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ON THE L_1 NORM OF THE DIRICHLET KERNEL ON THE GROUP OF 2-ADIC INTEGERS

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Abstract. The main result in this paper provides an estimate of the L_1 norm of the Dirichlet kernel on the group of 2-adic integers. As an application we derive a description of partial sums of Fourier series related to functions from the space H_1.

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1. Introduction

The L_1 norm of the generalized Walsh Dirichlet kernel D_n, on bounded Vilenkin groups was studied in [4] and [2], where it was estabilshed that it is dominated by some precisely identified unbounded sequences of natural numbers depending on the index n of the function D_n. Moreover, it known that the L_1 norm of partial sums of Fourier series related to some functions from the dyadic Hardy space H_1 diverges. Meanwhile, [2, Theorem 2] provides a sharp result proving that these partial sums can also be dominated for functions from H_1.

Our aim is to prove an analogue of [2, Theorem 2] on the group of 2-adic integers. Our techniques are different, since they are based on a decomposition of the Dirichlet kernel, as a linear combination of test functions with mutually disjoint supports proved in Lemma 1.

Denote by $I = [0, 1)$ the unit interval and for all $x \in I$ $n \in \mathbb{N}$, let $I_n(x) \subset I$ be the unit dyadic interval of the form $[\frac{i}{2^n}, \frac{i+1}{2^n})$ containing x, where i is a nonnegative integer depending on x. $I_n(0)$ is denoted by I_n. For every $x \in I$, let $x = \sum_{n=0}^{\infty} x_n 2^{-n-1}$ be its dyadic expansion, where $x_n \in \{0, 1\}$. The group $(I, +)$ is called the group of 2-adic integers.

For every nonnegative integer n and all $x = \sum_{n=0}^{\infty} x_n 2^{-n-1} \in I$, let

$$v_{2^n}(x) = \exp 2\pi i \left(\frac{x_n}{2} + \ldots + \frac{x_0}{2^{n+1}} \right).$$

If we define $\theta_n(x) = \sum_{i=0}^{n} x_i 2^i$, then it can be easily seen that

$$v_{2^n}(x) = \exp 2\pi i \left(\frac{\theta_n(x)}{2^{n+1}}\right). \tag{1}$$

If a nonnegative integer n has the dyadic expansion $n = \sum_{i=0}^{\infty} n_i 2^i$, then set $v_n = \prod_{i=0}^{\infty} (v_{2^i})^{n_i}$.

The Dirichlet kernel and the partial sums of the Fourier series of any integrable function f are respectively defined as follows

$$D_n = \sum_{k=0}^{n-1} v_k, \quad S_n f(y) = \int D_n(y-x) f(x) dx.$$

A function $a \in L_\infty$ is said to be an atom if it is supported on some interval $I_N(y)$, for some nonnegative integer N and $y \in I$, $\int_{I_N(y)} a = 0$, and such that $\|a\|_\infty \leq 2^N$. The space H_1 consists of functions f that can be put in the atomic decomposition $f = \sum\limits_{i=1}^{\infty} \lambda_i a_i$, where $\sum\limits_{i=1}^{\infty} |\lambda_i| < \infty$ and a_i is an atom for $i \geq 1$. The norm in H_1 is defined by $\|f\|_{H_1} = \inf \sum\limits_{i=1}^{\infty} |\lambda_i|$, where the infimum is taken over all the atomic decompositions.

For every nonnegative integer $j = \sum_{i=0}^{\infty} j_i 2^i$, define $z_j \in I$ in the form $z_j = \sum_{i=0}^{\infty} j_i 2^{-i-1}$. It can be seen that if $|j| = \lfloor \log_2 j \rfloor$, then

$$\theta_n(z_j) = j, \forall n \geq 2^{|j|+1}. \tag{2}$$

Moreover, for every nonnegative integer N, we have $I = \biguplus\limits_{j=0}^{2^N-1} I_N(z_j)$.

