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RWPRI and $(2T)_1$ Flag-transitive Linear Spaces

F. Buekenhout P.-O. Dehaye* D. Leemans†

Département de Mathématique, Université Libre de Bruxelles
C.P. 216 - Géométrie, Boulevard du Triomphe, B-1050 Bruxelles, Belgium

Abstract. The classification of finite flag-transitive linear spaces is almost complete. For the thick case, this result was announced by Buekenhout, Delandtsheer, Doyen, Kleidman, Liebeck and Saxl, and in the thin case (where the lines have 2 points), it amounts to the classification of 2-transitive groups, which is generally considered to follow from the classification of finite simple groups. These two classifications actually leave an open case, which is the so-called 1-*dimensional* case. In this paper, we work with two additional assumptions. These two conditions, namely $(2T)_1$ and RWPRI, are taken from another field of study in Incidence Geometry and allow us to obtain a complete classification, which we present at the end of this paper. In particular, for the 1-dimensional case, we show that the only $(2T)_1$ flag-transitive linear spaces are $AG(2,2)$ and $AG(2,4)$, with $A\Gamma L(1,4)$ and $A\Gamma L(1,16)$ as respective automorphism groups.

1. Introduction

Flag-transitive linear spaces[1] are very common objects appearing in the theory of Incidence Geometry [5]. A major goal of Incidence Geometry is to generalize J. Tits' theory of Buildings, which was itself developed in order to acquire a better understanding of the simple groups of Lie type.

*The address of this author is now: Department of Mathematics, Building 380, Stanford University, 94305-2125 Stanford, CA, U.S.A.
†e-mail: dleemans@ulb.ac.be

[1]See Section 2 for the definitions.

The study of flag-transitive linear spaces with line-size 2 is equivalent to the study of 2-transitive permutation groups, which is generally considered to follow from the classification of finite simple groups. Some work concerning the classification of 2-transitive groups goes back to the 19^{th} century, and an explicit list finally appears in Kantor [33]. The list includes a unique case remaining open to some extent. We call this case 1-*dimensional*, because the automorphism group is then a group of semilinear transformations of an affine line.

This 1-dimensional gap also appears in the study of the thick case (when the line-size is greater than 2). In this case, a basic result of Higman and McLaughlin [4] shows that all flag-transitive groups acting on a linear space are primitive on their point-set. Using the well-known O'Nan-Scott theorem, Buekenhout, Delandtsheer and Doyen[2] showed that the possible groups are either almost simple or affine [10]. In 1990, these three authors together with Kleidman, Liebeck and Saxl[3] announced a classification of the linear spaces for which the full automorphism group is not 1-dimensional and acts flag-transitively [11].

Proposition 1.1. (BDDKLS [11]) *Any flag-transitive thick linear space with non-*1*-dimensional automorphism group is one of the following:*

- *a projective space*
- *an affine space*
- *a Hering space*
- *a Witt-Bose-Shrikhande space*
- *a hermitian unital*
- *a Ree unital* □

The proof of this result was published in several papers, and only completed a short time ago. The proof in the affine case is due to Liebeck [40]. The almost simple case involved several authors. Delandtsheer took the case where the simple socle is an alternating group [20]. She also handled the case where the group G is one of the simple groups $L_2(q)$, $L_3(q)$, $U_3(q)$ and ${}^2B_2(q)$ [17]. In his paper [35], Kleidman solved the case where the socle of G is an exceptional group of Lie type. He gave a proof for three of the ten families of exceptional groups and some hints for the remaining cases. The case of the sporadic groups was ruled out by Buekenhout, Delandtsheer, and Doyen [8] and Davies [16]. Finally, Saxl completed the proof in a recent preprint [46], where he dealt with the remaining families of exceptional type together with the classical groups of Lie type.

The second aspect of this work is to use other properties, namely $(2T)_1$ and RWPri. Historically, these concepts arose from experimental and theoretical work of Buekenhout, Cara, Dehon and Leemans. They start with a group G, a collection of subgroups and a construction of Tits based on this data. This method produces too many geometries, and a natural task is to find additional properties fulfilled by the most "interesting" of those geometries. Prototypes for interesting geometries are the buildings. The theory of buildings and other interesting sporadic geometries lead to the RWPri and $(2T)_1$ conditions. The RWPri geometries of several infinite families of groups are now classified (see for instance Leemans [37, 39, 36, 38] for the Suzuki groups, Salazar Neumann [45] for the $PSL(2,q)$

[2]Following usual conventions, the names of these authors are abbreviated to BDD.

[3]This group of authors is, logically, referred to as BDDKLS.

groups and their rank 2 geometries). The RWPri geometries have also been studied for small groups such as $PSL(3,4)$ in Gottschalk and Leemans [29] or the small alternating and symmetric groups in Cara [14]. Atlases have also been collected [6, 7].

This interest in the RWPri and $(2T)_1$ conditions brought us to look once again at the classifications of BDDKLS and of the 2-transitive groups, in particular at the case left aside, namely the 1-dimensional case.

Let us observe that Anne Delandtsheer obtained in [19] a complete classification of the 1-dimensional case using as extra assumption that the automorphism group acts transitively on the unordered pairs of intersecting lines.

Section 2 presents the definitions and notation we use in this paper. In Section 3, we proceed very fast through the classification of BDDKLS, assuming complete knowledge of its introductory papers (see in particular the description of all known flag-transitive linear spaces in BDD [10]). Similarly, the classification of 2-transitive permutation groups provided by Kantor [33] is the basis for Section 4. We complete our study of flag-transitive and RWPri linear spaces with Section 5, in which we consider the 1-dimensional spaces. We summarize all our results in the Theorems 6.1 and 6.2 of Section 6.

2. Definitions and notation

2.1. Basic review of incidence geometry

Our main reference for incidence geometry is the Handbook of Incidence Geometry [5].

A *linear space* is an incidence structure made of *points* (elements of type 0) and *lines* (elements of type 1) such that there is exactly one line incident with any two points, any point is incident with at least two lines and any line has at least two points. Throughout this work, we further assume that the set of points is finite. Two elements of the same type are never considered to be incident. A *flag* of a linear space Γ is a set of pairwise incident elements. The *type* of a flag F, that is denoted by $t(F)$, is the set of types of the elements of F. We denote by $\mathrm{Aut}(\Gamma)$ the group of type-preserving automorphisms of Γ. Let $G \leq \mathrm{Aut}(\Gamma)$ be a group of automorphisms of Γ. We say that G is *flag-transitive* if it is transitive on flags of the same type.

If F is a flag of an incidence structure Γ, we denote by Γ_F the *residue* of F, i.e. the incidence structure made of the elements of Γ incident with all elements of F.

We define G_F to be the *stabilizer* in G of a flag F of Γ and K_F to be the subgroup of G_F fixing Γ_F elementwise. We write $\overline{G_F}$ for the quotient G_F/K_F, i.e. the group induced by G_F on Γ_F. Since in flag-transitive geometries stabilizers of flags of the same type are conjugate subgroups in G, we prefer to write $G_{t(F)}$ or $\overline{G_{t(F)}}$ instead of G_F or $\overline{G_F}$.

