Acta Universitatis Apulensis No. 63/2020
ISSN: 1582-5329 pp. 81-96
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2020.63.07

NEWTON-RICCI-BOURGUIGNON ALMOST SOLITONS ON LAGRAIGIAN SUBMANIFOLDS OF COMPLEX SPACE FORM

Mohd Danish Siddiqi

Abstract. The present discourse establish the geometrical bearing on Lagrangian submanifold of complex space form in terms of r-Newton-Ricci-Bourguignon almost soliton where $\rho \neq 0$. Moreover, we extensively study the conception of Lagragian immersion of Ricci-Bourguignon almost solitons and arrive at the sufficient conditions for H-minimal and totally geodesic under Newton transformation with the potential function $\psi : M^n \longrightarrow \mathbb{R}$. Finally, we conclude our paper with the study of 1-almost Newton-Ricci-Bourguignon almost solitons on Lagragian submanifold of complex space form immersed in a locally symmetric Einstein manifold.

2010 *Mathematics Subject Classification*: 53C44, 53B21, 53C15, 58J05.

Keywords: keywords, phrases. Newton-Ricci-Bourguignon almost solitons, Lagrangian submanifold, Complex space form, Locally symmetric spaces, Einstein manifold.

1. Introduction

In 1981, the conception of Ricci-Bourguignon flow as a extension of Ricci flow [16] has been initiated by J. P. Bourguinon [5] based on some unprinted work of Lichnerowicz and a paper of Aubin [1]. Ricci-Bourguignon flow are intrinsic geometric flow on pseudo-Riemannian manifolds, whose fixed points are solitons.
Ricci-Bourguignon solitons, which generates self-similar solution to the Ricci-Bourguignon flow [7]

$$\frac{\partial g}{\partial t} = -2(Ric - \rho R g), \quad g(0) = g_0, \tag{1}$$

where Ric is the Ricci curvature tensor , R is the scalar curvature with respect to the g and ρ is a real non zero constant. It should be noticed that for special values of the constant ρ in equation (1) we have obtain the following situations for the tensor $Ric - \rho Rg$ appearing in equation (1). This PDE system (1) defined the evolution equation is of special interest, in particular [7],

1. $\rho = \frac{1}{2}$, the Einstein tensor $Ric - \frac{R}{2}g$,
2. $\rho = \frac{1}{n}$, the traceless Ricci tensor $Ric - \frac{R}{n}g$,
3. $\rho = \frac{1}{2(n-1)}$, the Schouten tenosr $Ric - \frac{R}{2(n-1)}$,
4. $\rho = 0$, the Ricci tensor Ric.

In dimension two, the first three tensors are zero, hence the flow is static and in higher dimension the value of ρ are strictly ordered as above in descending order.

Short time existence and uniqueness for the solution of this geometric flow has been proved in [7]. In fact, for sufficiently small t the equation has a unique solution for $\rho < \frac{1}{2(n-1)}$.

In the other hand, quasi Einstein metrics or Ricci solitons serve as a solution to Ricci flow equation [8]. This motivates a more general type of Ricci soliton by considering the Ricci-Bourguignon flow. In fact, a pseudo-Riemannian manifold of dimension $n \geq 3$ is said to be Ricci-Bourguignon soliton [1] if

$$\frac{1}{2}\mathcal{L}_X g + Ric + (\lambda + \rho R)g = 0, \tag{2}$$

where $\mathcal{L}_X$ denotes the Lie derivative operator along vector field X and λ is an arbitrary real constant. Similar to Ricci solitons, a Ricci-Bourguignon soliton is called expanding if $\lambda > 0$, steady if $\lambda = 0$ and shrinking if $\lambda < 0$.

According to Pigola et al. [18] if we assume that the constant λ in (2) as a smooth function $\lambda \in C^\infty(M)$, called soliton function, then we say that (M, g) is Ricci-Bourguignon almost soliton, see ([2] [4], [7]). This concept drags the attention of many geometers. Therefore, in recent years much effort has been devoted to the classification of self-similar solutions of geometric flows.

In fact, the some axioms and physical application of *Ricci-Bourguignon flow* recently studied by Cantino and Mazzieri [7]. In this more general setting, we call (2) as being fundamental equation of an *Ricci-Bourguignon almost soliton.*

Definition 1. *[7] A Riemannian manifold (M, g) of dimension n is said to be the gradinet Ricci almost soliton or almost gradient Ricci soliton if*

$$Ric - \frac{1}{2}Rg + \nabla^2\psi = \lambda g, \tag{3}$$

for some function $\psi, \lambda \in C^\infty(M)$, where $\psi : M \longrightarrow \mathbb{R}$ and $X = \nabla\psi$.

A more general type gradient Eisntein soliton [8] has been deduced by considering the following *Ricci-Bourguignon flow* ([6], [16]).

Definition 2. *[6] A Riemannian manifold (M, g) of dimension $n \geq 3$ is said to be the gradient Ricci-Bourguignon almost soliton if*

$$Ric + \nabla^2 \psi = \lambda g + \rho R g, \quad \rho \in \mathbb{R}, \quad \rho \neq 0, \tag{4}$$

for some function $\psi, \lambda \in C^\infty(M)$. The function $\psi : M \longrightarrow \mathbb{R}$ is called Ricci-Bourguignon potential, when vector filed $X = \nabla\psi$ type. The gradient Ricci-Bourguignon almost soliton is called shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively. In some studies Ricci-Bourguignon almost soliton is also known as Ricci ρ-almost soliton [21] for some special values of ρ.

