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Projective Schemes with Degenerate General Hyperplane Section II

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Abstract. We study projective non-degenerate closed subschemes $X \subseteq \mathbf{P}^n$ having degenerate general hyperplane section, continuing our earlier work. We find inequalities involving three relevant integers, namely: the dimensions of the spans of X_{red} and of the general hyperplane section of X, and a measure of the "fatness" of X, which is introduced in this paper. We prove our results first for curves and then for higher dimensional schemes by induction, via hyperplane sections. All our proofs and results are characteristic free. We add also many clarifying examples.

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Introduction

We continue our study, started in [2], of non-degenerate projective schemes $S \subseteq \mathbf{P}^n_K$ (K an algebraically closed field), having degenerate general hyperplane section and we proceed in our attempt to classify them, continuing our earlier work [2].

There are several new contributions with respect to the previous paper.

First of all we consider systematically the dimension of the span of the general hyperplane sections, namely

$$s(S) := \dim(\langle H \cap S \rangle),$$

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where H is a general hyperplane and $\langle Z \rangle$ is the linear span of Z. (Observe that the general hyperplane section of S is degenerate if and only if $s(S) \leq n-2$).

Next, we introduce a suitable measure of the relevant part of the "fatness" of an irreducible scheme S, which we call "generic spanning increasing" and denote by z_S (Definitions 2.1 and 3.2); if S is non-degenerate with degenerate general hyperplane section one always has $z_S > 0$.

Our main results are inequalities involving $s(S)$, z_S and $m(S) := \dim(\langle S_{red} \rangle)$. The inequalities we get are often sharp, and we give several examples to clarify them.

The paper is organized as follows.

In Section 1 we collect several preliminary results concerning general hyperplane sections.

In Section 2 we deal with curves, namely pure one-dimensional schemes.

Some of our results can be summarized in the following:

Theorem A. *Let $Y \subseteq \mathbf{P}^n$ be a non-degenerate curve with degenerate general hyperplane section. Let $s := s(Y), m := m(Y), z := z_Y$.*

(i) *If Y_{red} is irreducible, then $z + 2m \leq s + 2 \leq n$ and $2m \leq s + 1$* (Lemma 2.4, Theorem 2.5).

(ii) *If Y_{red} is irreducible and Y is "almost minimal"* (Definition 2.9)*, then $(z+1)m \leq s+1$* (Theorem 2.10).

(iii) *If Y_{red} is connected then $m \leq s$* (Proposition 2.11).

We also show that if Y is a multiple line all possible values of s in (i) can occur (Example 2.8) and we discuss the extremal case $s = 1$ (Remark 2.7). Then we give a variant of Theorem A(i) when Y_{red} is assumed to be only connected (Proposition 2.13), and we also give some hints on how to attack the general case (i.e. Y_{red} not connected), by introducing the notion of "linearly connected" curve (Definition 2.15, Lemma 2.16).

We end Section 2 with some further steps in the classification of non-degenerate curves in $\mathbf{P}^5$ with degenerate general hyperplane section.

Section 3 deals with higher dimension. The general idea is to use induction on the dimension via general hyperplane section, starting from the results on curves. Our first result is Lemma 3.1, which shows the behavior of the integer $s(S)$ when passing to a general hyperplane section. It can be considered as one of the main new contributions of the present paper, mainly because it is characteristic free (in our previous paper [2] we had a much weaker result based on the Socle lemma, and hence needing characteristic zero). To use induction we show first that if S has property S_2 and S_{red} is connected (resp. irreducible), then a "general curve section" Y of S exists and is connected (resp. irreducible) and moreover $z_S = z_Y$ (Lemma 3.4, Proposition 3.5).

These results allow reduction to the 1-dimensional case. For example we have the following:

Theorem B. *Let $X \subseteq \mathbf{P}^n$ be a non-degenerate closed subscheme with degenerate general hyperplane section. Assume $d := \dim(X) \geq 2$ and that X has Serre's property S_2. Let $s := s(Y)$, $m := m(Y)$, $z := z_Y$.*

(i) *If X_{red} is irreducible, then $z+2m \leq s+2 \leq n$ and $2m \leq s+1$* (Theorem 3.6).
(ii) *If X_{red} is connected, then $m \leq s-d+1$* (Proposition 3.7.)

The above theorem provides immediately lower bounds for n and s in terms of d (Corollary 3.8). We show that the bound for n is achieved when n is odd or if $n \leq 6$, by producing suitable double structures on a linear space (Example 3.13). This implies that also the bound for s is achieved in some cases, while the general problem remains open.

From the above double structures we obtain also further examples of non-degenerate schemes with degenerate general hyperplane section (Example 3.15).

1. Notation and preliminaries

We work over an algebraically closed field $\mathbf{K}$, of arbitrary characteristic. By *curve* we will always mean a pure one-dimensional locally Cohen-Macaulay scheme.

If $S \subseteq \mathbf{P}^n$ is a non-degenerate closed subscheme we use the following notation:

(i) $\langle S \rangle :=$ the *linear span of S*, that is the least linear space containing S as a subscheme;
(ii) $m(S) := \dim\langle S_{red}\rangle$;
(iii) $s(S) := \dim\langle H \cap S\rangle$, where H is a general hyperplane (thus $H \cap S$ is degenerate if and only if $s(S) \leq n-2$).

We shall use frequently the notion of S_1-*image* of a closed subscheme $S \subseteq \mathbf{P}^n$ via a linear projection $\mathbf{P}^n \cdots \to \mathbf{P}^{n'}$. We refer to [2] 1.14 and 1.15 for the definition and the main properties of S_1-image.

The following lemma explains the behavior of the invariant s under linear projections.

Lemma 1.1. *Let $Y \subseteq \mathbf{P}^n$ be a non-degenerate curve and let $L \subseteq \mathbf{P}^n$ be a linear subspace of dimension ℓ. Put $n' := n-\ell-1$ and let $g(Y)$ be the S_1-image of Y via the linear projection $g : \mathbf{P}^n \cdots \to \mathbf{P}^{n'}$ with center L. Assume that $n' \geq 2$ and that $g(Y) \neq \emptyset$. Assume further that:*

(1) *$Y \cap L$ is zero-dimensional and $L = \langle Y \cap L\rangle$.*
(2) *L does not meet any line contained in Y.*

Then $s(g(Y)) \leq s(Y)-\ell-1$ and equality holds if and only if there is a hyperplane $H \subseteq \mathbf{P}^n$ such that $L \subseteq H$, H does not contain any component of Y_{red} and $\dim\langle H \cap Y\rangle = s(Y)$.