2.2. The RWPri and $(2T)_1$ properties

We define notation and properties used throughout this paper. Some of them are rather frequent: PPr stands for point-primitivity, (2iL)T for transitivity on pairs of intersecting lines and (2P)T for transitivity on ordered pairs of points. Assume now G is the automorphism group of a linear space Γ. Then, the action of G on Γ is said to be

LoLPr (***locally line-primitive***) if G is transitive on the set of points of Γ and if, for every point x of Γ, the stabilizer G_x acts primitively on the set of lines incident with x;

LoPPr (***locally point-primitive***) if G is transitive on the set of lines of Γ and if, for every line L of Γ, the stabilizer G_L acts primitively on the set of points incident with L;

LoPr (***locally primitive***) if G is both LoLPr and LoPPr. These three properties appeared in a previous paper of BDD [9].

(2iL)T if G is transitive on ordered pairs of intersecting lines;

WPri (***weakly primitive***) if there exists an element e of Γ such that G_e is a maximal subgroup of G;

RWPri (***residually weakly primitive***) if this action is weakly primitive and the action of G_e is residually weakly primitive on Γ_e for any element e of Γ;

$\mathbf{(2T)_1}$ if G is 2-transitive on all rank 1 residues of Γ (i.e. all residues with elements of only one type).

Lemma 2.1. (Dixon and Mortimer [22]) *A transitive permutation group G is primitive on its point-set if and only if the point-stabilizers are maximal subgroups of G.* □

Remarks.

- Lemma 2.1 explains the terminology for the RWPri property.
- The LoPr-property implies flag-transitivity. This is stated by BDD [10].
- A flag-transitive group acting on a linear space is primitive on its point-set. This is a well-known result due to Higman and McLaughlin [4]. This fact, together with the preceding remark, implies that a linear space is flag-transitive and RWPri if and only if it is LoPr, and also that a flag-transitive $(2T)_1$ linear space is always RWPri.

2.3. 1-dimensional affine groups

A permutation group G acting on a set of points Ω is called *affine* if Ω can be identified with $GF(p^d)$ or equivalently with the point-set of $AG(d,p)$ and if $G \leq AGL(d,p)$. If we denote by $A\Gamma L(n, p^{d/n})$ the group of all semilinear transformations[4] of $AG(n, p^{d/n})$, then the group G is called *n-dimensional affine* if n is the smallest positive integer such that $G \leq A\Gamma L(n, p^{d/n})$.

3. Six families of linear spaces from BDDKLS

Historical introductions, constructions and some properties are given for each of those families in [11] and [10]. We give a brief description of each case, and present the properties needed to determine whether the space considered is RWPri, $(2T)_1$ or does not have any of these properties[5].

[4]Semilinear transformations are defined using the field automorphisms. The general form of such a permutation is $x \mapsto ax^\sigma + b$ where $a, b \in GF(p^d), a \neq 0, \sigma \in \mathrm{Aut}(GF(p^d))$.

[5]Recall that $(2T)_1$ implies RWPri for flag-transitive linear spaces.

3.1. Projective spaces

We start with a proposition due to Cameron and Kantor.

Proposition 3.1. (Kantor [32, 34], Cameron and Kantor [13]) *Assume Γ is a linear space consisting of the points and lines of a projective space, and $G \leq \mathrm{Aut}(\Gamma)$ acts flag-transitively. Then, one of the following occurs:*

- *The point-set and line-set of Γ are those of $PG(n,q)$ and we have the inclusions $PSL(n+1,q) \trianglelefteq G \leq P\Gamma L(n+1,q)$, with $n \geq 2$ and q a power of a prime p. This is the* classical *case.*
- *$\Gamma = PG(3,2)$, with $G \cong A_7$.*
- *Γ is a projective plane of order m, m^2+m+1 is a prime and G is a sharply flag-transitive Frobenius group of order $(m^2+m+1)(m+1)$. This is the* 1-dimensional *case.* □

We first recall a result of Feit that restricts the last case of Proposition 3.1.

Proposition 3.2. (Feit [24]) *Assume Γ and G are as in the third case of Proposition* 3.1. *Then, the only Desarguesian spaces are $PG(2,2)$ and $PG(2,8)$. Moreover, any other example, if it exists, must have $m \equiv 0 \mod 8$, m not a power of 2 and $d^{m+1} \equiv 1 \mod (m^2+m+1)$ for every d dividing m.* □

Remark. Using his powerful result, Feit showed that all projective planes with a flag-transitive group of order $\leq 14{,}400{,}008$ are Desarguesian (the problem is heavily reduced to a number theory matter, which is much more suitable for computation). Feit asserted in his paper that his result could easily be extended. With the help of the computer algebra packages MAGMA [1] and GAP [27], we extended this bound up to $m \leq 10^{10}$.

We now prove the following results concerning RWPRI projective spaces and $(2T)_1$ projective spaces.

Proposition 3.3. *Assume Γ is a linear space consisting of the points and lines of a projective space, and G acts flag-transitively on Γ. If this action is also* RWPRI, *then one of the following occurs:*

- *$\Gamma = PG(n,q)$ and $PSL(n+1,q) \trianglelefteq G \leq P\Gamma L(n+1,q)$.*
- *$\Gamma = PG(3,2)$ and $G \cong A_7$.*
- *$\Gamma = PG(2,2)$ and $G \cong 7:3$.*

Moreover, the action is $(2T)_1$ only in the first two cases.

Proof. We consider each case of Proposition 3.1.

Classical case: Those spaces are typical examples of linear spaces satisfying $(2T)_1$ and RWPRI.

$\Gamma = PG(3,2)$*:* We can look at the action of A_7 described in Taylor [49]. We have the following relations: $G_0 = \overline{G_0} \cong PSL(3,2)$, $\overline{G_1} \cong S_3$ and $G_{01} \cong S_4$. The action of both stabilizers is 2-transitive, hence this action is $(2T)_1$.

1-*dimensional case:* This case is already reduced by Proposition 3.2. Assume now the action is RWPRI. This would imply that the action is also LOLPR, hence, since the group is sharply flag-transitive, that the point-degree $(m+1)$ is a prime.
Consider first the non-Desarguesian case. By Proposition 3.2, we get $m^{m+1} \equiv 1 \mod (m^2+m+1)$. Moreover, $m^3 = m(m^2+m+1) - (m^2+m+1) + 1$. Hence, $3|m+1$, a contradiction to $m+1$ prime and m the order of a non-Desarguesian projective plane.
An easy verification for the Desarguesian case shows that only $G = 7:3$ on $\Gamma = PG(2,2)$ has an RWPRI action. □

3.2. Affine spaces

We denote by q the power of a prime p (we assume $q \geq 3$, and we let $q^n = p^d$) and by $[Af_n(q)]$ (resp. $[AG(n,q)]$) the class of all geometries made of the points and lines of an affine space (resp. Desarguesian affine space). Finally, if $\Gamma \in [Af_n(q)]$, G must be a subgroup of $\mathrm{Aut}(\Gamma)$.

3.2.1. Affine spaces of dimension at least 3

We consider here $\Gamma \in [Af_n(q)]$ with $n \geq 3$. In this case, it is well-known that $\Gamma \in [AG(n,q)]$. Furthermore, BDDKLS showed that one of the following holds:

- G is 1-dimensional, but we delay the discussion of this case until Section 5.
- $\Gamma = AG(4,3)$ and the last term of the derived series of G_0 is $2 \cdot A_5$. Huybrechts [31] noticed that this case is impossible.
- G is 2-transitive. This case was restricted further by Huybrechts, as we state now.