Many geometers extensively studied the above mentioned solitons which is closely related to this paper, for further details see ([8], [9],[18], [20], [21], [22]).

In 2011, Barros and his co-authors [3] discussed isometric immersions of an almost Ricci soliton (M^n, g, X, λ) in to Riemannian manifold M^{n+p}. Particularly, if M^{n+p} has non-positive sectional curvature, they proved that an almost Ricci soliton is a Ricci soliton and the vector field has integrable norm on M^n, then M^n can not be minimal. Furthermore, in [24] Wylie proved that if (M^n, g, X, λ) is a shrinking Ricci soliton, with X having bounded norm on M^n, then M^n must be compact. In of case, if M^{n+p} is a space form of non-positive sectional curvature, then such an immersions can not be minimal. Recently, in 2018, Cunah and his co-authors [10] have studied the immersed almost Ricci solitons under Newton transformation P_r with second order differential operators L_r and developed a new concept of r-almost Newton-Ricci soliton, for some $0 \leq r \leq n$. In contrast Ejiri [15] studied totally real minimal immersions of n-dimensional Real Space Forms into n-dimensional complex Space.

Let M is an r-Newton-Ricci-Bourguignon almost soliton, for some $0 \leq r \leq n$, if there exist a smooth function $\psi : M^n \longrightarrow \mathbb{R}$ such that [6]

$$Ric + P_r \circ Hess\psi = \lambda g + \rho R g, \tag{5}$$

where λ is a smooth function on M^n and $P_r \circ Hess\psi$ stands for tensor given by

$$P_r \circ Hess\psi(X, Y) = g(P_r \nabla_X \nabla_\psi, Y), \tag{6}$$

for tangent vectors fields $X, Y \in \chi(M)$. For $r = 0$, equation (5) reduces to the definition of a gradient Ricci-Bourguignon almost soliton.

On the one hand, Lagrangian submanifolds of complex space forms have been deeply studied since the decade 1970's. The geometry of Lagrangian submanifolds have been important geometric objects of the study in symplectic geometry. Symplectic geometry covers different classes of symplectic manifolds [**?**]. The main study related to the Hamiltonian dynamics and some special types of submanifolds mainly

Lagrangian submanifolds (symplectic case). Symplectic geometry is a relatively new field in mathematics, and has connections to algebraic geometry, dynamical systems, geometric topology, and theoretical physics. In 1990's Oh [17] introduced the study of Hamiltonian minimal (H-minimal) Lagrangian submanifolds in complex Kähler manifold. This is a nice extensions of the notion of minimal submanifold, and has been studied by other geometers such as ([5], [6], [14], [25]). Also, a detailed survey about Lagrangian submanifolds can be found in [6].

Therefore, inspired by the above literature's in the present manuscript we have explore the study of r-Newton-Ricci-Bourguignon almost soliton on Lagrangian submanifolds of complex form. r-Newton-Ricci-Bourguignon almost soliton is the expansion of the previous research, therefore we can extensively furnish the previous results more exclusively.

2. Preliminaries

In 1993, Oh defined the Hamiltonian deformation of Lagragian submanifolds in complex manifold (Kähler manifold). Let $\bar{M}$ be a complex n-dimensional Kähler manifold with Kähler form ω , Riemannian metric g and complex structure J.

Let $\varphi : M^n \longrightarrow \bar{M}$ be a Lagrangian immersion from a real n-dimensional manifold M to $\bar{M}^{n+p}$. For a vector field X along ϕ we define a 1-form α_X on M as $\alpha_X = g(JX, .)$. Smooth family of embedding $e_t : M \longrightarrow P$ is called *Hamiltonian deformation* if for the variational vector field X, the 1-form α_X is exact. A Lagerangian submanifold M is Hamiltonian minimal (or H-minimal) if M is stationary for any Hamiltonian deformation Oh [17] proved that when M is compact, M is H-minimal if and only if α_H is co-closed, i.e, *Euler-Lagrange's* equation $\delta\alpha_H = 0$ where H is the mean curvature vector field of M. We have

$$M \quad is \quad Hamiltonian \quad minimal \Leftrightarrow divJH = 0.$$

Now, again let $\varphi : M^n \longrightarrow \bar{M}$ be a Lagrangian immersion into an n-dimensional Riemannian manifold $\bar{M}$.

Let $\bar{M}(4c)$ be an n-dimensional complex space form with constant holomorphic sectional curvature $4c$ and let $M = M^n$ be a Lagrangian submanifold in $\bar{M}^n(4c)$ [2].

The Gauss equation of the immersion is given by

$$R(X,Y)Z = (\bar{R}(X,Y)Z)^T) + g(AX,Z)AY - g(AY,Z)AX \tag{7}$$

for every tangent vector fields $X, Y, Z \in \chi(M)$, where $()^T$ denotes the tangential components of a vector field in $\chi(M)$ along M^n. $A : \chi(M) \longrightarrow \chi(M)$ stands for second fundamental form (or shape operator) of M^n in M^{n+1} with respect to a fixed orientation, $\bar{R}$ and R denotes the curvature tensors of M and $\bar{M}$, respectively. In particular, the scalar curvature τ of the submanifold M^n satisfies