The following formulae were proved in [5], for every positive integer k and nonnegative integer n

$$D_{2^k} = \begin{cases} 2^k, & x \in I_k; \\ 0, & x \in I \setminus I_k. \end{cases} \tag{3}$$

$$D_n(x) = v_n(x) \sum_{j=0}^{\infty} n_j (-1)^{x_j} D_{2^j}(x). \tag{4}$$

2. Main results

Lemma 1. *Let n be a positive integer having the dyadic representation $n = 2^{N_1} + \ldots + 2^{N_t}$, where $N_1 < N_2 < \ldots < N_t$ and $N_t = |n|$. Then, $D_n(x)$ can be written in the form*

$$D_n(x) = D_{2^{N_t}}(x) + v_{2^{N_t}}(x) \sum_{j=0}^{2^{N_t}-1} A_{n,j} D_{2^{N_t}}(x - z_j), \tag{5}$$

where

$$A_{n,0} := \sum_{i=1}^{t-1} 2^{N_i - N_t}, \tag{6}$$

and

$$A_{n,j} := v_{2^{N_{t-1}}}(z_j) \dots v_{2^{N_{i+1}}}(z_j) \left[2^{N_i - N_t} + v_{2^{N_i}}(z_j) \sum_{s=1}^{i-1} 2^{N_s - N_t} \right], \tag{7}$$

if $j = 0 \,(mod\, 2^{N_i})$ *and* $j \neq 0 \,(mod\, 2^{N_{i+1}})$ *for some* $i \in \{1, \dots, t-1\}$*. For* $j \neq 0 \,(mod\, 2^{N_1})$*,* $A_{n,j} = 0$*.*

Proof. Formula (4) can be written in the form

$$D_n(x) = v_n(x) \sum_{i=1}^{t} (-1)^{x_{N_i}} D_{2^{N_i}}(x) = \sum_{i=1}^{t} v_{2^{N_t}}(x) \dots v_{2^{N_i}}(x) (-1)^{x_{N_i}} D_{2^{N_i}}(x). \tag{8}$$

If for some $i \in \{1, \dots, t-1\}$, $x \in I_{N_i} \setminus I_{N_i+1}$, then $x_{N_i} = 1$ and $x_k = 0$ for all $k < N_i$. Therefore, in this case $v_{2^{N_i}}(x) = \exp 2\pi i (\frac{x_{N_i}}{2}) = \exp \pi i = -1 = (-1)^{x_{N_i}}$.

On the other hand, if $x \in I_{N_i+1}$, then $x_k = 0$ for all $k \leq N_i$, which means that $v_{2^{N_i}}(x) = 1 = (-1)^{x_{N_i}}$. Hence, (8) becomes

$$D_n(x) = D_{2^{N_t}}(x) + \sum_{i=1}^{t-1} v_{2^{N_t}}(x) \dots v_{2^{N_{i+1}}}(x) D_{2^{N_i}}(x)$$

$$= D_{2^{N_t}}(x) + \sum_{i=1}^{t-1} v_{2^{N_t}}(x) \dots v_{2^{N_{i+1}}}(x) \sum_{\substack{j \in \{0, \dots, 2^{N_t}-1\} \\ j = 0 \ (mod\ 2^{N_i})}} 2^{N_i - N_t} D_{2^{N_t}}(x - z_j)$$

$$= D_{2^{N_t}}(x) + \sum_{i=1}^{t-1} v_{2^{N_t}}(x) 2^{N_i - N_t} D_{2^{N_t}}(x) + \sum_{i=1}^{t-1} v_{2^{N_t}}(x) \dots v_{2^{N_{i+1}}}(x) \sum_{\substack{j \in \{1, \dots, 2^{N_t}-1\} \\ j = 0 \ (mod\ 2^{N_i})}} 2^{N_i - N_t} D_{2^{N_t}}(x - z_j)$$

$$= D_{2^{N_t}}(x) + \sum_{i=1}^{t-1} v_{2^{N_t}}(x) 2^{N_i - N_t} D_{2^{N_t}}(x) + \sum_{i=1}^{t-1} v_{2^{N_t}}(x) \dots v_{2^{N_{i+1}}}(x)$$

$$\sum_{\substack{j\in\{1,\ldots,2^{N_t}-1\}\\ j=0\ (mod\ 2^{N_i})\\ j\neq 0\ (mod\ 2^{N_{i+1}})}} \left[v_{2^{N_i}}(x)\ldots v_{2^{N_2}}(x)2^{N_1-N_t}+\ldots+2^{N_i-N_t}\right] D_{2^{N_t}}(x-z_j).$$