Proposition 3.4. (Huybrechts [31]) *If G is 2-transitive on $\Gamma \in [AG(n,q)]$, with $n \geq 3$, then $G = p^d : G_0$, where $G_0 \leq \Gamma L(n,q)$ and one of the following holds:*

1. $SL(u, q^{n/u}) \trianglelefteq G_0$, *for some integer $u \geq 1$ dividing n.*
2. $Sp(u, q^{n/u}) \trianglelefteq G_0$, *for some even integer $u \geq 4$ dividing n.*
3. $G_2'(q^{n/6}) \trianglelefteq G_0$, *for q even.*
4. $(n,q) = (6,3)$, $G_0 \cong SL(2,13)$ *and* $\overline{G_0} \cong PSL(2,13)$.
5. $(n,q) = (4,3)$ *and* $\overline{G_0} = 2^4.A$ *where* $A \in \{5, D_{10}, 5.4, A_5, S_5\}$.
6. $(n,q) = (4,3)$ *and* $G_0 = 4 \cdot A_5$ *or* $N \cdot S_5$, *with* $N \leq 4$. □

Using this classification, we determine all RWPRI affine spaces of dimension at least 3 for which the automorphism group is 2-transitive but not 1-dimensional. Actually, we already know by 2-transitivity that those spaces are LOPPR. Hence, the only property which remains to be considered is LOLPR.

A preliminary lemma will be very useful to our proof:

Lemma 3.5. *Let v be a positive integer and G be a permutation group. Consider the set S of all affine spaces of v points on which G acts faithfully and flag-transitively. Assume Γ has not the biggest line size in S. Then Γ is not* LOLPR.

Proof. Denote the affine space with biggest line size by Γ' (this space exists, since line size induces a total order on S, and the size of S is finite). Clearly, the lines of Γ' induce a partition of the lines of Γ. This partition is preserved under the action of G, since the automorphism group of Γ' is precisely G. Moreover, this partition is nontrivial by the non-degeneracy assumptions on Γ'. Hence, Γ is not LoLPr. $\square$

Proposition 3.6. *If G is flag-transitive on $\Gamma \in [AG(n,q)]$, with $n \geq 3$, then $G = p^d : G_0$, where $G_0 \preceq \Gamma L(n,q)$. Moreover, if G is 2-transitive,* RWPRI *but not 1-dimensional, one of the following holds:*

1. $SL(n,q) \trianglelefteq G_0$, *$n \geq 2$ and (Γ, G) is $(2\mathrm{T})_1$.*
2. $Sp(n,q) \trianglelefteq G_0$, *$n \geq 4$, n is even and (Γ, G) is not $(2\mathrm{T})_1$.*
3. $G_2'(q) \trianglelefteq G_0$, *for q even and (Γ, G) is not $(2\mathrm{T})_1$.*

Proof. We discuss each case according to the list given in Proposition 3.4.

Cases 1 *and* 2*:* Here, we have $G_0 \trianglerighteq SL(u, q^{n/u})$ or $Sp(u, q^{n/u})$; it implies that G acts 2-transitively on the following affine spaces of q^n points: $AG(n,q)$ and $AG(u, q^{n/u})$. Using Lemma 3.5, and considering the LoLPr-property, we deduce that u has to be equal to n, and using our knowledge of linear and symplectic groups, that those two actions are then LoLPr, hence RWPRI. It remains to check the $(2\mathrm{T})_1$ property on these two cases. For the first case, the action of $G_0 \trianglerighteq SL(u, q^{n/u})$ is 2-transitive on the lines intersecting 0 and so we have the $(2\mathrm{T})_1$ property. For the second case, the group $G_0 \trianglerighteq Sp(u, q^{n/u})$ is a rank 3 group and hence its action is not 2-transitive on the lines intersecting 0.

Case 3*:* We know that n has to be a multiple of 6. Say $n = 6t$. Using Proposition 3.4, we observe that the derived group $G_2'(q^t)$ can always act 2-transitively on both $AG(6t, q)$ and $AG(6, q^t)$. When $t \neq 1$, the second space has bigger line-size than the first one. This implies that all RWPRI spaces occuring in this case must be 6-dimensional affine spaces (or equivalently, we must have $n = 6$).
The action described here is closely related to generalized polygons and in particular to the Split Cayley hexagon $\mathbf{H}(q)$ (see Van Maldeghem [52]). If we fix a point and consider the action of this stabilizer at infinity, the action is the automorphism group of a particular embedding of the Split Cayley hexagon, i.e. the perfect symplectic embedding. This embedding can be achieved when the characteristic of the field is 2 and the field is perfect. Then, there is an embedding of $\mathbf{H}(q)$ into $PG(5,q)$. This $PG(5,q)$ is now our hyperplane at infinity. As a general consequence of Tits's theory of buildings, we know that the action of $G_2(q)$ is primitive on the points of this embedding. On the other hand, the action of $G_2(q)$ at infinity cannot be 2-transitive, since the hexagon contains non-collinear points. Hence this case does not satisfy the $(2\mathrm{T})_1$ property.

Cases 4, 5 *and* 6*:* A simple computation of the order of the point-stabilizers shows that all the groups are too small to act primitively on the set of lines intersecting in a given point. $\square$

3.2.2. Affine planes

Desarguesian affine planes

Proposition 3.7. (Foulser [25, 26]) *Let Γ be $AG(2,q)$, with q a prime power, and let G be a group acting flag-transitively on Γ. Then $G = p^d : G_0$ and one of the following holds:*

1. $G_0 \leq \Gamma L(1,q^2)$, *hence G is 1-dimensional,*
2. $SL(2,q) \trianglelefteq G_0$ *and* $PGL(2,q) \trianglelefteq \overline{G_0}$,
3. $q = 9$ *and* $\overline{G_0} \cong S_5$, *or* $q \in \{9, 11, 19, 29, 59\}$ *and* $\overline{G_0} \cong A_5$, *or* $q \in \{5, 7, 11, 13\}$ *and* $\overline{G_0} \cong S_4$, *or* $q \in \{5, 11\}$ *and* $\overline{G_0} \cong A_4$. □

The 1-dimensional automorphism groups will be discussed in Section 5. We now discuss the remaining cases in relation with the RWPRI and $(2T)_1$ properties.

Proposition 3.8. *Let Γ be $AG(2,q)$, with q a prime power, and let G be a non-1-dimensional flag-transitive group with a* RWPRI *action on Γ. Then $G = p^d : G_0$ and one of the following holds:*

1. $SL(2,q) \trianglelefteq G_0$ *and* $PGL(2,q) \trianglelefteq \overline{G_0}$,
2. $q = 9$ *and the action of the stabilizer of a point p on the lines intersecting in p is either* A_5 *or* S_5.

Moreover, this action is $(2T)_1$ only in the first case.

Proof. In the second case of Proposition 3.7, the action is trivially $(2T)_1$ and RWPRI.