$$\tau = \sum_{i,j}^{n} g(\bar{R}(E_i, E_j)E_j, E_i) + H^2 - |A|^2\,, \tag{8}$$

where $\{E_1, ...E_n\}$ is an orthonormal frame on TM and $|.|$ denotes the Hilbert-Schmidt norm. When M^{n+1} is a space form of constant sectional curvature c, we have the identity

$$\tau = n(n-1)c + nH^2 - \|A\|^2\,. \tag{9}$$

Associated to second fundamental form A of the M^n there are n algebraic invariants, which are the elementary symmetric functions τ_r of its principal curvatures $k_1, ...k_n$, given by

$$\tau_0 = 1, \quad \tau_r = \sum_{i_1<...<i_r} k_1, ...k_n. \tag{10}$$

The r-th mean curvature H_r of the immersion is define by $\binom{n}{r}H_r = \tau_r$.
If $r = 0$, we have $H_1 = \frac{1}{n}Tr(A) = H$ the mean curvature of M^n.

For each $0 \leq r \leq n$, we defines the Newton transformation $P_r : \chi(M) \longrightarrow \chi(M)$ of the M^n be setting $P_0 = I$ (the identity operator) and for $0 \leq r \leq n$, by the recurrence relation

$$P_r = \sum_{j=0}^{r}(-1)^{r-j}\binom{n}{j}H_j A^{r-j}, \tag{11}$$

where A^j denotes the composition of A with itself, j times ($A^0 = I$). Let us recall that associated to each Newton transformation P_r one has the second order linear differential operator $L_r : C^\infty(M) \longrightarrow C^\infty(M)$ defined by

$$L_r u = Tr(P_r \circ Hessu). \tag{12}$$

When $r = 0$, w note that L_0 is just the Laplacian operator. Moreover, it is not difficult to see that

$$div_m(P_r\nabla u) = \sum_{i=1}^{n} g(\nabla_{E_i}P_r)\nabla_u, E_i) + \sum_{i=1}^{n} g(P_r(\nabla_{E_i}\nabla_u), E_i) \tag{13}$$

$$= g(div_M P_r, \nabla_u) + L_r u,$$

where the divergence of P_r on M^n is given by

$$div_M P_r = Tr(\nabla P_r) = \sum_{i=1}^{n} (\nabla_{E_i} P_r) E_i. \tag{14}$$

In particular, when the ambient space has constant sectional curvature equation (13) reduces to

$$L_r u = div_M(P_r \nabla u), \tag{15}$$

because $div_M P_r = 0$ (see [19] for more details).

Our aim, it also will be appropriate to deal with the so called traceless second fundamental form of the submanifold M^n, which is is given by

$$\Phi = AHI, \qquad Tr(\Phi) = 0. \tag{16}$$

and

$$|\Phi|^2 = Tr(\Phi^2) = |A|^2 - nH^2 \geq 0. \tag{17}$$

with equality if and only if M^n is totally umbilical.

In order to establish our results let us mention the following maximum principle due to Caminha et al. for more details see [12]. We follows that, for each $p \geq 1$ use the notation

$$L^p(M) = \left\{ u : M^n \longrightarrow \mathbb{R}; \int_M |u|^p \, dm < +\infty \right\}. \tag{18}$$

Also, we have the following lemma:

Lemma 1. *Let X be a smooth vector field on the n-dimensional, complete, non compact, oriented Riemannian manifold M^n, such that $div_M X$ does not change sign on M^n. If $|X| \in L^1(M)$, then $div_M X = 0$.*

The following results further generalized Theorem 1.2 in [3].

Theorem 2. *Let the data (g, ψ, λ, r) be a complete r-Newton-Ricci-Bourguignon almost soliton on Lagrangian submanifold of complex form $\bar{M}^n(c)$ of constant sectional curvature c, with bounded second fundamental form and potential function $\psi : M^n \longrightarrow \mathbb{R}$ such that $|\nabla\psi| \in L^1(M)$. Then we have*

1. If $c \leq 0$, $\lambda > 0$ and $\rho > \frac{1}{n}$, then M^n can not be H-minimal.
2. If $c < 0$, $\lambda \geq 0$ and $\rho > \frac{1}{n}$, then M^n can not be H-minimal.

3. If $c = 0$, $\lambda \geq 0$, $\rho > \frac{1}{n}$ is H-minimal, then M^n is isometric to the $\mathbb{R}^n$.

Proof. We know that the ambient space has constant sectional curvature, by equation (15) the operator L_r is a divergent type operator. On the other side, since M^n has bounded second fundamental form it follows from (11) that the Newton transformation P_r has bounded norm. In particular,

$$|P_r \nabla \psi| \leq |P_r| \, |\nabla \psi| \in L^1(M), \tag{19}$$

Adopting (1) and (2), let us consider by contradiction that M^n is minimal. Then, equation (9) jointly with the considering $c \leq 0$ $(c < 0)$ imply that the scalar curvature of M^n satisfies $\tau \leq 0 (\tau < 0)$. Hence, contracting (5) we have $L_r \psi = n\lambda - (1 - n\rho)\tau > 0$ in both case, which contradicts Lemma (1), since the fact after mentioned. This completes the proof of the first two assertions.