Proof. We identify $\mathbf{P}^{n'}$ with a general linear subspace of $\mathbf{P}^n$. Let $H' \subseteq \mathbf{P}^{n'}$ be a general hyperplane, and let $H := \langle H' \cup L\rangle$. By (2) H does not contain any irreducible component of Y_{red}, whence $Y \cap H$ is zero-dimensional and by semicontinuity we have $\dim\langle H \cap Y\rangle \leq s(Y)$. Let M be any hyperplane containing $H \cap Y$. Then M contains L by (1), whence $M' := g(M)$ is a hyperplane containing $H' \cap g(Y)$. Conversely if $M' \subseteq \mathbf{P}^{n'}$ is a hyperplane containing $H' \cap g(Y)$, the hyperplane $M := \langle L, M'\rangle$ contains $H \cap Y$. It follows that $\operatorname{codim}_{\mathbf{P}^n}\langle Y \cap H\rangle = \operatorname{codim}_{\mathbf{P}^{n'}}\langle g(Y) \cap H'\rangle$ and the conclusion follows. ☐

The next result, probably well-known, shows that scheme-theoretical inclusions can be checked via general hyperplane section, under obvious assumptions. We give a proof for lack of a reference.

Proposition 1.2. *Let $X \subseteq \mathbf{P}^n$ be a closed subscheme with no zero-dimensional irreducible components (embedded or not) and let Y be a closed subscheme of X. Assume that $X \cap H = Y \cap H$ for every general hyperplane H. Then $X = Y$.*

Proof. Let $X_1, \ldots, X_t$ be the irreducible components of X, with the induced scheme structure. By our assumption on X, for each $i = 1, \ldots, t$ we can fix a closed point $x_i \in H \cap X_i$. Now fix i and let $x := x_i$. Put $A := \mathcal{O}_{X,x}$ and $\mathfrak{m} := \mathfrak{m}_x$. Since $X \cap H = Y \cap H$ we have $x \in Y$ and there is an ideal $\mathfrak{a} \subseteq A$ such that $A/\mathfrak{a} = \mathcal{O}_{Y,x}$. Let $h \in \mathfrak{m}$ be a local equation of $X \cap H$. Since H is general we may assume that x is not the support of an irreducible component of Y; this implies that $\operatorname{depth}(A/\mathfrak{a}) > 0$ and hence we may also assume that h is $A/\mathfrak{a}$-regular. Now $\mathfrak{a} \subseteq hA$, and since h is $(A/\mathfrak{a})$-regular it follows that $h\mathfrak{a} = \mathfrak{a}$, whence $\mathfrak{a} = 0$ by Nakayama's Lemma.

Let now $J \subseteq \mathcal{O}_X$ be the ideal sheaf corresponding to Y, and let z_i be the generic point of X_i. By the above argument we have $J_{x_i} = 0$, whence $J_{z_i} = 0$ for $i = 1, \ldots, t$. Then by using an affine covering and standard facts on primary decomposition and localizations, it follows that $J = 0$, i.e. $X = Y$ (see e.g. [4], Lemma 4 for details). ☐

Corollary 1.3. *Let $X \subseteq \mathbf{P}^n$ be a closed subscheme with no zero-dimensional components and let $Z \subseteq \mathbf{P}^n$ be any closed subscheme. If $X \cap H \subseteq Z$ for every general hyperplane H, then $X \subseteq Z$.*

Proof. Set $Y := X \cap Z \subseteq X$. Then $X \cap H = X \cap H \cap Z = Y \cap H$, whence $X = Y$ by 1.2 and the conclusion follows. ☐

For easy reference we include the following lemma concerning the integer $m(X)$.

Lemma 1.4. *Let $X \subseteq \mathbf{P}^n$ be a non-degenerate closed subscheme. Then:*
(i) *if X is reduced and connected we have $s(X) = n - 1$,*
(ii) *if H is a general hyperplane we have $(X \cap H)_{red} = X_{red} \cap H$,*
(iii) *if X_{red} is connected and H is a general hyperplane we have $m(X \cap H) = m(X) - 1$.*

Proof. (i) follows from [5], Proposition 1.1, and (ii) is just [2], Lemma 3.2.(a). The last statement is an easy consequence of (i) and (ii). ☐

Before we proceed, let us recall some notation given in [2].
If $Y \subseteq \mathbf{P}^n$ is a curve, for every irreducible component D of Y_{red}, D'' will denote the maximal subcurve of Y with $D''_{red} = D$.
For any curve Y we denote by $Y^{(1)}$ the maximal generalized rope contained in Y. We have $Y_{red} \subseteq Y^{(1)} \subseteq Y$, and if D is any irreducible component of Y_{red}, the corresponding component of $Y^{(1)}$ is the largest curve contained in the subscheme $D'' \cap D^{[1]}$, where $D^{[1]}$ denotes the first neighborhood of D (see [1] for more details).

2. Main results about curves

We begin by defining the main new concept in this paper.

Definition 2.1. *Let $Y \subseteq \mathbf{P}^n$ be a curve such that Y_{red} is irreducible.*

(i) *Fix a hyperplane H not containing Y_{red} and a point $P \in H \cap Y_{red}$.*
We denote by $Z_{P,H}$ the connected component of the scheme $Y \cap H$ supported on P.

(ii) *Let now H be a general hyperplane, let $P \in H \cap Y_{red}$ and put $Z := Z_{P,H}$. Notice that $n-1-\dim\langle Z\rangle$ is the dimension of the kernel of the restriction map $H^0(H, \mathcal{O}_H(1)) \to H^0(Z, \mathcal{O}_Z(1))$. Therefore the integer $\dim\langle Y_{red} \cup Z\rangle - \dim\langle Y_{red}\rangle$ does not depend on the choice of P and H; it will be denoted by z_Y and called generic spanning increasing of Y. We write z whenever no explicit reference to Y is needed.*

(iii) *We call generic fattening dimension of Y the integer $f_Y := z_{Y^{(1)}}$. We write f whenever no explicit reference to Y is needed.*
Observe that we have $f_Y = z_{Y^{(1)}} \le z_Y$, whence $f \le z$ and equality holds if $Y = Y^{(1)}$.

In the following Proposition 2.3 we collect some elementary properties of the above defined integers. In order to prove it we need a well-known lemma which we include for lack of a reference.

