SOME PROPERTIES OF TUBULAR SURFACES IN $\mathbb{E}^3$

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Abstract. In this article, we consider tubular surfaces in Euclidean 3-space. We obtain the necessary and sufficient conditions for tubular surfaces in Euclidean 3-space to be semi-parallel and of the first kind of pointwise 1-type Gauss map. Also we study the tubular surface in Euclidean 3-space such that its mean curvature vector $\vec{H}$ satisfies $\Delta\vec{H} = \lambda\vec{H}$ for some differentiable functions $\lambda$.

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1. Introduction

The notion of finite type submanifolds introduced by B. Y. Chen during the late 1970’s has become an useful tool for investigating and characterizing submanifolds of Euclidean or pseudo-Euclidean space ([1], [3]). Afterwards, the notion was extended to differential maps, in particular, to the Gauss map of submanifolds. Especially, if an oriented submanifold $M$ has 1-type Gauss map $G$, then $G$ satisfies $\Delta G = \lambda (G + C)$ for a non-zero constant $\lambda$ and a constant vector $C$, where $\Delta$ is the Laplace operator. Extending this kind of property which is a typical character valid on helicoids, catenoids and several rotational surfaces, Y. H. Kim defined the notion of submanifolds of Euclidean space with pointwise 1-type Gauss map as follows:

Definition 1. [10] An oriented submanifold $M$ of Euclidean space is said to have pointwise 1-type Gauss map if its Gauss map $G$ satisfies

$$\Delta G = f (G + C)$$  \hspace{1cm} (1)

for a non-zero smooth function $f$ and a constant vector $C$.

A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector $C$ in (1) is the zero vector. Otherwise, a submanifold with pointwise 1-type Gauss map is said to be of the second kind. Many interesting submanifolds
with pointwise 1-type Gauss map have been studied from different viewpoint and different spaces ([4], [5], [6], [8], [9]).

On the other hand, the submanifold \( M \) is called semi-parallel (semi-symmetric [11]) if \( R \cdot h = 0 \) where \( R \) denotes the curvature tensor of Vander Waerden-Bortoletti connection \( \nabla \) of \( M \) and \( h \) is the second fundamental form of \( M \). This notion is an extrinsic analogue for semi-symmetric spaces, i.e. Riemannian manifolds for which \( R \cdot R = 0 \) and a direct generalization of parallel submanifolds, i.e. submanifolds for which \( \nabla h = 0 \) [7], [12].

In the present paper, we consider tubular surfaces in Euclidean 3-space to be semi-parallel and to have the first kind of pointwise 1-type Gauss map. We prove the following theorems:

**Theorem 1.** Let \( M \) be a tubular surface in \( \mathbb{E}^3 \). Then \( M \) has the first kind of pointwise 1-type Gauss map if and only if \( M \) is a cylindrical surface.

**Theorem 2.** Let \( M \) be a tubular surface in \( \mathbb{E}^3 \). Then \( M \) is semi-parallel if and only if \( M \) is a cylindrical surface.

Also we consider the tubular surface in Euclidean 3-space such that its mean curvature vector \( \vec{H} \) satisfies \( \Delta \vec{H} = \lambda \vec{H} \) for some differentiable functions \( \lambda \) and we prove the following theorems:

**Theorem 3.** Let \( M \) be a tubular surface in \( \mathbb{E}^3 \). Then the mean curvature vector \( \vec{H} \) of \( M \) satisfying \( \Delta \vec{H} = \lambda \vec{H} \) for some differentiable functions \( \lambda \) if and only if \( M \) is a cylindrical surface.

2. Preliminaries

We recall some well-known formulas for the surfaces in \( \mathbb{E}^3 \). Let \( M \) be a surface of \( \mathbb{E}^3 \), the standard connection \( D \) on \( \mathbb{E}^3 \) induces the Levi-Civita connection \( \nabla \) on \( M \). We have the following Gauss formula

\[
D_X Y = \nabla_X Y + h(X,Y),
\]

and the Weingarten formula

\[
D_X \xi = -A_\xi X + \nabla_X \xi,
\]

where \( X, Y \in \Gamma(TM) \) and \( \xi \in \Gamma(TM^\perp) \). Then \( \nabla \) is the Levi-Civita connection of \( M \), \( h \) is the second fundamental form, \( A_\xi \) is the shape operator, and \( \nabla^\perp \) is the normal connection. We note that