Since for all $x\in I$ and $j\in\{1,\ldots,2^{N_t}-1\}$, we have that $D_{2^{N_t}}(x-z_j)\neq 0$ only if $x\in I_{N_t}(z_j)$, then if for some $i\in\{1,\ldots,t-1\}$ we have that $j=0\ (mod\ 2^{N_i})$, then in this case $D_{2^{N_t}}(x-z_j)\neq 0$ only if $x\in I_{N_i}$, which implies that $v_{2^{N_{i-1}}}(x)=\ldots=v_{2^{N_2}}(x)=1$. Therefore, the last formula becomes

$$D_n(x)=D_{2^{N_t}}(x)+\sum_{i=1}^{t-1} v_{2^{N_t}}(x)2^{N_i-N_t}D_{2^{N_t}}(x)+\sum_{i=1}^{t-1} v_{2^{N_t}}(x)\ldots v_{2^{N_{i+1}}}(x)$$

$$\sum_{\substack{j\in\{1,\ldots,2^{N_t}-1\}\\ j=0\ (mod\ 2^{N_i})\\ j\neq 0\ (mod\ 2^{N_{i+1}})}} \left[v_{2^{N_i}}(x)\sum_{s=1}^{i-1}2^{N_s-N_t}+2^{N_i-N_t}\right] D_{2^{N_t}}(x-z_j)$$

$$=D_{2^{N_t}}(x)+v_{2^{N_t}}(x)\sum_{i=1}^{t-1}2^{N_i-N_t}D_{2^{N_t}}(x)$$

$$+v_{2^{N_t}}(x)\sum_{i=1}^{t-1}\sum_{\substack{j\in\{1,\ldots,2^{N_t}-1\}\\ j=0\ (mod\ 2^{N_i})\\ j\neq 0\ (mod\ 2^{N_{i+1}})}} v_{2^{N_{t-1}}}(z_j)\ldots v_{2^{N_{i+1}}}(z_j)\left[v_{2^{N_i}}(z_j)\sum_{s=1}^{i-1}2^{N_s-N_t}+2^{N_i-N_t}\right]D_{2^{N_t}}(x-z_j),$$

because $D_{2^{N_t}}(x-z_j)\neq 0$ only if $v_{2^i}(z_j)=v_{2^i}(x)$, for all $i<N_t$

Remark 1. *Using the notations of Lemma 1, it can be easily seen that if for some* $i\in\{1,\ldots,t-1\}$*,* $N_{i+1}\geq N_i+2$*, then if* $j=0\ (mod\ 2^{N_i+1})$ *and* $j\neq 0\ (mod\ 2^{N_{i+1}})$*, we have*

$$2^{N_i-N_t}\leq |A_{n,j}|<2^{N_i-N_t+1},$$

because in this case $v_{2^{N_i}}(z_j)=1$*.*

Remark 2. *Let* $j=0\ (mod\ 2^{N_i})$ *and* $j\neq 0\ (mod\ 2^{N_i+1})$*. If* $s\leq i$ *is the least positive integer satisfying* $N_s+i-s=N_i$*, then*

$$2^{N_s-N_t-1}<|A_{n,j}|\leq 2^{N_s-N_t},$$

because in this case $v_{2^{N_i}}(z_j)=-1$*.*

Theorem 2. *For every positive integer n define $v(n) = n_0 + \sum_{j=0}^{\infty} |n_{j+1} - n_j|$, where $n = \sum_{j=0}^{\infty} n_j 2^j$, $n_j \in \{0,1\}$. Then,*

$$\frac{v(n)}{2} \leq \|D_n\|_1 \leq 2v(n).$$

Proof. Using the notations of Lemma 1, we define the numbers $r \in \{1,\ldots,t\}$, L_i, $L_i' \in \{1,\ldots,t\}$, where $i \in \{1,\ldots,r\}$ such that

$$1 = L_1 \leq L_1' < L_2 \leq L_2' < \ldots < L_r \leq L_r' = t,$$

$$\forall s \in \{1,\ldots,r\},\ \forall i \in \{L_s,\ldots,L_s'-1\},\ N_{i+1} = N_i + 1, \tag{9}$$

and

$$\forall s \in \{1,\ldots,r-1\},\ L_{s+1} = L_s' + 1,\ N_{L_{s+1}} \geq N_{L_s'} + 2.$$