Lemma 2.2. *Let $P \in \mathbf{P}^n$ be a point and let Z be a closed subscheme of the first neighborhood of P. Then:*

(i) *For every closed subscheme W of Z we have $\dim(\langle W\rangle) = \deg(W) - 1$;*

(ii) *the map $W \mapsto \langle W\rangle$ is a bijection between the set of closed subschemes of Z and the set of linear subspaces of $\langle Z\rangle$ containing P. The inverse map is: $L \to L \cap Z$.*

Proof. Replacing $\mathbf{P}^n$ with $\langle Z\rangle$ may assume that Z is non-degenerate. We may also assume that P is the origin of an affine chart with coordinates $X_1, \dots, X_n$. Put $R := K[X_1, \dots, X_n]$ and $M := (X_1, \dots, X_n)R$. Let $A := \mathcal{O}_{Z,P} = K[X_1, \dots, X_n]/Q$ and let $\mathfrak{m} = M/Q$ be the maximal ideal of A. By assumption A is artinian and $\mathfrak{m}^2 = 0$ whence the set of ideals of A coincides with the set of K-subspaces of A. Since Z is non-degenerate we have $Q = M^2$ and hence the K-subspaces of A correspond bijectively to the K-subspaces of the K-vector space generated by $X_1, \dots, X_n$. The conclusions follow easily from these remarks. We leave the details to the reader. □

Proposition 2.3. *Let $Y \subseteq \mathbf{P}^n$ be a curve, with Y_{red} irreducible. Put $s := s(Y)$, $m := m(Y)$, $z := z_Y$, $f := f_Y$. Fix a general $P \in Y_{red}$ and a general hyperplane H through P, and put $Z := Z_{P,H}$. Then:*

(i) *$z = \dim\langle Z\rangle - \dim(\langle Z\rangle \cap \langle Y_{red}\rangle) \le s$;*

(ii) *$f + m \le z + m \le s + 1$;*

(iii) *$z = 0$ if and only if $Y \subseteq \langle Y_{red}\rangle$ and $f = 0$ if and only if $Y^{(1)} \subseteq \langle Y_{red}\rangle$;*

(iv) *$z > 0$ if Y is non-degenerate and $s \le n-2$;*

(v) *$f = \dim(T_P(Y^{(1)})) - \dim(T_P(Y^{(1)} \cap \langle Y_{red}\rangle))$, where $T_P(X)$ denotes the embedded Zariski tangent space of the scheme $X \subseteq \mathbf{P}^n$ at the closed point $P \in X$;*

(vi) *$f \le \frac{\deg Y^{(1)}}{\deg Y_{red}} - 1$ and equality holds if and only if $\dim(\langle Z\rangle \cap \langle Y_{red}\rangle) = 0$.*

Proof. (i) We have: $\langle Y_{red} \cup Z\rangle = \langle\langle Z\rangle \cup \langle Y_{red}\rangle\rangle$, hence

$$\begin{aligned} z &= \dim\langle Y_{red}\rangle + \dim\langle Z\rangle - \dim(\langle Z\rangle \cap \langle Y_{red}\rangle) - \dim\langle Y_{red}\rangle \\ &= \dim\langle Z\rangle - \dim(\langle Y_{red}\rangle \cap \langle Z\rangle). \end{aligned}$$

(ii) Since $\langle H\cap Y\rangle \supseteq \langle Z\rangle \cup \langle H\cap Y_{red}\rangle$ we have:

$$\begin{aligned} s &\geq \dim\langle Z\rangle + \dim\langle H\cap Y_{red}\rangle - \dim(\langle Z\rangle \cap \langle H\cap Y_{red}\rangle) \\ &= \dim(\langle Z\rangle) + m - 1 - \dim\langle\langle Z\rangle \cap \langle Y_{red}\rangle) \\ &= z + m - 1. \end{aligned}$$

(iii) If $z=0$, then $\langle Z_i\rangle \subseteq \langle Y_{red}\rangle$ for all the components Z_i of $H\cap Y$. Then $H\cap Y \subseteq \langle Y_{red}\rangle$, whence $Y\subseteq \langle Y_{red}\rangle$ by Corollary 1.3. The converse is obvious. Similarly for f.

(iv) If $z=0$ then $Y\subseteq \langle Y_{red}\rangle$ by (iii). But Y_{red} is degenerate because $s\leq n-2$, and this is a contradiction.

(v) We may assume $Y=Y^{(1)}$, whence Z is contained in the first neighborhood of P. Then by Lemma 2.2 we have $T_P(Z)=\langle Z\rangle$. Now P is a smooth point of Y_{red}, and if r is the tangent line to Y_{red} at P it is easy to see that $T_P(Y)=\langle Z\cup r\rangle$. Since $r\subseteq \langle Y_{red}\rangle$ we have $\langle Z\cup Y_{red}\rangle = \langle T_P(Y)\cup Y_{red}\rangle$, and the conclusion follows.

(vi) As in (v) we may assume that Z is contained in the first neighborhood of P. Then by Lemma 2.2 we have $\dim\langle Z\rangle \leq \deg Z - 1$ and since $\deg Z = \frac{\deg Y^{(1)}}{\deg Y_{red}}$ the conclusion follows. □

Next, we want to prove an inequality relating the invariants m, s and z, which generalizes Theorem 2.1 of [2] (see Theorem 2.5 below). We begin with a weaker result.

Lemma 2.4. *Let $Y\subseteq \mathbf{P}^n$ be a non-degenerate curve with degenerate general hyperplane section and assume Y_{red} irreducible. Let $s:=s(Y)$ and $m:=m(Y)$. Then $2m\leq s+1$.*

Proof. We use induction on n. Observe first that $s\geq 1$, for otherwise $\deg Y=1$ and Y is degenerate. Then if $n=2$ the statement is empty. If $n=3$ we have $s=1$ and by [8] we have $m=1$ and the conclusion follows in this case.

Assume now $n\geq 4$. We argue by contradiction, by assuming $2m\geq s+2$. Since $s\geq 1$, this implies $m\geq 2$. Recall that since Y_{red} is irreducible and its general hyperplane section is degenerate, then Y_{red} itself is degenerate.

Take a general $P\in Y_{red}$ and a general $v\in T_P(Y)$. Since Y_{red} is degenerate and Y is not, we have $\langle v\rangle \not\subseteq \langle Y_{red}\rangle$.

Let $g:\mathbf{P}^n\cdots\to \mathbf{P}^{n-2}$ be the linear projection from the line $\langle v\rangle$ and let $g(Y)$ be the S_1-image (see [2], 1.14). Since $m\geq 2$ we have $g(Y)\neq\emptyset$ and $g(Y)$ is non-degenerate by [2], 1.15. Moreover $\langle v\rangle\cap\langle Y_{red}\rangle=\{P\}$ and then $\dim\langle g(Y)_{red}\rangle = m-1$. Clearly $\deg(\langle v\rangle\cap Y)\geq 2$ whence $\langle v\rangle = \langle\langle v\rangle\cap Y\rangle$. Then by Lemma 1.1 we have $s(g(Y))\leq s-2$. Then by induction we have $2(m-1)\leq s(g(Y))\leq s-2+1$, whence $2m\leq s+1$, a contradiction. □

Theorem 2.5. *Let $Y\subseteq \mathbf{P}^n$ be a non-degenerate curve with degenerate general hyperplane section and assume Y_{red} irreducible. Let $s:=s(Y), m:=m(Y), z:=z_Y, f:=f_Y$. Then $z+2m\leq s+2$; in particular $f+2m\leq s+2$.*

Proof. Let H be a general hyperplane, $P\in H\cap Y$ and put $Z:=Z_{P,H}$. By Proposition 2.3 (i) and (iii) we have $\dim\langle Z\rangle \geq z>0$. We need a preliminary result.