Now we concentrate on the third case of proposition 3.7 and restrict it as stated. Since (Γ, G) is LOLPR, $\overline{G_0}$ is primitive of degree $q+1$ on the set of lines containing 0. If $\overline{G_0} \cong A_4$ or S_4, the only primitive actions have degree at most 4, hence this case is ruled out. If $\overline{G_0} \cong A_5$, the only primitive actions have degree at most 10 hence we get $q = 9$ and so case 2 of our statement holds. □

We show that the second restriction is best possible.

Proposition 3.9. *Let Γ be $AG(2,9)$ and let G be a group acting flag-transitively on Γ with $\overline{G_0} \cong S_5$ or $\overline{G_0} \cong A_5$. Then (Γ, G) is* RWPRI.

Proof. Foulser [25, 26] classifies these groups and obtains ten of them namely three with $\overline{G_0} \cong A_5$ and seven with $\overline{G_0} \cong S_5$. He shows that there is a mimimal one, say F, appearing as a subgroup in each other. Hence we need only work with F that Foulser denotes by $3^4 : G_{60}^*$ and whose structure is $3^4 : 2 \cdot A_5$ where $3^4 : 2$ is the group of translations and point-symmetries. Clearly A_5 is acting at infinity inside $P\Gamma L(2,9)$ but actually inside $PSL(2,9)$ because A_5 is simple. Therefore F is contained in the affine group of the plane.
We need information on a 2-Sylow subgroup K of F. The order of K is 8 and it has a central involution i which is in the center of F_0 and which fixes a unique point 0. The quotient of K by $\{1, i\}$ is elementary abelian. Since K restricted to the line at infinity fixes two points there, K must leave two lines on 0 invariant and so K cannot be elementary abelian. Hence K has some element f of order 4. At infinity, f is an involution and fixes two points. Thus $f^2 = i$.

Now we want to check that (Γ, F) is RWPri. We first observe that it is LoLPr because $\overline{G_0} \cong A_5$ acts primitively on 10 points at infinity of the affine plane. Hence we need to check LoPPr. Therefore we consider a line l containing the point 0 and its stabilizer F_l. We need to show that $\overline{F_l}$, the group of degree 9 induced by F_l on the nine points of l is primitive. Assume it is imprimitive. Then there is a block B of imprimitivity containing 0 and two further points a and b. We may assume without loss of generality that l is invariant under the 2-Sylow subgroup K. Then f acts on $\{a, b\}$ hence $f^2 = i$ fixes 0, a, b a contradiction since i is the symmetry with respect to 0. □

Non-Desarguesian affine planes A short description of all the non-Desarguesian flag-transitive affine planes, together with some of their properties, is available in BDD [10]. We recall some of the facts concerning them. The *Lüneburg planes* are (2iL)T, but not LoPPr. Therefore, they are not RWPri. The *nearfield plane* $\mathcal{A}_9$ together with its automorphism group is (2P)T but not LoLPr and so is not RWPri. The *Hering plane of order* 27 is not LoLPr hence not RWPri. As to 1-*dimensional non-Desarguesian affine planes*, we refer to Section 5.

3.3. Hering spaces

Hering [30] constructed two nonisomorphic flag-transitive linear spaces on 3^6 points with line size 3^2 whose automorphism group is $3^6 : SL(2, 13)$. As stated in BDD [10], this group acts (2P)T, RWPri but not (2T)$_1$.

3.4. Witt-Bose-Shrikhande spaces

For any even prime power $q = 2^e$, with $e \geq 3$, we may define a *Witt-Bose-Shrikhande space* $W(q)$ [10]. This is a linear space of $2^{e-1}(2^e - 1)$ points and with line size equal to $q/2$. The first space of this family, $W(8)$, is rather special: it is isomorphic to the smallest Ree unital $U_R(3)$ (see Section 3.6), and it is the only Witt-Bose-Shrikhande space with 2-transitive automorphism group.

Kantor was the first to notice that $PSL(2, q)$ acts flag-transitively on $W(q)$, while it is already stated in BDD [10] that the full automorphism group of $W(q)$ is $P\Gamma L(2, 2^e)$. Let now $G = PGL(2, q)$. Huybrechts [31] showed that the stabilizer of a point-line flag is of order 2, and that the stabilizer of a line is an elementary abelian group of order 2^e. Hence, since this inclusion $G_{01} \leq G_1$ is never maximal, the action of $PGL(2, q)$ is never LoPPr. Huybrechts also showed that no novelties appear as we pass to $G = P\Gamma L(2, q)$. This proves that even the full automorphism group $P\Gamma L(2, q)$ of $W(q)$ has no LoPPr, hence no RWPri action.

3.5. Hermitian unitals

In this section, q denotes any prime power. Assume you are given a non-degenerate hermitian polarity π of $PG(2, q^2)$ (since all hermitian polarities are conjugate in $\mathrm{Aut}(PG(2, q^2))$, the construction is equivalent for all π). Then, define the *hermitian unital* $U_H(q)$ in the following way: the points of $U_H(q)$ are the absolute points of π and its lines the non-absolute lines of π. The incidence is symmetrized inclusion. This space is a unital, i.e. it has $q^3 + 1$ points and line-size $q+1$. Moreover, there are $q^2(q^2-q+1)$ lines, q^2 through each point. When $q=2$, this construction actually gives $AG(2,3)$.

The full automorphism group of $U_H(q)$ is isomorphic to $P\Gamma U(3,q)$. This was first published by O'Nan in [43] and Taylor found later a shorter proof [48]. Apparently, Tits knew this independently, but his result is more general: he did not use the finiteness assumption (see [50, 51]). A complete description of the automorphism group, its stabilizers and its action in general is available in Huybrechts [31]. It is based on the matrix representation of the elements of $PGU(3,q)$. She shows that no novelties about group inclusions appear between $PSU(3,q)$ and $P\Gamma U(3,q)$.

This allows us to say that results about $P\Gamma U(3,q)$ stated in [10] are valid for $PSU(3,q)$ also, i.e. this space is (2P)T and (2iL)T. Hence, it is also $(2T)_1$ and RWPRI.

3.6. Ree unitals

The Ree unitals form another class of flag-transitive linear spaces. To any $q = 3^{(2e+1)}$ $(e \geq 0)$, is associated a unital $U_R(q)$. Various constructions of these spaces exist (see [10] for references), and they all use the Ree group ${}^2G_2(q)$. Actually, the full automorphism group of the unital is $\mathrm{Aut}^2 G_2(q)$, and the action of this group is (2P)T, but not LOLPR [10], hence not RWPRI.

4. Circles

We now discuss *circles*, namely linear spaces with lines of 2 points. Obviously, a group is acting flag-transitively on a circle Γ if and only if it acts 2-transitively on the point-set of Γ. These groups have been almost completely classified. We refer to Kantor [33] and recall the result.