\[
\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle.
\]
The normal curvature tensor $\perp R$ is defined by
$$\perp R(X,Y)\xi = \perp \nabla_X \perp \nabla_Y \xi - \perp \nabla_Y \perp \nabla_X \xi - \perp \nabla_{[X,Y]}\xi,$$
where $X,Y \in \Gamma(TM)$ and $\xi \in \Gamma(TM^\perp)$. Taking the normal part of the following equation
$$DXDY\xi - DYDX\xi - D[X,Y]\xi = 0$$
where $X,Y \in \Gamma(TM)$ and $\xi \in \Gamma(TM^\perp)$, we get the Ricci equation
$$\left< \perp R(X,Y)\xi, \eta \right> = \left< A_\eta X, A_\xi Y \right> - \left< A_\xi X, A_\eta Y \right>$$
where $\eta \in \Gamma(TM^\perp)$.

The mean curvature vector field $-\vec{H}$, the mean curvature $H$ and the Gauss curvature of $M$ are given respectively by
$$-\vec{H} = \frac{1}{2} \text{trace} h \quad \text{and} \quad K = \det A.$$
A surface is called minimal if $H = 0$ identically. A surface is called flat if $K = 0$ identically ([2]).

Let $\overline{R} \cdot h$ be the product tensor of the curvature tensor $\overline{R}$ with the second fundamental form $h$. The surface $M$ is said to be semi-parallel if $\overline{R} \cdot h = 0$, i.e. $\overline{R}(X_i, X_j) \cdot h = 0$ ([11]). Now, we give the following result.

**Lemma 4.** ([7]) Let $M \subset \mathbb{E}^n$ be a smooth surface given with the patch $M(u,v)$. Then the following equalities hold:

$$\left( \overline{R}(X_1, X_2) \cdot h \right)(X_1, X_1) = \left( \sum_{\alpha=1}^{n-2} h_{11}^{\alpha} (h_{22}^{\alpha} - h_{11}^{\alpha}) + 2K \right) h(X_1, X_2)$$

$$+ \sum_{\alpha=1}^{n-2} h_{11}^{\alpha} h_{12}^{\alpha} (h(X_1, X_1) - h(X_2, X_2)),$$

$$\left( \overline{R}(X_1, X_2) \cdot h \right)(X_1, X_2) = \left( \sum_{\alpha=1}^{n-2} h_{12}^{\alpha} (h_{22}^{\alpha} - h_{11}^{\alpha}) \right) h(X_1, X_2)$$

$$+ \left( \sum_{\alpha=1}^{n-2} h_{12}^{\alpha} h_{12}^{\alpha} - K \right) (h(X_1, X_1) - h(X_2, X_2)),$$

$$\left( \overline{R}(X_1, X_2) \cdot h \right)(X_2, X_2) = \left( \sum_{\alpha=1}^{n-2} h_{22}^{\alpha} (h_{22}^{\alpha} - h_{11}^{\alpha}) - 2K \right) h(X_1, X_2)$$

$$+ \sum_{\alpha=1}^{n-2} h_{22}^{\alpha} h_{12}^{\alpha} (h(X_1, X_1) - h(X_2, X_2))$$
where $K$ is the Gauss curvature of the surface.

The Laplacian $\triangle$ on $M$ is given by

$$
\triangle = -\frac{1}{\sqrt{\det (g^{ij})}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_{i}} \left( \sqrt{\det (g^{ij})} g^{ij} \frac{\partial}{\partial x_{j}} \right)
$$

where $(g^{ij})$ is the inverse matrix of $(g_{ij})$, which is the local components of the metric on $M$.

3. Tubular Surface in $\mathbb{E}^3$

In this section, we study some geometrical properties of tubular surfaces in $\mathbb{E}^3$. We prove the main theorems theorem 1 and theorem 2 and related results.

A canal surface $M$ in $\mathbb{E}^3$ is an immersed surface swept out by a sphere moving along a curve $\alpha = \alpha(s)$ or by a particular circular cross-section of the sphere along the same path ([13]). Due to the generating process of canal surfaces, the parametric formula of $M$ can be given as follows:

$$
M(s,u) = \alpha(s) + r'((\cos u)N(s) + (\sin u)B(s))
$$

where the curve $\alpha(s)$ is called the spine curve (center curve) parametrized by arc-length $s$ and $r(s)$ is called the radial function of $M$. Here $\{T,N,B\}$ is Frenet frame of $\alpha(s)$. In particular, if $r(s)$ is a constant, then $M$ is called a tubular surface.