Claim. For $i = 0, \dots, z$ there is a linear space M_i such that:

1) $M_i \subseteq \langle Z \rangle$
2) $M_i \cap \langle Y_{red} \rangle = \{P\}$
3) $M_i = \langle M_i \cap Y \rangle$
4) $\dim M_i = i$.

Proof of Claim. We use induction on i. If $i = 0$ just take $M_0 = \{P\}$. Assume that M_i exists for some i with $0 \le i < z$. Then $M_i \neq \langle Z \rangle$ by 4) and by 1), 2), 3) there is a line r with the following properties:

i) $r \subseteq \langle Z \rangle$
ii) $r \cap \langle Y_{red} \rangle = \{P\}$
iii) $r \cap Y \not\subseteq M_i$.

Then $M_{i+1} := \langle M_i \cup r \rangle$ has the required properties.

Now we can prove the theorem. Observe first that by Proposition 2.3(i) we may assume $m \ge 2$. Put $M := M_z$ and let $\pi : \mathbf{P}^n \cdots \to \mathbf{P}^{n-z-1}$ be the linear projection from M. Observe that since $s \le n-2$ we have $n - z - 1 \ge m \ge 2$ by Proposition 2.3(ii). Let Y' be the S_1-image of Y under the projection π.

Since $M \cap \langle Y_{red} \rangle = \{P\}$ and $\dim\langle Y_{red} \rangle \ge 2$, we have that $Y' \neq \emptyset$ and Y'_{red} is the image of Y_{red} under the projection from the point P. Hence $\dim\langle Y'_{red} \rangle = m - 1$. Moreover Y' is non-degenerate by [2], 1.15.

Put $s' := s(Y')$. By Lemma 1.1 we have $s' \le s - z - 1$ and by Lemma 2.4 applied to $Y' \subseteq \mathbf{P}^{n-z-1}$ we have $2(m-1) \le s' + 1$. It follows $2(m-1) \le s - z$, whence our claim. ☐

Remark 2.6. From 2.4 we have that $s \ge 2m - 1$, whence $s \ge 1$. Therefore the extremal case is $s = 1$. From 2.5 and 2.3(iii) when $s = 1$ it follows that $z + 2m \le 3$, whence $2m \le 3 - z \le 2$ which implies that $m = 1$. It follows that if $Y \subseteq \mathbf{P}^n$ is a non-degenerate curve whose general hyperplane section spans a line, then Y_{red} is a line and $z_Y = 1$.

The easiest curves with this property are double lines and it is easy to show that in any $\mathbf{P}^n$ with $n \ge 3$ there are non-degenerate double lines (e.g. [2], Example 1.6).

There are also non-degenerate multiple lines of degree ≥ 3 with collinear general hyperplane section, but they can occur only in positive characteristic, as follows from Hartshorne Restriction Theorem (see [8] for $\mathbf{P}^3$ and [6] for arbitrary $\mathbf{P}^n$).

A complete classification of the non-degenerate multiple lines of degree ≥ 3 in $\mathbf{P}^3$ having collinear general hyperplane section is given by Hartshorne [8]. It is not difficult to produce examples of such lines also in higher dimensional spaces, starting from examples in $\mathbf{P}^3$, but a general classification is not known.

Example 2.7. Let $Y \subseteq \mathbf{P}^5$ be a non-degenerate curve with degenerate general hyperplane section and assume Y_{red} irreducible, with $\deg Y \ge 2$. Then from 2.5 we have immediately $m = 2$ and $z = 1$. We shall see later that such curves do exist (see Example 3.15).

Example 2.8. If Y_{red} is a line, it is clear from Definition 2.1 that $z_Y = s(Y)$, and hence for these curves the bound $z + 2m \le s + 2$ of Theorem 2.5 is sharp. We want to show that such curves Y do exist, and that every compatible value of z can occur.

More precisely we will show that for any $n \geq 3$ and any s with $1 \leq s \leq n-1$ there is a non-degenerate curve Y such that Y_{red} is a line and $s = s(Y) = \deg(Y) - 1$.

For this let $S \subseteq \mathbf{P}^n$ be a smooth surface scroll of minimal degree $n-1$. If $n = 3$, then S is a smooth quadric and one can take as Y the scheme corresponding to the divisor $d\ell$, where ℓ is a line on S and $2 \leq d \leq 3$.

If $n > 3$, S can be constructed as follows: let $L \subseteq \mathbf{P}^n$ be a linear subspace of dimension $n-2$ and let $C \subseteq L$ be a rational normal curve spanning L. Let ℓ be a line skew with L, and let $\varphi : \ell \to C$ be an isomorphism. Then the required S is the union of the lines joining P and $\varphi(P)$, for each $P \in \ell$. Observe that S is a non-degenerate integral surface of degree $n-1$. It follows that $H \cap S$ is a curve of degree $n-1$ spanning H, hence is a rational normal curve in H. Let now d be an integer with $2 \leq d \leq n$ and let Y be the curve corresponding to the divisor $d\ell$ on S. Then $\deg(Y) = d$ and Y is non-degenerate. Moreover, if H is a general hyperplane, $H \cap Y$ is a zero-dimensional subscheme of $H \cap S$ of degree d. Since $H \cap S$ it is a rational normal curve it follows that $\dim\langle H \cap Y\rangle = d-1$, that is $s(Y) = d-1$. The conclusion follows.

Now we give a result for a class of curves which include minimal curves as defined in [2].

Definition 2.9. *Let $Y \subseteq \mathbf{P}^n$ be a non-degenerate curve. We will say that Y is almost minimal if $Y = Y^{(1)}$ and moreover for every subcurve Y' of Y, with $Y_{red} \subseteq Y'$, we have either $Y' \subseteq \langle Y_{red}\rangle$ or $\langle Y'\rangle = \mathbf{P}^n$. A minimal curve (as defined in* [2], *Definition* 1.12*) is almost minimal by* [2], *Remark* 1.13.

