in a position to state the following:

Theorem 8. *A ϕ-concircularlly semisymmetric, h-concircularlly semisymmetric and concircularlly semisymmetric η-Ricci soliton on 3-dimensional N(k)-contact metric manifold is either Sasakian, flat or of ξ-sectional curvature $k < -1$ and constant ϕ-sectional curvature $-k$.*

7. Example

We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, (x, y, z) \neq (0, 0, 0)\}$, where (x, y, z) are the standard coordinate in $\mathbb{R}^3$. Then e_1, e_2, e_3 are three linearly independent vector fields in $\mathbb{R}^3$ and

$$[e_1, e_2] = (1 + \alpha)e_3, \quad [e_2, e_3] = 2e_1 \quad \text{and} \quad [e_3, e_1] = (1 - \alpha)e_2,$$

where $\alpha \neq \pm 1$ is a real number.
Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by

$$\eta(U) = g(U, e_1)$$

for any $U \in \chi(M)$. Let ϕ be the $(1, 1)$-tensor field defined by

$$\phi e_1 = 0, \ \ \phi e_2 = e_3, \ \ \phi e_3 = -e_2.$$

Using the linearity of ϕ and g we have

$$\eta(e_1) = 1,$$

$$\phi^2(U) = -U + \eta(U)e_1$$

and

$$g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$$

for any $U, W \in \chi(M)$. Moreover

$$he_1 = 0, \quad he_2 = \alpha e_2 \ \textit{and} \ he_3 = -\alpha e_3.$$

The Riemannian connection ∇ of the metric tensor g is given by Koszul's formula which is given by,

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Using Koszul's formula we get the following:

$$\begin{aligned} &\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0, \\ &\nabla_{e_2} e_1 = -(1 + \alpha)e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = (1 + \alpha)e_1, \\ &\nabla_{e_3} e_1 = (1 - \alpha)e_2, \quad \nabla_{e_3} e_2 = -(1 - \alpha)e_1, \quad \nabla_{e_3} e_3 = 0. \end{aligned}$$

In view of the above relations we have

$$\nabla_X \xi = -\phi X - \phi h X, \quad \text{for} \quad e_1 = \xi$$

Therefore the manifold is a contact metric manifold with the contact structure (ϕ, ξ, η, g).

Now, we find the curvature tensors as follows:

$$R(e_1, e_2)e_2 = (1-\alpha^2)e_1, \quad R(e_3, e_2)e_2 = -(1-\alpha^2)e_3,$$
$$R(e_1, e_3)e_3 = (1-\alpha^2)e_1, \quad R(e_2, e_3)e_3 = -(1-\alpha^2)e_2,$$
$$R(e_2, e_3)e_1 = 0, \quad R(e_1, e_2)e_1 = -(1-\alpha^2)e_2, \quad R(e_3, e_1)e_1 = (1-\alpha^2)e_3.$$

In view of the expressions of the curvature tensors we conclude that the manifold is a $N(1-\alpha^2)$-contact metric manifold.

Using the expressions of the curvature tensor we find the values of the Ricci tensors as follows:

$$S(e_1, e_1) = 2(1-\alpha^2), \quad S(e_2, e_2) = 0, \quad S(e_3, e_3) = 0.$$

From (28) we obtain $S(e_1, e_1) = -(\lambda + \mu)$ and $S(e_2, e_2) = S(e_3, e_3) = -\lambda$. Therefore $\lambda = 0$ and $\mu = 2(\alpha^2 - 1)$. The data (g, ξ, λ, μ) defines an η-Ricci soliton on 3-dimensional $N(k)$-contact metric manifold.

References

[1] Blair, D. E., *Contact manifolds in Riemannian geometry*, Lecture note in Math., 509, Springer-Verlag, Berlin-New York, 1976.

[2] Blair, D. E., *Riemannian Geometry of contact and symplectic manifolds*, Birkhäuser, Boston, 2002.

[3] Blair, D. E., Koufogiorgos, T. and Sharma, R., *A classification of 3-dimensional contact metric manifolds with $Q\phi = \phi Q$*, Kodai Math. J., 13(1990), 391-401.

[4] Blaga, A. M., *η-Ricci solitons on Lorentzian para-Sasakian manifolds*, Filomat 30 (2016), no. 2, 489-496.

[5] Blaga, A. M., *η-Ricci solitons on para-Kenmotsu manifolds*, Balkan J. Geom. Appl. 20 (2015), 1-13.