It can be easily seen that $v(n) = 2r$. Moreover, according to (5), we get

$$\|D_n\|_1 = \sum_{j=0}^{2^{|n|}-1} |A_{n,j}| = \sum_{l=0}^{|n|-1} \sum_{\substack{j\in\{0,\ldots,2^{|n|}-1\} \\ j=0\ (mod\ 2^l) \\ j\neq 0\ (mod\ 2^{l+1})}} |A_{n,j}|.$$

Since $A_{n,j} = 0$ for $j \neq 0(mod\ 2^{N_1})$, we get

$$\|D_n\|_1 = \sum_{l=N_1}^{|n|-1} \sum_{\substack{j\in\{0,\ldots,2^{|n|}-1\} \\ j=0\ (mod\ 2^l) \\ j\neq 0\ (mod\ 2^{l+1})}} |A_{n,j}|$$

$$= \sum_{s=1}^{r-1} \sum_{i=L_s}^{L_s'} \sum_{l=N_i}^{N_{i+1}-1} \sum_{\substack{j\in\{0,\ldots,2^{|n|}-1\} \\ j=0\ (mod\ 2^l) \\ j\neq 0\ (mod\ 2^{l+1})}} |A_{n,j}| + \sum_{i=L_r}^{L_r'-1} \sum_{l=N_i}^{N_{i+1}-1} \sum_{\substack{j\in\{0,\ldots,2^{|n|}-1\} \\ j=0\ (mod\ 2^l) \\ j\neq 0\ (mod\ 2^{l+1})}} |A_{n,j}| + |A_{n,0}|.$$

Applying (9) we get

$$\|D_n\|_1 = \sum_{s=1}^{r} \sum_{i=L_s}^{L_s'-1} \sum_{\substack{j\in\{0,\ldots,2^{|n|}-1\} \\ j=0\ (mod\ 2^{N_i}) \\ j\neq 0\ (mod\ 2^{N_i+1})}} |A_{n,j}| + \sum_{s=1}^{r-1} \sum_{l=N_{L_s'}}^{N_{L_{s+1}}-1} \sum_{\substack{j\in\{0,\ldots,2^{|n|}-1\} \\ j=0\ (mod\ 2^l) \\ j\neq 0\ (mod\ 2^{l+1})}} |A_{n,j}| + |A_{n,0}|$$

$$= \sum_{s=1}^{r} \sum_{i=L_s}^{L'_s} \sum_{\substack{j \in \{0,\dots,2^{|n|}-1\} \\ j=0\ (mod\ 2^{N_i}) \\ j \neq 0\ (mod\ 2^{N_i+1})}} |A_{n,j}| + \sum_{s=1}^{r-1} \sum_{l=N_{L'_s}+1}^{N_{L_{s+1}}-1} \sum_{\substack{j \in \{0,\dots,2^{|n|}-1\} \\ j=0\ (mod\ 2^{l}) \\ j \neq 0\ (mod\ 2^{l+1})}} |A_{n,j}| + |A_{n,0}|.$$

According to Remark 2 we have

$$\forall s \in \{1,\dots,r\},\ \forall i \in \{L_s,\dots,L'_s\},\ \forall j = 0\ (mod\ 2^{N_i}),\ j \neq 0\ (mod\ 2^{N_i+1}):$$
$$2^{N_{L_s}-1-|n|} < |A_{n,j}| \leq 2^{N_{L_s}-|n|}.$$

Similarly, by Remark 1 we have

$$\forall s \in \{1,\dots,r-1\},\ \forall l \in \{N_{L'_s}+1,\dots,N_{L_{s+1}}-1\},\ \forall j = 0\ (mod\ 2^{l}),\ j \neq 0\ (mod\ 2^{l+1}):$$
$$2^{N_{L'_s}-|n|} < |A_{n,j}| \leq 2^{N_{L'_s}-|n|+1}.$$