For the (3) assertion, since the ambient space has constant sectional curvature $c = 0$ and M^n is minimal, then the equation (9) becomes as

$$\tau = - \|A\|^2 \leq 0. \tag{20}$$

So, since $\lambda \geq 0$ and $n\rho > 1$ we have that $L_r(\psi) = n\lambda - (1 - n\rho)\tau \geq 0$. Now, using the fact that $L_r u = div_M(P_r \nabla u)$ and $|P_r \nabla \psi| \in L^1(M)$, we have again from Lemma 1 that $L_r \psi = 0$ on M^n. Hence, we conclude that $0 \geq \tau = \frac{n\lambda}{(1-n\rho)} \geq 0$, that is, $\tau = \frac{\lambda}{(1-n\rho)} = 0$. This implies that $|A|^2 = 0$. Therefore, the r-Newton-Ricci-Bourguignon almost soliton M^n must be geodesic and flat.

I order to prove our next theorems we will need the following lemmas, which corresponds to Theorem 3 [3].

Lemma 3. *Let u be a non-negative smooth subharmonic function on a complete Riemannian manifold M^n. If $u \in L^p(M)$, for some $p > 1$, the u is constant.*

Further, we are in condition to establish the following result, which holds when the ambient space is an arbitrary Riemannian manifold.

Theorem 4. *Let the data (g, ψ, λ, r) be a complete r-Newton-Ricci-Bourguignon almost soliton on Lagrangian submanifold of complex form $\bar{M}^n(c)$ of sectional curvature K, such that P_r is bounded from above (in the sense of quadratic forms) and its potential function $\psi : M^n \longrightarrow \mathbb{R}$ is non-negative and $\psi \in L^p(M)$ for some $p > 1$. Then we have*

1. *If $K \leq 0$, $\lambda > 0$ and $\rho > \frac{1}{n}$, then M^n can not be H-minimal,*
2. *If $K < 0$, $\lambda \geq 0$ and $\rho > \frac{1}{n}$, then M^n can not be H-minimal,*

3. *If $K \leq 0$, $\lambda \geq 0$, $\rho > \frac{1}{n}$ and M^n is H-minimal, then M^n is flat and totally geodesic.*

Proof. For proving (1), we begin with a contradiction that M^n is minimal our assumption on the sectional curvature of the ambient space and equation (8) imply that $\tau \leq 0$. Hence, contracting equation (5) we have

$$L_r\psi = n\lambda - (1 - n\rho)\tau > 0. \tag{21}$$

Thus, since we are considering that P_r is bounded from above, there exists a positive constant β such that

$$\beta\Delta\psi \geq L_r\psi > 0. \tag{22}$$

In particular, from Lemma (3) we get that ψ must be constant, which gives a contradiction. Finally, reasoning as in the proof of Theorem (2) we can easily obtain (2) and (3).

In our next results we generalized Theorem 1.5 of [3] for the case when $X = \nabla\psi$, giving conditions for a r- Newton-Ricci-Bourguignon almsot soliton immersed be totally umbilical since it has bounded second fundamental form. Therefore, we prove the following theorem:

Theorem 5. *Let the data (g, ψ, λ, r) be a complete r-Newton-Ricci-Bourguignon almost soliton on Lagrangian submanifold of complex form $\bar{M}^n(c)$ of constant sectional curvature c, with bounded second fundamental form and potential function $\psi : M^n \longrightarrow \mathbb{R}$ such that $|\nabla\psi| \in L^1(M)$. Then we have*

1. *$\rho > \frac{1}{n}$ and $\lambda \geq (1 - n\rho)(n-1)c + nH^2$, then M^n is totally geodesic, with $\lambda = (1 - n\rho)(n-1)c$, $\rho = \frac{1}{2}$ and scalar curvature $\tau = n(n-1)c$,*

2. *If M^n is compact $\rho > \frac{1}{n}$ and $\lambda \geq (1-n\rho)(n-1)(c+H^2)$, then M^n is isometric to a Euclidean sphere,*

3. *If $n\rho > 1$ and $\lambda \geq (1 - n\rho)(n-1)(c + H^2)$, then M^n is totally umbilical. In particular, the scalar curvature $\tau = n(n-1)K_M$ is constant, where $K_M = \frac{\lambda}{(1-n\rho)(n-1)}$ is the sectional curvature of M^n.*

Proof. To prove (1), using the equations (5) and (9), we obtain

$$L_r\psi = n[\lambda + (n-1)(1 - n\rho)c - H^2] + \|A\|^2. \tag{23}$$

Then, for our consideration on λ, we get that $L_r\psi$ is non-negative function on M^n. By Lemma (1) we find that $L_r\psi$ vanishes identically. Hence, from equation (23) we arrive at that M^n is totally geodesic and $\lambda = (1-n\rho)(n-1)c$, $n\rho = 1$. Moreover, it is clear form (9) that $\tau = n(n-1)c$, which complete the proof of (1).

If M^n is compact, as it is totally geodesic, then the ambient space must be necessarily a sphere $\mathbb{S}^n$ and M^n is isometric to the Euclidean sphere $\mathbb{S}^n$, proving (2).

For the assertion (3), we start with equation (23) that can be written in terms of the traceless second fundamental form Φ as

$$L_r\psi = n[\lambda + (1-n\rho)(n-1)c + H^2] + \|\Phi\|^2 . \tag{24}$$

Therefore, our assumption on λ and ρ gives $L_r\psi \geq 0$. Then by applying Lemma (1) once again we have $L_r\psi = 0$. This implies that $|\Phi|^2$, that is, M^n is a totally umbilical submanifold. In particular κ of M^n is constant and M^n has constant sectional curvature given by $K_M = c + \kappa^2$. This combined with (24) , we obtain that

$$\lambda = (1-n\rho)(n-1)(c+H^2) = (1-n\rho)(n-1)(c+\kappa^2) \tag{25}$$

$$= (1-n\rho)(n-1)K_M,$$

which implies that $\tau = n(n-1)K_M$, as desired.