[6] Calin, C. and Crasmareanu, M., *From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds*, Bull. Malays. Math. Soc. 33(3)(2010), 361-368.

[7] Cartan, E., *Sur une classe remarqable d' espaces de Riemannian*, Bull. Soc. Math. France., 54(1962), 214-264.

[8] Chave, T. and Valent, G., *Quasi-Einstein metrics and their renoirmalizability properties*, Helv. Phys. Acta. 69(1996) 344-347.

[9] Chave, T. and Valent, G., *On a class of compact and non-compact quasi-Einstein metrics and their renoirmalizability properties*, Nuclear Phys. B. 478(1996), 758-778.

[10] Cho, J. T. and Kimura, M., *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J. 61, no. 2 (2009), 205-212.

[11] De, U. C., and Majhi, P., *On a type of contact metric manifolds*, Lobacheviskii Journal of Mathematics, 34(2013), 89-98.

[12] De, U. C., Murathan, C. and Arsalan, K., *On the Weyl projective curvature tensor of an $N(k)$-contact metric manifold*, Mathematica Panonoica, 21, 1 (2010), 129-142.

[13] De, Avik and Jun, J. B., *On N(k)-contact metric manifolds satisfying certain curvature conditions*, Kyungpook Math. J. 51, 4 (2011), 457-468.

[14] Deshmukh, S., *Jacobi-type vector fields on Ricci solitons*, Bull. Math. Soc. Sci. Math. Roumanie 55(103), 1 (2012), 41-50.

[15] Deshmukh, S., Alodan, H. and Al-Sodais, H., *A Note on Ricci Soliton*, Balkan J. Geom. Appl. 16, 1 (2011), 48-55.

[16] Friedan, D., *Non linera models in $2 + \epsilon$ dimensions*, Ann. Phys. 163(1985), 318-419.

[17] Hamilton, R.S., *The Ricci flow on surfaces, Mathematics and general relativity* (Santa Cruz, CA, 1986), 237-262.Contemp. Math.**71**, American Math. Soc., 1988.

[18] Ivey, T., *Ricci solitons on compact 3-manifolds*, Diff. Geom. Appl. 3(1993),301-307.

[19] Kowalski, O., *An explicit classification of* 3- *dimensional Riemannian spaces satisfying* $R(X,Y).R = 0$, Czechoslovak Math. J. 46(121)(1996), $427 - 474$.

[20] Kuhnel, W., *Conformal transformations between Einstein spaces*, Conform. Geometry. Aspects Math., **12**(1988), Vieweg, Braunschweig, 105-146.

[21] Özgür, C., *Contact metric manifolds with cyclic-parallel Ricci tensor*, Diff. Geom. Dynamical systems, 4 (2002), 21-25.

[22] Özgür, C. and Sular, S., *On N(k)-contact metric manifolds satisfying certain conditions*, SUT J. Math. 44, 1 (2008), 89-99.

[23] Prakasha, D. G. and Hadimani, B. S., *η-Ricci solitons on para-Sasakian manifolds*, J. Geometry 108(2017), 383-392.

[24] Szabó , Z. I., *Structure theorems on Riemannian spaces satisfying* $R(X,Y).R = 0$*, the local version*, J. Diff. Geom., 17(1982), 531-582.

[25] Tanno, S., *Ricci curvature of contact Riemannian manifolds,* Tohoku Math. J., 40(1988), 441-448.

[26] Yildiz, A. and De, U. C., *A classification of* (k,μ)*-contact metric manifolds*, Commun. Korean Math. Soc. 27(2012)(2), 327-339.

[27] Yano, K. and Bochner, S., *Curvature and Betti numbers*, Annals of mathematics studies, 32, Princeton university press, 1953.

[28] Yano, K., *Concircular geometry I. concircular transformations*, Proc. Imp. Acad. Tokyo 16 (1940), 195-200.

Debabrata Kar,
Department of Pure Mathematics,
University of Calcutta,
35, Ballygaunge Circular Road,
Kolkata 700019,
West Bengal, India
email: *debabratakar6@gmail.com*

Pradip Majhi
Department of Pure Mathematics,
University of Calcutta,
35, Ballygaunge Circular Road,
Kolkata 700019,
West Bengal, India
email: *mpradipmajhi@gmail.com*

Uday Chand De
Department of Pure Mathematics,
University of Calcutta,
35, Ballygaunge Circular Road,
Kolkata 700019,
West Bengal, India
email: *uc_de@yahoo.com*