Proposition 4.1. (Kantor [33]) *Assume (G,Ω) is a 2-transitive permutation group, with $|\Omega| = v$. Then one of the following holds.*

A. *G is an almost simple group listed below:*

1. $G = A_v, G = S_v$, $v \geq 5$.
2. $PSL(n,q) \trianglelefteq G \leq P\Gamma L(n,q)$, $v = (q^n-1)/(q-1)$, $n \geq 2$.
3. $PSU(3,q) \trianglelefteq G \leq P\Gamma U(3,q)$, $v = q^3+1$, $q > 2$.
4. ${}^2B_2(q) \trianglelefteq G \leq \mathrm{Aut}({}^2B_2(q))$, $v = q^2+1$, $q = 2^{2e+1}$, $e > 0$.
5. ${}^2G_2(q) \trianglelefteq G \leq \mathrm{Aut}({}^2G_2(q))$, $v = q^3+1$, $q = 3^{2e+1}$, $e \geq 0$. [6]
6. $G = Sp(2n,2)$, $v = 2^{2n-1} \pm 2^{n-1}$, $n \geq 3$.
7. $G = PSL(2,11)$, *v=11.*
8. $G = M_v$, $v = 11, 12, 22, 23, 24$ *or* $G = \mathrm{Aut}(M_{22})$, $v = 22$.
9. $G = M_{11}$, $v = 12$.

[6]The $G = {}^2G_2(3)$-case is special: it is the only case where G has no simple normal subgroup N such that N is 2-transitive.

10. $G = A_7$, $v = 15$.
11. $G = HS$ *(Higman-Sims group)*, $v = 176$.
12. $G = Co_3$ *(Conway's third group)*, $v = 276$.

B. *G is of affine type, i.e. G has an elementary abelian normal subgroup T of order $v = p^d$ regular on Ω and $G = T : G_0$ where $G_0 \leq GL(d,p)$. Moreover, one of the following occurs:*

1. $G_0 \leq \Gamma L(1,v)$.
2. $G_0 \trianglerighteq SL(n,q)$, $q^n = p^d$, $n \geq 2$.
3. $G_0 \trianglerighteq Sp(n,q)$, $q^n = p^d$, *n even*, $n \geq 4$.
4. $G_0 \trianglerighteq G_2'(q)$, $q^6 = p^d$, *q even.*
5. $G_0 \cong A_6$ *or* A_7, $v = 2^4$.
6. $G_0 \trianglerighteq E$, *where E is an extraspecial group of order 2^{d+1} and* $v = 3^4, 3^2, 5^2, 7^2, 11^2$ *or* 23^2.
7. $G_0 \trianglerighteq SL(2,5)$, $v = 9^2, 11^2, 19^2, 29^2$ *or* 59^2.
8. $G_0 \cong SL(2,13)$, $v = 3^6$.

All groups given above are necessarily 2-transitive, except possibly in the following cases:

Case B1: The complete list is not known. This explains why the classification is not complete yet, and why later work concerning flag-transitive linear spaces always had difficulties with the 1-dimensional case.

Cases B6 and B7: The exact list is known and can be found in Foulser [25].

We now state a result which will be very useful while studying RWPRI circles.

Lemma 4.2. *Let (Γ, G) be a thick linear space, with G acting 2-transitively on the point-set of Γ. Let Ω denote the circle obtained from the point-set of Γ. Then (Ω, G) is flag-transitive but it is never* LOLPR.

Proof. This is just a restatement of Lemma 3.5 in the particular case of circles. □

We now give the complete lists of the RWPRI circles and the $(2T)_1$ circles.

Proposition 4.3. *Assume a group G acts flag-transitively on a circle Γ of v points. Then, if (Γ, G) is* RWPRI*, one of the following occurs:*

1. $G = A_v, G = S_v$, $v \geq 5$. [7]
2. $PSL(2,q) \trianglelefteq G \leq P\Gamma L(2,q)$, $v = q+1$.
3. $G = Sp(2n,2)$, $v = 2^{2n-1} \pm 2^{n-1}$ *with* $n \geq 3$.
4. $G = PSL(2,11)$, $v = 11$.
5. $G = M_v$, $v = 11, 12, 22, 23, 24$ *or* $G = \mathrm{Aut}(M_{22})$, $v = 22$.

[7] As a matter of fact, the values $v = 3$ and $v = 4$ work as well but they are included in case 2 for small values of q.

6. *$G = M_{11}$, $v = 12$.*
7. *$G = A_7$, $v = 15$.*
8. *$G = HS$ (Higman-Sims group), $v = 176$.*
9. *$G = Co_3$ (Conway's third group), $v = 276$.*
10. *$G = ea(2^n) : G_0$, $v = 2^n$, $G_0 \trianglerighteq SL(n, 2)$ with $n \geq 2$.*
11. *$G = ea(2^n) : G_0$, $v = 2^n$, $G_0 \trianglerighteq Sp(n, 2)$ with n even and $n \geq 4$.*
12. *$G = ea(2^4) : G_0$, $v = 16$, $G_0 \cong A_6$ or A_7.*

Moreover, if (Γ, G) is $(2T)_1$, G is 3-transitive and one of the following occurs:

1. *$G = A_v, G = S_v$, $v \geq 5$.* [8]
2. *$G \trianglerighteq PSL(2, q)$, $v = q + 1$ and G normalizes a sharply 3-transitive permutation group.*
3. *$G = M_v$, $v = 11, 12, 22, 23, 24$ or $G = \mathrm{Aut}(M_{22})$, $v = 22$.*
4. *$G = M_{11}$, $v = 12$.*
5. *$G = A_7$, $v = 15$.*
6. *$G = ea(2^n) : G_0$, $v = 2^n$, $G_0 \trianglerighteq SL(n, 2)$ with $n \geq 2$.*
7. *$G = ea(2^4) : A_7$, $v = 16$.*

Proof. We start with Proposition 4.1 and discuss each of its cases. Since we have the assumption of 2-transitivity (line-size is 2), LoPPr is granted. Therefore, we only need to check the LoLPr-property in order to select RWPri spaces.

A.1 This case obviously is RWPri and $(2T)_1$.

A.2 $n \geq 3$: We can apply Lemma 4.2 with $\Gamma = PG(n - 1, q)$ and deduce that it is not LoLPr.

$n = 2$: In order to apply Lemma 4.2, we would have to use $\Gamma = PG(1, q)$, which consists of only one line and is not a linear space in the sense of the present paper. Therefore, we cannot apply this lemma, and have to distinguish the present case. The action of G obviously is primitive on intersecting lines. Since the $(2T)_1$-property is equivalent to 3-transitivity for circles, we now wish to determine the 3-transitive projective groups normalizing $PSL(2, q)$ and acting on $q + 1$ points. In fact, as shown in [28], Theorem 2.1, each group normalizes a sharply 3-transitive permutation group. For most of the cases, this sharply 3-transitive group is simply $PGL(2, q)$. However, when q is an even power of an odd prime, the 3-transitive group we are looking for may normalize the Mathieu-Zassenhaus-Tits group instead (denoted by M in Theorem 2.1 of [28]).

A.3 Use Lemma 4.2 with $\Gamma = U_H(q)$.

A.4 Use Lemma 4.2 with $\Gamma = L\ddot{u}(q^2)$.

A.5 Use Lemma 4.2 with $\Gamma = U_R(q)$.

A.6 Some information on this case is given in [5], but the main sources are Buekenhout [2, 3]. Assume we fix a point, say x. Then G_x is $O^{\pm}(2n, 2)$, which is of rank 3 on the remaining points. The orbits of $G_{(x,y)}$ are of order $2(2^{n-1} \mp 1)(2^{n-2} \pm 1)$ and 2^{2n-2}.

[8] See the footnote 7.