Now, we have the following consequence of the Theorem (5):

Theorem 6. *Let the data (g, ψ, λ, r) be a complete r-Newton-Ricci-Bourguignon almost soliton on Lagrangian submanifold of complex form $\bar{M}^n(c)$ of constant sectional curvature c. If $\lambda = (1-n\rho)(n-1)H^2$, then M^n is isometric to $\mathbb{S}^n$.*

From Theorem 1.6 of [2] which states that a nontrivial almost Ricci soliton M^n, minimally immersed in $\mathbb{S}^{n+1}$ with $\tau \geq n(n-2)$ and such that the nor of the second fundamental form obtain its maximum, must be isometric to $\mathbb{S}^n$. Now, applying Theorem (5) we obtain an generalization of this results.

Theorem 7. *Let the data (g, ψ, λ, r) be a complete r-Newton-Ricci-Bourguignon almost soliton on Lagrangian submanifold of complex form $\bar{M}^n(c)$ of constant sectional curvature c. Consider that $\tau \geq n(n-2)$, the norm of the second fundamental form attains its maximum and $\lambda \geq \lambda = (1-n\rho)(n-1)$, $\rho > \frac{1}{n}$. Then, M^n is isometric to $\mathbb{S}^n$.*

Proof. Since the immersions is minimal with $\tau \geq n(n-2)$, from (9) we arrive at

$$\|A\|^2 = n(n-1) - \tau \leq n.$$

From Simons's formula [20], we obtain

$$\Delta \|A\|^2 = \|\nabla A\|^2 + (n - \|A\|^2) \|A\|^2 \geq 0. \tag{26}$$

Thus, we can apply Hopf's strong maximum principle to get that $\nabla A = 0$ on M^n. Therefore, Proposition 1 of [13] assures that M^n must be compact and, hence, the results from Theorem (5).

Another application of Theorem (4), we can also obtain the following theorem:

Theorem 8. *Let the data (g, ψ, λ, r) be a complete r-Newton-Ricci-Bourguignon almost soliton on Lagrangian submanifold of complex form $\bar{M}^n(c)$ of constant sectional curvature c, such that P_r is bounded from above and its potential function $\psi : M^n \longrightarrow \mathbb{R}$ is non-negative and $\psi \in L^p(M)$ for some $p > 1$. Then we have*

1. *$\rho > \frac{1}{n}$ and $\lambda \geq (1 - n\rho)(n-1)(c + H^2)$, then M^n is totally geodesic, with $\lambda = (1 - n\rho)(n-1)c$, $n\rho = 1$ and scalar curvature $\tau = n(n-1)c$.*

2. *If $\rho > \frac{1}{n}$ and $\lambda \geq (1 - n\rho)(n-1)(c + H^2)$, then M^n is totally umbilical. In particular, the scalar curvature $\tau = n(n-1)K_M$ is constant, where $K_M = \frac{\lambda}{(1-n\rho)(n-1)}$ is the sectional curvature of M^n.*

Proof. Let us begin observing that by equation (23) and assumption on λ we get

$$L_r\psi = n[\lambda + (n-1)(1 - n\rho)c - nH^2] + |A|^2 \geq 0. \tag{27}$$

Since we are assuming that P_r is bounded from above, there is a positive constant β such that

$$\beta\Delta\psi \geq L_r\psi \geq 0. \tag{28}$$

Using Lemma (3), we have that ψ must constant. Therefore $L_r\psi = 0$, and equation (27) we conclude that M^n is totally geodesic, $\lambda = (1 - n\rho)(n-1)c$, $\rho = \frac{1}{n}$ and $\tau = n(n-1)c$, proving assertion (1), reasoning as in Theorem (5), it is easy to prove assertion (2).

3. 1-Newton-Ricci-Bourguignon almost soliton

This section, explores the study of 1-Newton-Ricci-Bourguignon almost soliton on Lagragain submanifold of complex space form immersed into a locally symmetric space.

We know that a complex space is called locally symmetric if all the covariant derivative components of its curvature tensor vanishes identically. In this aspect, such spaces exhibit an specific extension of constant curvature spaces.

Let M^n be a Lagrangian submanifold of complex manifold $\bar{M}^n$. In what follows we initiated our curvature constraint, which will be consider in the prime results of this segment. More precisely, we will consider that there is a constant μ such that the sectional curvature C_K of the ambient space $\bar{M}^n$ satisfies the following equality:

$$C_K(\eta, t) = \frac{\mu}{n}, \tag{29}$$

where the vectors $\eta \in T^{\perp}(M)$ and $t \in T(M)$.

A Riemannian manifold $\bar{M}^n$ of constant sectional curvature c is a locally symmetric space and it is easy to observe that the curvature condition (29) is satisfies for every hypersurface $\mathbb{H}^n$ immersed into $\bar{M}^n$, with $\frac{\mu}{n} = c$. Therefore in some extent our consideration is natural generalization of the case where the ambient space has constant sectional curvature. Moreover, when the ambient manifold is a Riemannian product of two Riemannian manifolds of constant sectional curvature, say $M = M_1(k_1) \times M_2(k_2)$, the M us locally symmetric and, if $k_1 = 0$ and $k_2 \geq 0$, then every hypersurface of the type $\mathbb{H} = \mathbb{H}_1 \times M_2(k_2)$, where $\mathbb{H}_1$ is an orientable and connected hypersurface immersed in $M_1(k_2)$, satisfied the curvature constraint (29) with $\mu = 0$.