Theorem 2.10. *Let the notation and assumptions be as in 2.5 and assume further Y almost minimal. Then $(f+1)m \leq s+1$.*

Proof. Since Y is almost minimal we have $Y = Y^{(1)}$, whence $f = z_Y$. If $m = 1$ the conclusion follows from Proposition 2.3(ii). So we assume that $m \geq 2$ and we will show that for $i = 1, \dots, m$ the following statement holds (see Definition 2.1 for notation):

(α_i) There are general $P_1, \dots, P_i \in Y_{red}$ and a general hyperplane H containing $P_1, \dots, P_i$ such that $\dim\langle Z_1 \cup \ldots \cup Z_i \cup Y_{red}\rangle = if + m$, where $Z_i := Z_{P_i,H}$.

Our conclusion will follow from (α_m). Indeed if (α_m) holds we have:

$$\langle Z_1 \cup \ldots \cup Z_m \cup Y_{red}\rangle \cap H \subseteq H \cap Y$$

and hence $mf + m - 1 \leq s$, which is our claim.

We prove (α_i) by induction on i. (α_1) follows immediately from Definition 2.1. Now we assume that $(\alpha_1), \dots, (\alpha_i)$ are true and (α_{i+1}) is false for some i with $1 \leq i \leq m-1$ and we get a contradiction. Fix $P_1, \dots, P_i$ such that (α_i) holds. Then for each general H and $P \in H \cap Y_{red}$, (α_{i+1}) does not hold for $P_1, \dots, P_i, P$, that is if $M_i := \langle Z_{P_1} \cup \ldots \cup Z_{P_i} \cup Y_{red}\rangle$ then $\dim\langle M_i\rangle = if + m$ and $\dim M_{P,H} \leq (i+1)f + m - 1$, where $M_{P,H} := \langle Z_1 \cup \ldots \cup Z_i \cup Z_{P,H} \cup Y_{red}\rangle$. Since $M_{P,H} = \langle M_i \cup \langle Z_{P,H} \cup Y_{red}\rangle\rangle$ we have:

$$\begin{aligned}
\dim(M_i \cap \langle Z_{P,H} \cup Y_{red}\rangle) &= \dim M_i + \dim\langle Z_{P,H} \cup Y_{red}\rangle + \\
&\quad - \dim\langle (M_i \cup \langle Z_{P,H} \cup Y_{red}\rangle)\rangle \\
&= if + m + f + m - \dim M_{P,H} \\
&\geq if + m + f + m - [(i+1)f + m - 1] \\
&\geq m + 1.
\end{aligned}$$

Since $M_i \cap \langle Z_{P,H} \cup Y_{red} \rangle \supseteq \langle Y_{red} \rangle$ it follows that $M_i \cap \langle Z_{P,H} \rangle \not\subseteq \langle Y_{red} \rangle$. Then there is a point $Q \in M_i \cap \langle Z_{P,H} \rangle$, $Q \notin \langle Y_{red} \rangle$. Let D be the line QP. Since $D \subseteq \langle Z_{P,H} \rangle$ and $Y = Y^{(1)}$, by Lemma 2.2 D contains a subscheme $W_{P,H}$ of $Z_{P,H}$, supported at P and spanning D. It follows that $W_{P,H}$ is a subscheme of Y, supported at P and of length at least 2. Now we have: $D \subseteq M_i$ and $D \not\subseteq \langle Y_{red} \rangle$ (because $D \cap \langle Y_{red} \rangle = \{P\}$ scheme-theoretically), whence $W_{P,H} \not\subseteq \langle Y_{red} \rangle$. Let $Y' \subseteq Y$ be the least subcurve of Y containing all the subschemes $W_{P,H}$ for general P. Then $Y_{red} \subseteq Y' \subseteq Y \cap M_i$, whence $Y' \subseteq M_i$. Moreover $Y' \not\subseteq \langle Y_{red} \rangle$ by construction, and since Y is almost minimal we have $\langle Y' \rangle = \mathbf{P}^n$. Then $M_i = \mathbf{P}^n$. On the other hand we have

$$\begin{aligned} \dim(M_i) &\leq \dim(\langle H \cap Y \rangle \cup \langle Y_{red} \rangle) \\ &= s + m - \dim\langle H \cap Y_{red} \rangle \\ &= s + m - (m-1) \\ &= s + 1 \\ &\leq n - 1. \end{aligned}$$

and this is a contradiction. □

The results obtained so far need the assumption that Y_{red} is irreducible. If we drop this assumption we can get weaker results, which still generalize some statements in [2]. The next result e.g. generalizes Proposition 2.3 of [2].

Proposition 2.11. *Let $Y \subseteq \mathbf{P}^n$ be a non-degenerate curve, with degenerate general hyperplane section and with Y_{red} connected. Put $m := m(Y)$ and $s := s(Y)$. Then $m \leq s$.*

Proof. We argue by contradiction by assuming $m \geq s+1$. Let H be a general hyperplane. Then $\dim(\langle Y \cap H \rangle) = s$ and $\dim(\langle Y_{red} \cap H \rangle) = m - 1$ because Y_{red} is connected. It follows that $\dim(\langle Y_{red} \cap H \rangle) \geq \dim(\langle Y \cap H \rangle)$, and since $Y_{red} \cap H \subseteq Y \cap H$ it follows that $\langle Y \cap H \rangle = \langle Y_{red} \cap H \rangle$. The proof can be concluded exactly as in [2], proof of Proposition 2.3. □

Definition 2.12. *Let Y be a non-degenerate curve, and let Y_i, $1 \leq i \leq r$ be the irreducible components of Y_{red}. For each i put $z_i := z_{Y_i}$ Set $z_Y := \min_{1 \leq i \leq r}\{z_i\}$. The integer z_Y will be called minimal spanning increasing of Y. Similarly we can define the minimal fattening dimension f_Y of Y (see Definition 2.1).*

Observe that if Y_{red} is irreducible the above notation is consistent with the one given in Definition 2.1.

Proposition 2.13. *Let Y be a non-degenerate curve, with degenerate general hyperplane section and assume Y_{red} connected. Assume further that $\langle Y_{red} \rangle = \langle D \rangle$ for every irreducible component D of Y_{red}. Let $z = z_Y$, $f = f_Y$ and put $m := m(X)$. Then $z + 2m \leq s + 2$; in particular $f + 2m \leq s + 2$.*

Proof. Let $Y_1, \ldots, Y_r$ be the irreducible components of Y_{red}. We use induction on r. For $r = 1$ we apply Theorem 2.5. Assume that $r > 1$ and assume that the statement is true for every r' with $1 \leq r' < r$. Since Y_{red} is connected we can always assume that $Y_1 \cup \ldots \cup Y_{r-1}$ is connected.