Let $\alpha : I \to \mathbb{E}^3$ be a unit-speed planar curve satisfying

$$
T'(s) = \kappa(s)N(s),
$$
$$
N'(s) = -\kappa(s)T(s)
$$

and $M$ be a tubular surface whose spine curve is $\alpha$ as follows

$$
M(s,u) = \alpha(s) + r((\cos u)N(s) + (\sin u)B)
$$

where $B$ is constant vector in $\mathbb{E}^3$. Differentiating (3) with respect to $s$ and $u$, respectively, we get

$$
M_{s}(s,u) = (1 - r\kappa \cos u)T,
$$
$$
M_{u}(s,u) = -r(\sin u)N + r(\cos u)B.
$$
Here without lost of generality, we assume that $1 - r\kappa \cos u > 0$ for the regularity of the surface $M$. Thus, an orthonormal tangent bases on $M$ is given by

$$e_1 = \frac{M_s}{\|M_s\|} = T(s), \quad (6)$$

$$e_2 = \frac{M_u}{\|M_u\|} = -(\sin u) N(s) + (\cos u) B. \quad (7)$$

From (6) and (7), we find

$$e_3 = e_1 \times e_2 = -(\cos u) N(s) - (\sin u) B. \quad (8)$$

By covariant differentiation with respect to $e_1$ and $e_2$, a straightforward calculation gives

$$D_{e_1}e_1 = \frac{1}{\|M_s\|} D_{M_s}e_1 = \frac{\kappa}{1 - r\kappa \cos u} N,$$

$$D_{e_1}e_2 = \frac{1}{\|M_u\|} D_{M_u}e_2 = \frac{\kappa \sin u}{1 - r\kappa \cos u} T,$$

$$D_{e_2}e_2 = \frac{1}{\|M_u\|} D_{M_u}e_2 = \frac{1}{r} (-\cos u N - \sin u B).$$

Then we find,

$$h_{11} = \langle D_{e_1}e_1, e_3 \rangle = \frac{-\kappa \cos u}{1 - r\kappa \cos u}, \quad h_{12} = \langle D_{e_1}e_2, e_3 \rangle = 0$$

$$h_{22} = \langle D_{e_2}e_2, e_3 \rangle = \frac{1}{r}.$$ 

Then we have the following theorem.

**Theorem 5.** Let $M$ be a tubular surface given by (3) in $\mathbb{E}^3$. Then the Gauss curvature and mean curvature of $M$ is found as follows

$$K = \frac{-\kappa \cos u}{r (1 - r\kappa \cos u)} \quad \text{and} \quad H = \frac{1 - 2r\kappa \cos u}{2r (1 - r\kappa \cos u)}. \quad (9)$$

Now we define the Gauss map $G(s,u)$ of $M$ by

$$G(s,u) = -(\cos u) N(s) - (\sin u) B. \quad (10)$$

By using (10) and (2), we have

$$\Delta G = \frac{-\kappa' \cos u}{(1 - r\kappa \cos u)^3} T + \frac{-2 \cos u + r(1 + 3 \cos 2u) \kappa - 4r^2\kappa^2 \cos^3 u}{2r^2 (1 - r\kappa \cos u)^2} N \quad (11)$$

$$+ \frac{-\sin u + r\kappa \sin 2u}{r^2 (1 - r\kappa \cos u)} B.$$ 

Then we give the following theorem.
Theorem 6. Let $M$ be a tubular surface given by (3) in $\mathbb{E}^3$. Then $M$ has the first kind of pointwise 1-type Gauss map if and only if $M$ is a cylindrical surface.

Proof. Let $M$ be a tubular surface given by (3) in $\mathbb{E}^3$ and assume that $M$ has the first kind of pointwise 1-type Gauss map, that is, the following equation holds

$$\Delta G = \lambda G$$

where $\lambda$ is a real valued $C^\infty$ function. By using (10) and (11) in (12), we get

$$\lambda = \frac{1}{r^2} \text{ and } \kappa = 0,$$

which implies that the surface $M$ is a cylindrical surface.