Let $\bar{M}^n$ be a locally symmetric complex manifold satisfying condition (29) and let $\{E_1, ...E_n\}$ be an orthonormal frame on $T(M)$. Then, its scalar curvature $\bar{\tau}$ is given by

$$\bar{\tau} = \sum_{i=1}^{n} Ric(E_i, E_i) \tag{30}$$

$$= \textstyle\sum_{i,j=1}^{n} g(Ric(E_i, E_j)E_j, E_j) + 2\sum_{i=1}^{n} g(Ric(E_{n+1}, E_i)E_{n+1}, E_i)$$

$$\bar{\tau} = \sum_{i,j=1}^{n} g(Ric(E_i, E_j)E_j, E_j) + 2\mu. \tag{30}$$

Moreover, it is well know fact that scalar curvature of a locally symmetric complex manifold is constant. Thus $= \sum_{i,j=1}^{n} g(Ric(E_i, E_j)E_j, E_j)$ is a constant naturally attached to a locally symmetric complex manifold satisfying (29). Therefore, for the sake of simplicity, we choose the following notation $\bar{\tau}_S := \frac{1}{n(n-1)} \sum_{i,j=1}^{n} g(Ric(E_i, E_j)E_j, E_j)$. It is worth pointing out that when M^{n+1} is a space of constant sectional curvature, the the constant $\bar{\tau_S}$ agrees with its sectional curvature.

The following results are the generalization of Theorem (2) for the context of r-Newton-Ricci-Bourguinon almost soliton on Lagrangian submanifold of complex manifold.

Theorem 9. *Let the data $(g, \psi, \lambda, 1)$ be a complete r-Newton-Ricci-Bourguignon almost soliton on Lagrangian submanifold of complex form $\bar{M}^n(c)$ of constant sectional curvature c, with bounded second fundamental form and potential function $\psi : M^n \longrightarrow \mathbb{R}$ such that $|\nabla\psi| \in L^1(M)$ and let M^{n+1} be a locally symmetric Einstein manifolds satisfying the curvature condition (29). Then we have*

1. If $\bar{\tau_S} \leq 0$, $\lambda > 0$ and $\rho > \frac{1}{n}$, then M^n can not be H-minimal.
2. If $\bar{\tau_S} < 0$, $\lambda \geq 0$ and $\rho > \frac{1}{n}$, then M^n can not be H-minimal.
3. If $\bar{\tau_S} = 0$, $\lambda \geq 0$,$\rho > \frac{1}{n}$, and M^n is H-minimal, then M^n is totally geodesic.

Proof. For the proof of (1) considering the proof of theorem (2) by contradiction that M^n is minimal. Then by our assumption on the constant $\rho > \frac{1}{n}$ we get from the equation (8) that the scalar curvature of M^n satisfies $\tau \leq 0$, which implies that $L_r(\psi) = n\lambda - (1 - n\rho)\tau \geq 0$.

On the other side, we have the differential operator L_1 satisfies

$$L_1\psi = div_M(P_1\nabla\psi) - g(div_M P_1, \nabla\psi). \tag{31}$$

In particular, taking an orthonormal frame $\{E_1, ..., E_n\}$ in $T(M)$ and denoting by N the orientation of M^n, it follows from Lemma 25 of [2] that

$$g(div_M P_1, \nabla\psi) = \sum_{i=1}^{n} g(R(N, E_i)\nabla\psi, E_i) = Ric(N, \nabla\psi). \tag{32}$$

Since M^n is consider to be Einstein we conclude by equation (31) combined with the equation (32), we arrive at

$$L_1\psi = div_M(P_1\nabla\psi). \tag{33}$$

Moreover, as we have observed from Theorem (2) we obtain from our consideration on second fundamental form that $|\nabla\psi| \in L^1(M)$. Therefore, we are in position to apply Lemma (1) to conclude that $L_r\psi = 0$, which gives a contradiction. Finally, reasoning as above it is easy to prove (2) and (3).

Now, we obtaining the analogous results to Theorem (5) in the case where $r = 1$ and the ambient space is locally symmetric. Particularly, we obtain the following theorem:

Theorem 10. *Let the data $(g, \psi, \lambda, 1)$ be a complete r-Newton-Ricci-Bourguignon almost soliton on Lagrangian submanifold of complex form $\bar{M}^n(c)$ of constant sectional curvature c, with bounded second fundamental form and potential function $\psi : M^n \longrightarrow \mathbb{R}$ such that $|\nabla\psi| \in L^1(M)$ and let M^{n+1} be a locally symmetric Einstein manifolds satisfying the curvature condition (29). Then we have*

1. *$\rho > \frac{1}{n}$ and $\lambda \geq (1 - n\rho)(n-1)\bar{\tau_S} + nH^2$, then M^n is totally geodesic, with $\lambda = (1 - n\rho)(n-1)c$, $n\rho = 1$ and scalar curvature $\tau = n(n-1)c$,*
2. *If $\rho > \frac{1}{n}$ and $\lambda \geq (1 - n\rho)(n-1)(\bar{\tau_S} + H^2)$, then M^n is totally umbilical. In particular, the scalar curvature $\tau = n(n-1)(\bar{\tau_S} + \kappa^2)$ is constant, where κ is the principal curvature of M^n.*

Proof. The proof is similar as in the proof of Theorem (5). For the sake of completeness, we give the following argument that proves (1). Taking trace in (5) and using definition of the constant $\bar{\tau_S}$, we obtain equation (8) that

$$L_r\psi = n[\lambda + (n-1)(1 - n\rho)\bar{\tau_S} - nH^2] + |A|^2, \tag{34}$$

which implies that $L_1\psi \geq 0$ because our assumption on λ and ρ. Then from Lemma (1) we obtain that $L_r\psi = 0$. Therefore, we conclude from equation (34) that M^n is totally geodesic with $\lambda = (1 - n\rho)(n-1)\bar{\tau_S}$, $n\rho = 1$ and scalar curvature $\tau = n(n-1)\bar{\tau_S}$. This complete the prove of the result.