Let $X_1 := Y_1'' \cup \ldots \cup Y_{r-1}''$ and $X_2 := Y_r''$. Then $Y = X_1 \cup X_2$ and $\langle X_1 \rangle \cap \langle X_2 \rangle \neq \emptyset$. Hence by [2], Lemma 1.1.(b) at least one among X_1, X_2, say X_1, has degenerate general hyperplane section in its span.

Then we can apply the induction assumption to X_1. We have $s(X_1) \leq s$, $m(X_1) = m$ and $z \leq z_{X_1}, f \leq z$. Thus by induction we have:

$$z + 2m(X) \leq z_{X_i} + 2m(X_i) \leq s(X_i) + 2 \leq s(X) + 2.$$

which is our claim. □

Example 2.14. Let $Y \subseteq \mathbf{P}^5$ be a non-degenerate curve with degenerate general hyperplane section and $m = 2$. Then for every irreducible component D of Y_{red}, with $\deg D \geq 2$ we have $z \leq 1$. Indeed, set $w := \dim(\langle D'' \rangle)$. If $w = 2$ we have $z = 0$. If $w = 3$ we have $z \leq 1$. If $w = 5$ we have $z = 1$ by Example 2.6.

Moreover it cannot be $w = 4$. Indeed, assume $w = 4$. Then: if D'' has degenerate general hyperplane section, we conclude by 2.5 applied to $D'' \subseteq \langle D'' \rangle = \mathbf{P}^4$; if D'' has non-degenerate general hyperplane section, then for every general hyperplane H we have $\dim(\langle (Y \cap H) \rangle) = 3$, whence $\langle (D'' \cap H) \rangle = \langle (Y \cap H) \rangle$, which implies $Y \cap H \subseteq \langle (D'' \cap H) \rangle = \langle D'' \rangle \cap H$, and $Y \subseteq \langle D'' \rangle$ by Corollary 1.3. This is a contradiction, being Y non degenerate.

Note that this example is more precise than Proposition 2.13: indeed it allows components of Y_{red} not spanning $\langle Y_{red} \rangle$).

We conclude this section by some hints on how to deal with the case when Y_{red} is non-connected. In this situation there is a trivial case, namely $Y = A \cup B$ where A and B are curves with $\langle A \rangle \cap \langle B \rangle = \emptyset$. Indeed if this holds, then the general hyperplane section of Y is degenerate, as remarked in [2], Lemma 1.1.

In view of the above it is natural to give the following:

Definition 2.15. *Let $Y \subseteq \mathbf{P}^n$ be a curve and let Y_j, $(1 \leq j \leq p)$ be the irreducible components of Y_{red}. Then Y is said to be linearly connected if it is possible to order the Y_j's in such a way that*

$$\langle Y_1 \cup Y_2 \cup \ldots \cup Y_{j-1} \rangle \cap \langle Y_j \rangle \neq \emptyset \qquad (2 \leq j \leq p).$$

Observe that if Y_{red} is connected, then Y is linearly connected. It is easy to show that the converse is false.

For linearly connected curves we have the following:

Lemma 2.16. *Let $Y \subseteq \mathbf{P}^n$ be a reduced linearly connected curve. Then $s(Y) = m(Y) - 1$. In particular if Y is non-degenerate, then the general hyperplane section of Y is non-degenerate.*

Proof. Let $Y_1, \ldots, Y_p$ be the irreducible components of Y. We use induction on p, the case $p = 1$ being well known (see e.g. Lemma 1.4).

Let $p > 1$ and set $Z_1 = Y_1 \cup Y_2 \cup \ldots \cup Y_{p-1}$, $Z_2 = Y_p$. Since Y is linearly connected we may assume that Z_1 is linearly connected, and hence by the inductive hypothesis the claim holds for Z_1 and Z_2. Assume the claim false for Y, that is assume $Y \cap H$ is degenerate (with respect to $\langle Y \rangle$.

Since Y is linearly connected we may assume that $\langle Z_1 \rangle \cap \langle Z_2 \rangle \neq \emptyset$, whence by [2], Lemma 1.1 either Z_1 or Z_2 has degenerate general hyperplane section (with respect to its span), a contradiction. □

Remark 2.17. Using Lemma 2.16 it is easy to show that 2.11 and 2.13 hold under the weaker assumption of Y being linearly connected. Indeed the proofs of 2.11 and 2.13 work also in this case. We leave the details to the reader.

3. Results in higher dimension

We begin with a generalization of Lemma 3.1 in [2], which allows to work in arbitrary characteristic and to bring into play the integer $s(X)$ (see Section 1).

Lemma 3.1. *Let $X \subseteq \mathbf{P}^n$ be a non-degenerate closed subscheme with degenerate general hyperplane section. Assume that X has no zero-dimensional components and that* $\dim(X) \geq 2$. *Let Y be a general hyperplane section of X. Then $s(Y) \leq s(X) - 2$.*
In particular the general hyperplane section $Y \cap H$ of Y, considered as a closed subscheme of $\langle Y \rangle \cap H$, is degenerate.

Proof. Let H_1, H_2, H_3 be three general hyperplanes, and put $Y_i := X \cap H_i$ and $Z_{i,j} := Y_i \cap H_j$, for $i, j = 1, 2, 3$ and $i \neq j$. By definition we have $\dim\langle Y_i \rangle = s(X)$ and $\dim\langle Z_{i,j} \rangle = s(Y)$ for all i, j as above. Clearly $s(Y) \leq s(X) - 1$, so if the conclusion is false we have $\dim(\langle Z_{1,3} \rangle) = \dim(\langle Z_{2,3} \rangle) = \dim(\langle Y_3 \rangle) - 1$, and since $\langle Z_{1,3} \rangle \neq \langle Z_{2,3} \rangle$ we must have $\langle \langle Z_{1,3} \rangle \cup \langle Z_{2,3} \rangle \rangle = \langle Y_3 \rangle$. Then an easy calculation shows that $\langle Y_3 \rangle \subseteq \langle \langle Y_1 \rangle \cup \langle Y_2 \rangle \rangle$. Now let H_1 and H_2 be fixed, and let H_3 vary. Then by Corollary 1.3 we have that $X \subseteq \langle \langle Y_1 \rangle \cup \langle Y_2 \rangle \rangle$, and since X is non-degenerate it follows $\dim(\langle \langle Y_1 \rangle \cup \langle Y_2 \rangle \rangle) = n$. Now $\langle \langle Y_1 \rangle \cap \langle Y_2 \rangle \rangle = \langle Z_{1,2} \rangle$ and from the above it follows:

$$\begin{aligned} s(Y) &= \dim(\langle Z_{1,2} \rangle) \\ &= \dim(\langle Y_1 \rangle) + \dim(\langle Y_2 \rangle) - n \\ &\leq s(X) - 2 \end{aligned}$$

the last inequality because $s(X) - n \leq -2$ by assumption. This is a contradiction, and the conclusion follows. □