This shows that the action of $Sp(2n, 2)$ is 2-primitive, hence LOLPR and RWPRI. Since the stabilizer of a point is a rank 3 group, this action is not $(2T)_1$.

A.7 Here, the stabilizer of a point p is A_5. It acts transitively on the 10 lines intersecting in p. Since A_5 has only one transitive action on ten points, and that this action is primitive, we deduce that this space is RWPRI. However, A_5 is far too small to act 2-transitively on the 10 lines intersecting in a point.

A.8 All of these groups are at least 3-transitive and so the stabilizer of a point is at least 2-transitive. Hence, the action is LOPR, and $(2T)_1$.

A.9 Again, such a group is 3-transitive, and so the action is LOPR and $(2T)_1$.

A.10 Thanks to the Atlas [15], the action of $G_0 = U_3(5) : 2$ on the 175 remaining points is primitive but not 2-transitive.

A.11 Again, the Atlas shows $G_0 = McL : 2$ and $G_{01} = U_4(3) : 2$. Since this inclusion is maximal, the action is LOLPR, hence RWPRI. However, it is not $(2T)_1$.

B.1 We will discuss this case in Section 5.

B.2 If $q \geq 3$, we can apply Lemma 4.2, with an affine space as Γ. If $q = 2$, then the circle is itself an affine space, and $SL(n, 2) = GL(n, 2)$. Hence, a point-stabilizer is 2-transitive, because $n \geq 2$. We deduce from the 3-transitivity of G that this space is $(2T)_1$.

B.3 Again, when $q \geq 3$, it is not LOLPR. On the other hand, when $q = 2$, we have one of the cases presented in Buekenhout [2, 3]. Since a point-stabilizer is a rank 3 group with orbits of length 1, $2^{n-1} - 2$ and 2^{n-1}, we can show that the action of a point-stabilizer is primitive but not 2-transitive on the lines intersecting in that point.

B.4 This space arises from the affine space $AG(6, q)$ (see Section 3.2.1). Hence, by using Lemma 4.2, we see that it is not LOLPR.

B.5 The maximal subgroups of A_6 and A_7 provided by the Atlas shows that both actions are RWPRI. Also, $2^4 : A_7$ is $(2T)_1$, since this group is 3-transitive. Indeed, its point-stabilizer occurs in the table of 2-transitive groups on position A.10.

B.6 Since $G_0 \leq GL(d, p)$, we can apply Lemma 4.2 with $\Gamma = AG(d, p)$ and reject all these cases.

B.7 Same as B.6.

B.8 Same as B.6. □

5. 1-dimensional spaces

Delandtsheer has studied these spaces in relation with the LOLPR property. We refer to [18] for a full description of each of the known spaces, and state two results.

Proposition 5.1. (Delandtsheer [19]) *Let Γ be a finite linear space. If $G \leq \mathrm{Aut}(\Gamma)$ is flag-transitive and $G \leq A\Gamma L(1, v)$, then*
(i) *G is* LOLPR *if and only if the point-degree r is a prime number,*
(ii) *G is not transitive on the unordered pairs of intersecting lines, except in the following cases:*

1. *S is the trivial $2-(v,2,1)$ design with $v = 3$, 4 or 8 and $G = AGL(1,3)$, $AGL(1,4)$, $A\Gamma L(1,4)$, or $A\Gamma L(1,8)$,*
2. *$S = PG(2,2)$ and $G = AG^2L(1,7)$,*
3. *$S = AG(2,4)$ and $G = A\Gamma L(1,16)$.* □

This result is very useful in order to classify LOPR hence RWPRI 1-dimensional spaces.

Proposition 5.2. (Delandtsheer [18]) *Assume Γ is a 1-dimensional space. Then, the point-set of Γ is the point-set of an affine space $AG(d,p)$, and G is a group of affine transformations of this space. Moreover, the group $G \leq A\Gamma L(1,p^d)$ contains the translation group $T \cong p^d$ and two cases need to be distinguished, according to the fact that a line-stabilizer is trivial or not.* □

5.1. Spread case

This case occurs when the stabilizer of a line in T, say T_L, is not reduced to the identity element. Actually, L is a point-orbit of T_L, and so the lines of the linear space are subspaces of dimension l (with $l|d$) of the affine space $AG(d,p)$. Hence, the line-size k is p^l. Lines through a given point form a spread, i.e. they induce a spread of $(l-1)$-subspaces on the hyperplane at infinity $PG(d-1,q)$.

Our discussion of this case is in several steps. We first consider the property LOLPR alone. Then, we consider it together with LOPPR, i.e. we look at the LOPR or RWPRI property. Then, we add the $(2T)_1$ property.

5.1.1. Property LOLPR

In this section, we get strong conditions on spread 1-dimensional spaces with property LOLPR. We first state a result of Number Theory, which is of particular importance in this section.

Proposition 5.3. (after Ribenboim [44]) *Let r be a prime, let $x > 1$, $m \geq 1$, $u \geq 3$ be integers satisfying*

$$\frac{x^u - 1}{x - 1} = r^m. \tag{1}$$

Then,

(a) *The exponent u is a prime, equal to the order of x modulo r, and $r \equiv 1 \mod u$.*

(b) *If $x = s^b$, $b \geq 1$, then $b = u^e$, $e \geq 0$, $r \equiv 1 \mod u^{e+1}$, and r is not a Fermat prime.* □

Remark. Equations of this type have been studied by various authors, the first one being Suryanarayana [47]. For complete reviews of the literature on this equation, see also Edgar [23] and Ribenboim [44].

We are now ready to state the result of this section.

Proposition 5.4. *Assume a* 1*-dimensional automorphism group* G *has a* LOLPR *action on a linear space* Γ *of spread type. Then,* Γ *has* $p^{u^{(e+1)}}$ *points and each line has* p^{u^e} *points, where* p *and* u *are primes and* $e \geq 0$.

Proof. Using Proposition 5.1, we may easily select the spread 1-dimensional spaces with property LOLPR. We put $u = d/l$ and get that

$$r = \frac{v-1}{k-1} = \frac{p^d - 1}{p^l - 1} = p^{(u-1)l} + p^{(u-2)l} + \cdots + p^l + 1 \tag{2}$$

must be a prime, and that this is sufficient.

When $u \geq 3$, equation (2) is a particular case of equation (1): we may impose in equation (1) the additional conditions $m = 1$ and $x = p^l$, where p is a prime. When $u = 2$, the relation $r = p^l + 1$ shows that r is a Fermat prime. In both cases, v and k have the properties of our statement. □

5.1.2. Property RWPRI

Under the RWPRI-condition, we prove the following statement.

Proposition 5.5. *Assume a* 1*-dimensional automorphism group* G *has a* RWPRI *action on a linear space* Γ *of spread type. Then, the point-degree is a prime,* Γ *has* $p^{u^{(e+1)}}$ *points and each line has* p^{u^e} *points, where* p *and* u *are primes and* $e \geq 0$*. Moreover, for any* p *and* u *satisfying all those arithmetic conditions, there exists a* RWPRI *affine space,* $AG(u, p^{u^e})$.