We completing our paper mentioning the following theorem, which can be furnish from the similar manner used in the proof of Theorem (8) and (10).

Theorem 11. *Let the data $(g, \psi, \lambda, 1)$ be a complete r-Newton-Ricci-Bourguignon almost soliton on Lagrangian submanifold of complex form $\bar{M}^n(c)$ of constant sectional curvature c such that P_r is bounded from above, its potential function $\psi : M^n \longrightarrow \mathbb{R}$ is nonnegative and such that $|\nabla\psi| \in L^1(M)$ for some $p > 1$ and let M^{n+1} be a locally symmetric Einstein manifolds satisfying the curvature condition (29). Then we have*

1. *$\rho > \frac{1}{n}$ and $\lambda \geq (1 - n\rho)(n-1)\bar{\tau_S} + nH^2$, then M^n is totally geodesic, with $\lambda = (1 - n\rho)(n-1)c$, $n\rho = 1$ and scalar curvature $\tau = n(n-1)c$,*
2. *If $\rho > \frac{1}{n}$ and $\lambda \geq (1 - n\rho)(n-1)(\bar{\tau_S} + H^2)$, then M^n is totally umbilical. In particular, the scalar curvature $\tau = n(n-1)(\bar{\tau_S} + \kappa^2)$ is constant, where κ is the principal curvature of M^n.*

We observe that from Theorems (9), (10) and (11) we can replace that the hypothesis that the ambient space M^{n+1} is Einstein under the condition, Ricci curvature tensor identically vanishes.

4. EXAMPLES

Example 1. *For example in an even dimensional Cartesian space $X = \mathbb{R}^{2n}$ equipped with its canonical symplectic form $\omega = \sum_{i=1}^{n} dq_i \wedge dp^i$, standard Lagrangian submanifolds are the submanifolds $\mathbb{R}^n \hookrightarrow \mathbb{R}^{2n}$ of fixed values of the $\{q_i\}_{i=1}^{n}$ coordinates. Indeed locally, every Lagrangian submanifold looks like this.*

Example 2. *Let us consider the standard immersion of $\mathbb{S}^n$ in $\mathbb{S}^{n+1}$, which we know that its is totally geodesic. In particular, $P_r = 0$ for all $1 \leq r \leq n$, and choosing $\lambda = \frac{(n-1)}{n}$, where scalar curvature $R = (n-1)$ and $\rho = \frac{1}{n}$, we obtain that the immersion satisfies equation (5).*

Example 3. *Let $\mathbb{S}^{2n}(1)$ be the unit sphere in the Euclidean space $\mathbb{R}^{2n}$ and $\psi : \mathbb{S}^{2n}(1) \hookrightarrow \mathbb{R}^{2n}$ the natural embedding with induced metric g on $\mathbb{S}^{2n}(1)$, then $(\mathbb{S}^{2n}(1), J, g)$ is a complex structure. It is well known that this complex structure gives a complex space form $\mathbb{S}^{2n}(1)$ and its a complex space form space form with constant sectional curvature $c = 1$. Let $i : M^n \longrightarrow \mathbb{S}^{2n}(1) \subset \mathbb{R}^n \cong \mathbb{R}^{2n}$ an immersion of a smooth n-dimensional manifold M^n in to unit sphere.*

For a constant $t \in \mathbb{R}^n$, according to [4], by choosing the functions $\bar{f}_t$ on $\mathbb{R}^{2n}$ such that

$$\bar{f}_l(t) = -g(t,l) + 2m - 1 \quad \text{and} \quad \psi_l(t) = -\bar{f}_l + c, \quad \bar{f}_l := i * \tilde{f}_l \in C^{\infty}(S^{2n})$$

where $l \in \mathbb{S}^{2n}(1)$, $t \neq 0$, $c \in \mathbb{R}^{2n}$ and $t = (t_1, ...t_{2n}) \in \mathbb{S}^{2n}$ is the position vector, we have that $(\mathbb{S}^{2n}, g, \nabla\psi_l, \lambda_l)$ satisfies

$$Ric + Hess\psi_l = (\lambda_l - \rho R)g. \tag{35}$$

On the other hand, it is well know that $\mathbb{S}^{m+1}$ is totally umbilical with r-th mean curvature $H_r = 1$ and second fundamental form $B = I$. In particular, for every $0 \leq m$ the Newton tensor are given by

$$P_r = \alpha I, \tag{36}$$

where $\alpha = \sum_{j=0}^{r}(-1)^{r-j}\binom{m}{j}$. Hence, taking smooth function $\psi = \alpha^{-1}\psi_l$ we get that subamnifold satisfied equation (7).