Therefore, we obtain

$$\|D_n\|_1 \leq \sum_{s=1}^{r} \sum_{i=L_s}^{L'_s} 2^{|n|-N_i} 2^{N_{L_s}-|n|} + \sum_{s=1}^{r-1} \sum_{l=N_{L'_s}+1}^{N_{L_{s+1}}-1} 2^{|n|-l} 2^{N_{L'_s}-|n|+1} + 1$$
$$\leq \sum_{s=1}^{r} 2^{|n|-N_{L_s}+1} 2^{N_{L_s}-|n|} + \sum_{s=1}^{r-1} 2^{|n|-N_{L'_s}} 2^{N_{L'_s}-|n|+1} + 1 \leq 4r.$$

In a similar way we get

$$\|D_n\|_1 \geq \sum_{s=1}^{r} \sum_{i=L_s}^{L'_s} 2^{|n|-N_i} 2^{N_{L_s}-|n|-1} + \sum_{s=1}^{r-1} \sum_{l=N_{L'_s}+1}^{N_{L_{s+1}}-1} 2^{|n|-l} 2^{N_{L'_s}-|n|}$$
$$\geq \sum_{s=1}^{r} 2^{|n|-N_{L_s}} 2^{N_{L_s}-|n|-1} + \sum_{s=1}^{r-1} 2^{|n|-N_{L'_s}-1} 2^{N_{L'_s}-|n|} \geq r.$$

Theorem 3. *1. There exists a positive constant C such that for every $f \in H_1$ and $n \in \mathbb{N}$,*

$$\|S_n f\|_{H_1} \leq C v(n) \|f\|_{H_1}.$$

2. *If $(b_n)_n$ is an increasing sequence of positive numbers such that $b_n \to \infty$ and $\left(\frac{v(n)}{b_n}\right)_n$ is unbounded, then there exists some $f \in H_1$ such that $\left(\frac{\|S_n f\|_1}{b_n}\right)_n$ is unbounded.*

Proof. (1) From Theorem 2 we get

$$\|S_n f\|_1 \leq \|D_n\|_1 \|f\|_1 \leq 2v(n)\|f\|_{H_1}.$$

Besides, as noticed in the proof of [2, Theorem 2],

$$\|S_n f\|_{H_1} \leq \|f\|_{H_1} + \|S_n f\|_1.$$

Hence,

$$\|S_n f\|_{H_1} \leq Cv(n)\|f\|_{H_1}.$$

(2) We use the same construction made in [2, Theorem 2]. From the assumptions made on the sequence $(b_n)_n$, it contains a subsequence $(b_{n_k})_k$ such that

$$\sum_{k=1}^{\infty} \frac{\sqrt{b_{n_k}}}{\sqrt{v(n_k)}} < +\infty. \tag{10}$$

Define $f = \sum_i \lambda_i a_i$, where

$$\lambda_i = \frac{\sqrt{b_{n_i}}}{\sqrt{v(n_i)}},$$

and

$$a_i = D_{2^{n_i+1}} - D_{2^{n_i}}.$$

Since each a_k, $k \in \mathbb{N}$, is an atom then from (10) we can see that $f \in H_1$.

From the definition of Fourier series we get that

$$S_{n_k} f = S_{2^{|n_k|}} f + S_{n_k} f - S_{2^{|n_k|}} f.$$

By the construction of f we get that

$$S_{n_k} f - S_{2^{|n_k|}} f = \lambda_k \left(D_{n_k} - D_{2^{|n_k|}}\right)$$

and

$$S_{2^{|n_k|}} f = \sum_{i=1}^{k-1} \lambda_i a_i.$$

It follows that

$$\|S_{n_k} f\|_1 \geq \lambda_k \|D_{n_k}\|_1 - \lambda_k \|D_{2^{|n_k|}}\|_1 - \sum_{i=1}^{k-1} \lambda_i \|a_i\|_1.$$

Hence, according to Theorem 2

$$\|S_{n_k} f\|_1 \geq \lambda_k \frac{v(n_k)}{2} - \lambda_k - \sum_{i=1}^{k-1} \lambda_i \geq \lambda_k \frac{v(n_k)}{2} - \sum_{i=1}^{k} \lambda_i.$$

Therefore,

$$\left\| \frac{S_{n_k} f}{b_{n_k}} \right\|_1 \geq \frac{\sqrt{v(n_k)}}{2\sqrt{b_{n_k}}} - \sum_{i=1}^{\infty} \lambda_i \to \infty, \ k \to \infty.$$

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