Definition 3.2. *Let $X \subseteq \mathbf{P}^n$ be a closed subscheme of dimension $d \geq 2$, with X_{red} irreducible.*

(i) *We define the generic spanning increasing z_X of X as in the case of curves (cf. Def. 2.1), by replacing the general hyperplane H with a general linear space L of codimension d.*

(ii) *We define the general fattening dimension of* X *to be the integer*

$$f_X := \dim(T_P(X^{(1)})) - \dim(T_P(X^{(1)} \cap \langle X_{red} \rangle)$$

where $X^{(1)}$ *is the largest subscheme of* X*, without embedded components, contained in the first neighborhood of* X_{red} *and* P *is a general point in* X_{red}*.*

Definition 3.3. *Let* $X \subseteq \mathbf{P}^n$ *be a closed subscheme of dimension* $d \geq 2$*. Let* L *be a general linear space of codimension* $d-1$*. If* $X \cap L$ *is a curve according to our definition we call it a general curve section of* X*.*

The following Lemma gives sufficient conditions for the existence of a general curve section.

Lemma 3.4. *Let* $X \subseteq \mathbf{P}^n$ *be a closed subscheme of dimension* $d \geq 2$*. Assume further that* X *has Serre's property* S_2 (see e.g. [7]). *Then we have:*
(i) *If* X_{red} *is irreducible, then a general curve section* Y *of* X *exists and* Y_{red} *is irreducible;*
(ii) *if* X_{red} *is connected, then* X *is equidimensional, a general curve section* Y *of* X *exists and* Y_{red} *is connected.*

Proof. (i) By [2], Lemma 3.2(b) it follows that Y has property S_2, and is equidimensional, hence it is a curve according to our definition. The irreducibility of Y_{red} follows easily from Lemma 1.4 and Bertini's Theorem.

(ii) Since X_{red} is connected, property S_2 implies that X is equidimensional, as follows easily from [7], Corollary. 5.10.9. The conclusion follows from [2], Lemma 3.2(c). □

We want to study the behavior of the integer z_X when passing to a general curve section.

Proposition 3.5. *Let* $X \subseteq \mathbf{P}^n$ *be a non-degenerate* S_2 *closed subscheme of dimension* $d \geq 2$*, with* X_{red} *irreducible and let* Y *be a general curve section of* X*. Then:*
(i) $z_X = z_Y$ *and* $f_X = f_Y$*;*
(ii) *if the general hyperplane section of* X *is degenerate, then* $z_X > 0$*.*

Proof. (i) Put $Y = X \cap L$, where L is a general linear space of codimension $d-1$. Then Y_{red} is an irreducible curve by Lemma 3.4, and hence the statement makes sense.

By Lemma 1.4 we have $Y_{red} = X_{red} \cap L$ and the first equality follows immediately. Moreover one can prove easily that $Y^{(1)} = X^{(1)} \cap L$ and $T_P(Y^{(1)}) = T_P(X^{(1)}) \cap L$. The conclusion follows from Proposition 2.3 and a direct calculation.

(ii) By Lemma 3.1 the general hyperplane section of Y is degenerate, whence $z_Y > 0$ by Proposition 2.3(iii). The conclusion follows from (i). □

Our next result generalizes Theorem 2.5 to higher dimension.

Theorem 3.6. *Let* $X \subseteq \mathbf{P}^n$ *be an* S_2 *closed subscheme of dimension* d*, with* X_{red} *irreducible. Assume that the general hyperplane section of* X *is degenerate. Put* $s := s(X)$*,* $m := m(X)$*,* $z := z_X$ *and* $f := f_X$*. Then*

(i) $f + 2m \leq z + 2m \leq s + 2 \leq n$
(ii) $2m \leq s + 1 \leq n - 1$.

Proof. If $d = 1$ this is just Theorem 2.5. Assume $d \geq 2$ and let Y be a general curve section of X (Lemma 3.4). Then we have: $m(X) = m(Y) + d - 1$ by Lemma 1.4, and $s(X) \geq s(Y) + 2(d - 1)$ by Lemma 3.1. The conclusion follows from Proposition 3.5 and Theorem 2.5 applied to Y. □

Now we turn our attention to reducible schemes. Our first result generalizes [2], Theorem 3.5(a).

Proposition 3.7. *Let $X \subseteq \mathbf{P}^n$ be a non-degenerate S_2 closed subscheme with degenerate general hyperplane section. Assume X_{red} connected. Put $d := \dim(X)$, $m := m(X)$, $s := s(X)$. Then $m \leq s - d + 1$.*

Proof. If $d = 1$ the statement is Proposition 2.11. Assume $d > 1$. By Lemma 3.4 we can consider a general curve section Y of X and Y_{red} is connected. By Lemma 3.1 we have $s(Y) \leq s(X) - 2(d - 1)$ and by Lemma 1.4 we have $m(Y) = m(X) - (d - 1)$. The conclusion follows from Proposition 2.11 and a trivial calculation. □

Corollary 3.8. *Let the notation and the assumptions be as in Proposition 3.7. Then $d \leq \frac{s+1}{2}$ and if equality holds then X_{red} is a linear space. In particular $d \leq \frac{n-1}{2}$.*

Proof. We have $d \leq m$ and $d = m$ if and only if X_{red} is a linear space. The conclusion follows from Proposition 3.7. □

As an immediate consequence of Corollary 3.8 we have the following statement, which was proved in characteristic zero ([2], Theorem 3.3(a)) .

Corollary 3.9. *Let $X \subseteq \mathbf{P}^n$ be a non-degenerate S_2 closed subscheme with X_{red} connected. Assume that $\dim(X) \geq \frac{n}{2}$. Then the general hyperplane section of X is non-degenerate.*

Now we want to generalize Proposition 2.13 to higher dimension.