Example 4. *We recall the Gaussian soliton is the Euclidean space $\mathbb{R}^m$ endowed with its standard metric $|.|$ admits the standard complex space form and the potential function $\psi(x) = \frac{\lambda}{4}|x|^2$. It is well know that the spheres of the hyperbolic space $\mathbb{H}^{m+1}$ are totally umbilical subamnifold isometric to $\mathbb{R}^m$, having r-th mean curvature $H_r = 1$ and second fundamental form $B = I$. Hence, we can reason as in example (2) to verify that the spheres $\mathbb{S}^n \hookrightarrow \mathbb{H}^{m+1}$ satisfies equation (7).*

Example 5. *Since the canonical immersion $\mathbb{S}^n \hookrightarrow \mathbb{S}^n \times \mathbb{R}$ is totally geodesic, proceeding as in Example (2) we see that this immersion satisfies equation (5) for all $1 \leq r \leq n$ and $\lambda = \frac{\rho}{n}$.*

References

[1] T. Aubin, *Métriques Riemanniennes et courbure*, J. Differential Geometry 4, (1970), 383-424.

[2] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*. 2nd ed. Progress in Math., 203, Birkhauser, (2010).

[3] A. Barros, J. N. Gomes, E. Jr. Ribeiro, *Immersion of almost Ricci solitons into a Riemannian manifold*, Mat. Contemp. 40 (2011), 91-102.

[4] A. Barros, E. Jr. Ribeiro, *Some characterizations for compact almost Ricci solitons*, Proc. Amer. Math. Soc. 140 (2012), 1033-1040.

[5] B. Y. Chen, *Riemanian Geometry of Lagrangian submanifolds*, Taiwanse J. Math. Vol. 5, 4 (2001), 681-723.

[6] B. Y. Chen, *Lagrangian H-umbilical submanifolds of para-Kähler manifolds*, Taiwanse J. Math., 15, 6 (2011), 2483-2502.

[7] G. Catino, L. Cremaschi, Z. Djadli, C. Mantegazza, L. Mazzieri, *The Ricci-Bourguignon flow, Pacific J. Math.*, 287 (2017) 337-370.

[8] G. Catino and L. Mazzieri, *Gradient Einstein solitons*, Nonlinear Anal., 132 (2016), 66-94.

[9] J. T. Cho and M. Kimura, *Ricci solitons and Real hypersurfaces in a complex space form*, Tohoku Math.J., 61 (2009), 205-212.

[10] A. W. Cunha, E. L. de Lima, H. F. de Lima, *r-Almost Newton-Ricci soliton immersed into a Riemannian manifold*, J. Math. Anal. App., 464 (2018), 546-556.

[11] H. Cao, D. Zhou, *On complete gradient shrinking Ricci solitons*, J.Differential Geom. 85 (2010) 175-185.

[12] A. Caminha, P. Sousa, F. Camargo, *Complete foliations of space forms by hypersurfaces*, Bull. Braz. Math. Soc. 41 (2010) 339-353.

[13] H. Cao, *Recent progress on Ricci solitons*, Adv. Lect. Math. (ALM) 11 (2009) 1-38.

[14] F. Dillen and L. Vrancken, *C-totally real submanifolds of Sasakian space forms*, J. Math. Pures Appl. 69 (1990) 85–93.

[15] N. Ejiri, *Totally Real Minimal Immersions of n-dimensional Real Space Forms into n-dimensional Complex Space Forms*, Proc. of American Math. Society. 84, 2 (1982) 243-246.

[16] R. S. Hamilton, *The Ricci flow on surfaces, Mathematics and general relativity*, (Santa Cruz. CA, 1986), Contemp. Math. 71, Amer. Math. Soc., (1988), 237-262.

[17] Y. G. Oh, *Volume minimization of Lagrangian submanifolds under Hamiltonian deformations*, Math. Zeit. 212(1993), 175-192.

[18] S. Pigola, M. Rigoli, M. Rimoldi, A. Setti, *Ricci almost solitons*, Ann. Sc. Norm. Super. Pisa Cl. Sci. 5, 10 (2011) 757-799.

[19] H. Rosenberg, *Hypersurfaces of constant curvature in space forms*, Bull. Soc. Math. 117 (1993) 217-239.

[20] M. D. Siddiqi, *η-Einstein soliton in a δ- Lorentzian Trans Sasakian manifolds, Mathematical Advances in Pure and Applied Sciences*, 1, 1 (2018) 27-38.

[21] M. D. Siddiqi, *Ricci ρ-soliton and geometrical structure in a dust fluid and viscous fluid sapcetime*, Bulg. J. Phys. 46 (2019), 163-173.

[22] M. D. Siddiqi, S. A. Siddqui, S. A, *Conformal Ricci soliton and Geometrical structure in a perfect fluid spacetime,* Int. J. Geom. Methods Mod. Phys, (2020) 2050083, (18 pages) https://doi.org/10.1142/S0219887820500838.

[23] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. 88 (1968) 62-105.

[24] W. Wylie, *Complete shrinking Ricci solitons have finite fundamental group*, Proc. Amer. Math. Soc. 136 (2008) 1803-1806.

[25] K. Yano, M. Kon, *Structures on manifolds, Series in Pure Mathematics*, Vol 3, World Scientific, Singapore, 1984.

Mohd Danish Siddiqi
Department of Mathematics, Faculty of Science,
Jazan University, Jazan
Kingdom of Saudi Arabia
email: *msiddiqi@jazanu@edu.sa*