Conversely, let $M$ be a cylindrical surface. We will show that $M$ has the first kind of pointwise 1-type Gauss map. Let us assume that the following holds

$$\Delta G = \lambda (G + C)$$

where $C = c_1 T(s) + c_2 N(s) + c_3 B$. Substituting (10) and (11) in (13), we obtain

$$\Delta G = -\frac{\cos u}{r^2} N - \frac{\sin u}{r^2} B,$$

$$\lambda (G + C) = \lambda c_1 T + \lambda (-\cos u + c_2) N + \lambda (-\sin u + c_3) B$$

Since the set $\{1, \sin u, \cos u\}$ is linearly independent, we get $\lambda = 1/r^2$ and $C = 0$, which means that $M$ has the first kind of pointwise 1-type Gauss map.

Then we give the following corollaries.

Corollary 7. Let $M$ be a tubular surface given by (3) in $\mathbb{E}^3$. Then $M$ has the first kind of pointwise 1-type Gauss map if and only if the spine curve of $M$ is a straight line.

Corollary 8. Let $M$ be a tubular surface given by (3) in $\mathbb{E}^3$. Then $M$ does not have a harmonic Gauss map.

Now, we consider the mean curvature vector $\vec{H}$ of $M$. The mean curvature vector $\vec{H}$ is given by

$$\vec{H} = \frac{1 - 2r\kappa \cos u}{2r (1 - r\kappa \cos u)} e_3.$$  

Then we have

$$\Delta \vec{H} = \frac{(-1 + 4r\kappa \cos u) \kappa' \cos u}{2r (1 - r\kappa \cos u)^4} T + P(s,u) N + Q(s,u) B$$
where \( P(s,u) \) and \( Q(s,u) \) are differentiable functions.

Assume that \( \Delta \vec{H} = \lambda H \) for some differentiable functions \( \lambda \). Then from the coefficients of \( T \), we have

\[
\kappa' = 0
\]

which implies that

\[
P(s,u) = \frac{(2 + 9\kappa^2 r^2) \cos u - 8\kappa^3 r^3 \cos^4 u + \kappa r (-2 - 8 \cos 2u + 5\kappa r \cos 3u)}{-4r^3 (1 - r\kappa \cos u)^3}
\]

and

\[
Q(s,u) = \frac{2 \sin u + 2\kappa r (\kappa r (6 - 4\kappa r \cos^3 u + 5 \cos 2u) \sin u - 4 \sin 2u)}{-4r^3 (1 - r\kappa \cos u)^3}.
\]

Since \( \Delta \vec{H} = \lambda H \), we get

\[
\kappa = 0 \quad \text{and} \quad \lambda = \frac{1}{r^2}.
\]

Then we get the following theorem.

**Theorem 9.** Let \( M \) be a tubular surface given by (3) in \( \mathbb{E}^3 \). Then the mean curvature vector \( \vec{H} \) of \( M \) satisfying \( \Delta \vec{H} = \lambda H \) for some differentiable functions \( \lambda \) if and only if \( M \) is a cylindrical surface.

**Theorem 10.** Let \( M \) be a tubular surface given by (3) in \( \mathbb{E}^3 \). Then \( M \) is semi-parallel if and only if \( M \) is a cylindrical surface.

**Proof.** Let \( M \) be a tubular surface given by (3) in \( \mathbb{E}^3 \). Assume that \( M \) is semi-parallel. Namely, for \( 1 \leq i,j \leq 2 \),

\[
(\vec{R}(e_1, e_2) \cdot h)(e_i, e_j) = 0.
\]

By a straightforward calculation, from Lemma 4, we have

\[
(\vec{R}(e_1, e_2) \cdot h)(e_1, e_1) = \frac{\kappa^2 (3 - 2r\kappa \cos u) \cos^2 u}{r (1 - r\kappa \cos u)^3} e_3,
\]

\[
(\vec{R}(e_1, e_2) \cdot h)(e_1, e_2) = \frac{-\kappa \cos u}{r^2 (1 - r\kappa \cos u)^2} e_3,
\]

\[
(\vec{R}(e_1, e_2) \cdot h)(e_2, e_2) = 0.
\]

From our assumption, we get \( \kappa = 0 \) which means that \( M \) is a cylindrical surface. The converse of the proof is clear.
As a result, we have the following corollary.

**Corollary 11.** Let $M$ be a tubular surface given by (3) in $E^3$. Then the followings are equivalent:

i. $M$ is a cylindrical surface,

ii. $M$ has the first kind of pointwise 1-type Gauss map,

iii. The mean curvature vector $\vec{H}$ of $M$ satisfying $\Delta \vec{H} = \lambda \vec{H}$ for some differentiable functions $\lambda$,

iv. $M$ is semi-parallel.

**References**


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