Definition 3.10. *Let $X \subseteq \mathbf{P}^n$ be a closed subscheme. We define the minimal spanning increasing z_X and the minimal fattening dimension f_X in the obvious way, similar to Definitions 2.12 and 3.3. If X is S_2 and equidimensional (in particular connected) and Y is a general curve section of X (Definition 3.4) one can prove that $z_Y = z_X$ and $f_Y = f_X$, with the same argument as in Proposition 3.5.*

Proposition 3.11. *Let $X \subseteq \mathbf{P}^n$ be a non-degenerate S_2 closed subscheme with degenerate general hyperplane section. Assume X_{red} connected and $\langle X_{red} \rangle = \langle X_i \rangle$ for every irreducible component of X_{red}. Put $d := \dim(X)$, $m := m(X)$, $s := s(X)$, $z = z_X$, $f = f_X$. Then $2m + f \leq 2m + z \leq s + 2 \leq n$.*

Proof. If $d = 1$ the statement is Proposition 2.13. The general case can be proved by reduction to the general curve section as in the proof of Proposition 3.7. We leave the details to the reader. □

Remark 3.12. Let $X \subseteq \mathbf{P}^n$ be a non-degenerate S_2 closed subscheme of dimension d with degenerate general hyperplane section. We wish to understand what can be the maximal possible degeneration for the general hyperplane section of X (i.e. the least value of $s(X)$ with respect to 3.6 and 3.8). Recall that if X_{red} is connected then $s \geq 2d-1$ by 3.8.

1) If $s = 2d-1$ then X_{red} is a linear space by 3.8 and $z = 1$ by 3.5 and 3.6. Then X must be a double structure on X_{red}.
2) If $s = 2d$ then $d \leq m \leq d+1$ by 3.6. Then:

2a) if $m = d$ then X_{red} is a linear space and $1 \leq z \leq 2$ by 3.5 and 3.6;

2b) if $m = d+1$ and we can apply 3.11 then $1 \leq z \leq 2$. Otherwise X_{red} has at least a linear irreducible component and a non-linear one.

We will see in Example 3.13 that in some situation the extremal cases described above can actually occur. We don't know what happens in general.

Example 3.13. We show that the bound $d \leq \frac{n-1}{2}$ in 3.8 is sharp when n is odd. This is clear if $d = 1$ (take any non-degenerate double line in $\mathbf{P}^3$), hence we assume $d \geq 2$. We must have $n = 2d+1$ and X_{red} must be a linear subspace, say L, of dimension d. As we have seen in Remark 3.12, we have that $z = 1$ and X is a double structure on L. We will show that such double structures do exist, also with property S_2. Set $I = I_L$ and recall that X is a double structure on L with property S_2 if and only if the ideal sheaf $J := I_X$ of X fits into an exact sequence of $\mathcal{O}_L$-modules of the form

$$(*) \qquad 0 \to J/I^2 \to I/I^2 \to \mathcal{O}_L(a) \to 0$$

for a suitable a (see e.g. [3], Theorem 5.2 for details).

Now we have $I/I^2 \cong (n-d)\mathcal{O}_L(-1)$, whence the above sequence exists if and only if there is a surjective morphism $\phi : (n-d)\mathcal{O}_L \to \mathcal{O}_L(a+1)$. Any such ϕ is determined by $f_1, \ldots, f_{n-d} \in H^0(\mathcal{O}_L(a+1))$ with no common zeroes. Since $n-d = d+1$ this can happen if and only if $a \geq -1$.

Now we study the non-degeneracy of X. Since the closed subscheme corresponding to the ideal sheaf I^2 is aCM, from the exact sequence:

$$0 \to I^2 \to J \to J/I^2 \to 0$$

we get the exact sequence

$$0 = H^0(I^2(1)) \to H^0(J(1)) \to H^0((J/I^2)(1)) \to 0.$$

It follows $h^0(J(1)) = h^0(J/I^2)(1)$. From $(*)$ we deduce the exact sequence:

$$0 \to (J/I^2)(1) \to (n-d)\mathcal{O}_L \to \mathcal{O}_L(a+1) \to 0$$

whence $H^0(J/I^2)(1) = 0$ if and only if the map $H^0((n-d)\mathcal{O}_L) \to H^0(\mathcal{O}_L(a+1))$ is injective, which is equivalent to say that $f_1, \dots, f_{n-d}$ are linearly independent; this can happen if and only if $a \geq 0$.

Therefore from now on we assume $a \geq 0$ and $f_1, \dots, f_{n-d}$ linearly independent.

Let now H be a general hyperplane and let $L' := L \cap H$. Let $X' := X \cap H$ and put $I' := I_{L',H} = I_{|H}$, $J' := I_{X',H} = J_{|H}$.

We have the exact sequences:

$$0 \to J'/I'^2 \to I'/I'^2 \to \mathcal{O}_{L'}(a) \to 0$$

$$0 \to I'^2 \to J' \to J'/I'^2 \to 0$$

As above we see that X' is degenerate if and only if $h^0((J'/I'^2)(1)) \neq 0$ and this is equivalent to say that the restriction of ϕ

$$\phi' : H^0((I'/I'^2)(1)) \to H^0(\mathcal{O}_{L'}(a+1))$$

is not injective.

Now $I'/I'^2 = (n-d)\mathcal{O}_{L'}(-1)$ whence $h^0((I'/I'^2)(1)) = n - d = d + 1$

Since $h^0\mathcal{O}_{L'}(a+1) = \binom{a+d}{d-1}$, ϕ' is certainly non injective if $\binom{a+d}{d-1} < d+1$ i.e. if $a = 0$.

Thus we get examples for $a = 0$.

Remarks 3.14.

(i) Do we get examples as in 3.13 also for $a > 0$? If so, this would depend on the choice of $f_1, \dots, f_{n-d}$.

(ii) We don't know how to produce examples if n is even. For example if $n = 6$ we have $d \leq 2$ and $m \leq 3$. If $d = 1$ it is easy to give examples. If $d = 2$ the argument in 3.13 can be adapted by taking $a = 1$ and produces a double plane as in the odd case. However we don't know if the case $m = 3$ can actually occur. For higher even n the problem remains open.

Example 3.15. Let $n \geq 5$ be an odd integer and let $n = 2d + 1$. Let $X \subseteq \mathbf{P}^n$ be the scheme of dimension d constructed in Example 3.13. Let F be a hypersurface of degree > 1 not containing X_{red}, and let $W := F \cap X$. From the exact sequence $0 \to I_W \to I_F \oplus I_X$, we get the exact sequence $0 \to H^0(I_W(1)) \to H^0(I_F(1)) \oplus H^0(I_X(1))$ which readily implies that $H^0(I_W(1)) = 0$, that is W is non-degenerate. Moreover the general hyperplane section of W is obviously degenerate. Observe also that if F is general we can have that W_{red} is smooth (and irreducible). Finally it is easy to see that W is a double structure on W_{red} and that $m(W) = d$. From Theorem 3.6 it follows that $z_W = 1$ and the bound given by the same theorem is sharp.

In particular if $n = 5$ we have examples of curves as described in Example 2.7.

Remark 3.16. When dealing with schemes X with X_{red} non-connected one could try to use the notion of linearly connected (Definition 2.15). But this definition has the obvious drawback of not being preserved by general hyperplane sections. This makes the usual induction methods not immediate to apply.

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