Proof. Since RWPRI implies LOLPR, the first part of our statement is already in Proposition 5.4. We still need to show the existence. Assume p and u are primes, and that the point-degree of $AG(u, p^{u^e})$ is also a prime. Then, this space is LOLPR by Proposition 5.1. Moreover, its full automorphism group G is 2-transitive, hence (Γ, G) is LOPPR and RWPRI. □

However, we cannot be sure that there is only one space, namely $AG(u, p^{u^e})$, which is associated with each set of parameters. Hence, the question is now:

Are all flag-transitive RWPRI spaces of spread type Desarguesian?

A positive answer would totally reduce the classification of flag-transitive linear spaces of spread type to Number Theory.

5.2. Property $(2T)_1$

We now look at the $(2T)_1$ property, for which we prove the following proposition.

Proposition 5.6. *There are only two* $(2T)_1$ 1*-dimensional linear spaces of spread type*[9]*:* $AG(2, 2)$ *and* $AG(2, 4)$ *with respective automorphism groups* $A\Gamma L(1, 4) \cong S_4$ *and* $A\Gamma L(1, 16)$.

[9] As we will prove in Section 5.3, the "spread case" restriction is not needed.

Proof. Since $(2\mathrm{T})_1$ implies LOLPR, the conditions obtained for the LOLPR-property in Proposition 5.4 are a good start.

Since the stabilizer of a point x is 2-transitive on the lines intersecting in x, its order is divisible by $r(r-1)$. Hence, we deduce the following relations:

$$\begin{array}{ccl} & r(r-1) & \text{divides } \#G_x \\ \text{implies} & \frac{(v-1)}{(k-1)}\left(\frac{(v-1)}{(k-1)}-1\right) & \text{divides } (v-1)u^{(e+1)} \\ \Leftrightarrow & \frac{(v-k)}{(k-1)^2} & \text{divides } u^{(e+1)} \\ \text{implies} & k & \text{divides } u^{(e+1)} \\ \Leftrightarrow & p^{u^e} & \text{divides } u^{(e+1)} \end{array}$$

And this last relation implies $p = u$, then $u^e \leq e+1$, and finally $0 \leq e \leq 1$. Let us look at each case separately:

$e = 0$: We have $v = p^p$, $k = p$. Since we assume $(2\mathrm{T})_1$, we get $r(r-1)|((p^p-1)p)$, or equivalently $\frac{(p^p-p)}{(p-1)^2}|p$. Then, $\frac{p^{(p-1)}-1}{(p-1)^2} = 1$, and this in turn implies $p = 2$. Now, Γ is $AG(2,2)$ and the group $A\Gamma L(1,4)$ is isomorphic to S_4. This isomorphism grants us the $(2\mathrm{T})_1$ property.

$e = 1$: Then $u \leq 2$, and $u = p = 2$. The value of u and p is then 2. Therefore, Γ is isomorphic to $AG(2,4)$, and $A\Gamma L(1,16)$ acts on Γ. It is then a short computation (for instance in MAGMA) to show that $A\Gamma L(1,16)$ has a $(2\mathrm{T})_1$ action. Moreover, none of its subgroups has this type of action. Indeed, the 2-transitivity of a point-stabilizer implies that the order of a $(2\mathrm{T})_1$ group is divisible by 16.5.4, while the 2-transitivity of a line-stabilizer forces the order to be a multiple of 20.4.3. This forces the order of such a group to be a multiple of 960, which is precisely the order of $A\Gamma L(1,16)$. □

5.3. Nonspread case

From now on, any 1-dimensional flag-transitive linear space which is not of spread type is called a *nonspread linear space*. This case is studied by Delandtsheer in [19, 18], along with the only known examples of this class, the *Generalized Netto Systems*, an extension of a family found by Netto [42, 41].

Proposition 5.7. (Delandtsheer [18]) *Assume we have a geometry Γ with point-set $\Omega = AG(d,p)$ together with a group $G \leq A\Gamma L(1,p^d)$ acting flag-transitively on Γ. Assume further that G contains the translation group $T \cong p^d$ and that $T_L = 1$ for any line L of Γ. Then, the following conditions apply on the number of lines b and the line size k:*

1. *$p^d|b$, where $b = p^d(p^d-1)/k(k-1)$, hence $(p,k) = 1$.*
2. *$p \geq 3$.* □

We now prove the following result.

Proposition 5.8. *Assume Γ is a linear space of nonspread type and G is a 1-dimensional automorphism group with a* RWPri *action on Γ. Then, $\Gamma = PG(2,2)$ and $G = 7:3$. Moreover, this action is not* $(2T)_1$.

Proof. We use Proposition 5.1, i.e. we look for nonspread spaces having prime point-degree. The point-degree r is $(p^d-1)/(k-1)$. Using the first property stated in Proposition 5.7, we see that $k|r$. Since r has to be a prime, we deduce that $k=r$, hence Γ is a projective space and $p^d-1 = k(k-1)$. Let $l = k-1$. The equation becomes $l^2+l+1 = p^d$. This shows that Γ is a projective plane. We discussed that case in Section 3.1, and we showed that the only 1-dimensional RWPri projective plane is $PG(2,2)$ with the Frobenius group $7:3$. Moreover, the stabilizer of a line in this space being cyclic, we may reject the $(2T)_1$-property. □

6. Conclusions on linear spaces

We now summarize the results of Sections 3 to 5. We give in Theorem 6.1 the list of linear spaces satisfying RWPri, then in Theorem 6.2 the list of linear spaces satisfying $(2T)_1$.

In these tables, q denotes a power of a prime p (with $q^n = p^d$).

Theorem 6.1. *Let Γ be a finite linear space of v points. Let G be a group acting flag-transitively and faithfully on Γ. If Γ is* RWPri *then one (at least[10]) of the following occurs:*

1. *The 1-dimensional spread case[11]: $G \leq A\Gamma L(1,v)$, with $v = p^{(u^{(e+1)})}$ and $k = p^{(u^e)}$. Moreover, the point-degree and u are primes.*
2. $\Gamma = PG(n,q)$, $v = \frac{q^{n+1}-1}{q-1}$, $PSL(n+1,q) \trianglelefteq G \leq P\Gamma L(n+1,q)$ *with* $n \geq 2$.
3. $\Gamma = PG(3,2)$, $v = 15$ *with* $G \cong A_7$.
4. $\Gamma = PG(2,2)$, $v = 7$ *with* $G \cong 7:3$.
5. $\Gamma = AG(n,q)$, $v = p^d = q^n$, $G = p^d : G_0$ *with* $SL(n,q) \trianglelefteq G_0$, $q \geq 3$ *and* $n \geq 2$.
6. $\Gamma = AG(n,q)$, $v = p^d = q^n$, $G = p^d : G_0$, *with* $Sp(n,q) \trianglelefteq G_0$, $q \geq 3$ *and* $n \geq 4$.
7. $\Gamma = AG(6,q)$, $v = p^d = q^6$, $G = p^d : G_0$ *with* $G_2'(q) \trianglelefteq G_0$ *and q even.*
8. $\Gamma = AG(2,9)$, $v = 81$, *G is one of the 3 (resp. 7) groups acting at infinity as A_5 (resp. S_5) and presented in Foulser [25].*
9. *Γ is one of the two Hering spaces built using 1-spreads in $PG(5,3)$, $v = 3^6$, $G = 3^6 : SL(2,13)$.*
10. *Γ is a hermitian unital $U_H(q)$, $v = q^3+1$, $PSU(3,q) \trianglelefteq G \leq P\Gamma U(3,q)$.*
11. *Γ is a circle:*
    (a) $G = A_v, G = S_v$, $v \geq 5$. [12]

[10] As far as we know, there is one non-empty intersection. Indeed, since $U_H(2)$ is isomorphic to $AG(2,3)$ and since the normalizer of $PSU(3,2) \simeq M_9$ is $AGL(2,3)$, cases 5 and 10 share $AGL(2,3)$ and $ASL(2,3)$.

[11] Desarguesian affine spaces occur here, but we do not know of the existence of non-Desarguesian affine spaces.

[12] As a matter of fact, the values $v=3$ and $v=4$ work as well but they are included in case 11b for small values of q.

(b) $PSL(2,q) \trianglelefteq G \leq P\Gamma L(2,q)$, $v = q+1$.

(c) $G = Sp(2n,2)$, $v = 2^{2n-1} \pm 2^{n-1}$ *with* $n \geq 3$.

(d) $G = PSL(2,11)$, $v = 11$.

(e) $G = M_v$, $v = 11, 12, 22, 23, 24$ *or* $G = \mathrm{Aut}(M_{22})$, $v = 22$.

(f) $G = M_{11}$, $v = 12$.

(g) $G = A_7$, $v = 15$.

(h) $G = HS$ *(Higman-Sims group)*, $v = 176$.

(i) $G = Co_3$ *(Conway's third group)*, $v = 276$.

(j) $G = ea(2^n) : G_0$, $v = 2^n$, $G_0 \trianglerighteq SL(n,2)$ *with* $n \geq 2$.

(k) $G = ea(2^n) : G_0$, $v = 2^n$, $G_0 \trianglerighteq Sp(n,2)$ *with* n *even and* $n \geq 4$.

(l) $G = ea(2^4) : G_0$, $v = 16$, $G_0 \cong A_6$ *or* A_7.

Proof. The proof of this theorem is divided in three parts:

Thick case: Here, we go through the list of BDDKLS as stated in Proposition 1.1 and discussed in Section 3. We simply check each case for the RWPRI condition.

In the first case of Proposition 1.1, Γ is a *projective space*. We apply Proposition 3.3, and get cases 2 to 4 of our statement.

If (Γ, G) are as in the second case of Proposition 1.1, we need to consider two subcases. For *affine spaces of dimension at least* 3, we apply Proposition 3.6 and get cases 5, 6 or 7 of this statement. For *affine planes*, we use Proposition 3.8 and results of BDD [10] (see end of Section 3.2.2) about non-Desarguesian planes and get cases 5 and 8 of our statement.

If (Γ, G) is either a *Hering space*, a *Witt-Bose-Shrikhande space*, a *Ree unital* or a *Hermitian unital*, we can apply BDD [10] as seen in Sections 3.3, 3.4, 3.6 and 3.5. Hering spaces and Hermitian unitals give us cases 9 and 10 of this statement. There is no RWPRI Witt-Bose-Shrikhande space, nor any RWPRI Ree unital.

Circles: We apply Proposition 4.3 to deduce the list appearing in case 11.

1-*dimensional:* This is the first case of our statement. It is discussed in Section 5 and so we distinguish two subcases. In the spread case, we apply Proposition 5.5. In the nonspread case, Proposition 5.8 shows that there is only one RWPRI linear space. This linear space is case 4 of our statement. □

We now present all $(2T)_1$ linear spaces.

Theorem 6.2. *Let* Γ *be a finite linear space of* v *points. Let* G *be a group acting flag-transitively and faithfully on* Γ*. If* Γ *is* $(2T)_1$ *then one (at least[13]) of the following occurs:*

1. $\Gamma = AG(2,4)$, *with* $G = A\Gamma L(1,16)$.
2. $\Gamma = PG(n,q)$, $v = \frac{q^{n+1}-1}{q-1}$, $PSL(n+1,q) \trianglelefteq G \leq P\Gamma L(n+1,q)$ *with* $n \geq 2$.
3. $\Gamma = PG(3,2)$, $v = 15$ *with* $G \cong A_7$.

[13] See the footnote 10.

4. $\Gamma = AG(n,q)$, $v = p^d = q^n$, $G = p^d : G_0$ *with* $SL(n,q) \trianglelefteq G_0$, $q \geq 3$ *and* $n \geq 2$.
5. Γ *is a hermitian unital* $U_H(q)$, $v = q^3+1$, $PSU(3,q) \trianglelefteq G \leq P\Gamma U(3,q)$.
6. Γ *is a circle and* G *is a* 3*-transitive permutation group:*
   (a) $G = A_v, G = S_v$, $v \geq 5$. [14]
   (b) $G \trianglerighteq PSL(2,q)$, $v = q+1$ *and* G *normalizes a sharply* 3*-transitive permutation group.*
   (c) $G = M_v$, $v = 11, 12, 22, 23, 24$ *or* $G = \mathrm{Aut}(M_{22})$, $v = 22$.
   (d) $G = M_{11}$, $v = 12$.
   (e) $G = A_7$, $v = 15$.
   (f) $G = ea(2^n) : G_0$, $v = 2^n$, $G_0 \trianglerighteq SL(n,2)$ *with* $n \geq 2$.
   (g) $G = ea(2^4) : A_7$, $v = 16$.

Proof. In view of the remark in Section 2.2, we may assume that (Γ, G) is RWPRI, and use the list of Theorem 6.1 as a starting point. We now give the arguments to use for each case of this list.

Case 1*:* We apply Proposition 5.6 and so we obtain type 1 of the present statement. Observe that we could also have applied Proposition 5.1 to obtain this result.

Cases 2, 3 *and* 4*:* We apply Proposition 3.3 and so we obtain types 2 and 3 of the present statement.

Cases 5, 6, 7 *and* 8*:* We apply Propositions 3.6 and 3.8 and get type 4 of the present statement.

Cases 9 *and* 10*:* We use results stated in BDD [10] (see Sections 3.3 and 3.5) and get type 5 of this statement.

Case 11*:* We apply Proposition 4.3 and get type 6 of this statement. □

Remark. If (Γ, G) is a flag-transitive linear space as in cases 1 to 11 of Theorem 6.1, we convinced ourselves that (Γ, G) is indeed RWPRI, which means that the converse statement of Theorem 6.1 is true. If (Γ, G) is a flag-transitive linear space as in cases 1 to 6 of Theorem 6.2, we convinced ourselves that (Γ, G) is indeed $(2T)_1$, which means that the converse statement of Theorem 6.2 is also true. The proofs are lenghty in view of the number of cases and not given here in full detail. An exception is Proposition 3.9. Here is a shortcut showing that it is sufficient to provide the details for a given Γ and a "minimal" G.

Lemma 6.3. *If* (Γ, H) *and* (Γ, K) *are flag-transitive linear spaces such that* H *is a subgroup of* K *and if* (Γ, H) *is* RWPRI *(resp.* $(2T)_1$*), then* (Γ, K) *is* RWPRI *(resp.* $(2T)_1$*).*

Proof. It suffices to recall that a permutation group (Ω, X) containing a primitive (resp. 2-transitive) subgroup (Ω, Y) is primitive (resp. $(2T)_1$). □

[14] See the footnote 12.

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