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Even Lattices with Covering Radius $< \sqrt{2}$

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1. Introduction

Let L be a lattice in Euclidean space $V := \mathbb{R} \otimes L$. Then the *covering radius* of L is the smallest number $r \in \mathbb{R}$ such that the spheres with radius r around all lattice points cover the whole space V.

The famous Leech lattice Λ_{24}, the unique even unimodular lattice of rank 24 with minimal distance 4, has covering radius $\sqrt{2}$, as shown in [2] (see [3, Chapter 23]). This is the main observation that enables Conway to calculate the automorphism group of the 26-dimensional even unimodular Lorentzian lattice $\Lambda_{24} \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ([1], see [3, Chapter 27]). The present article is motivated by a question of Richard Parker, who wants to have a list of all even lattices L with covering radius $\leq \sqrt{2}$ to construct examples of Lorentzian lattices $L \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for which he can calculate the automorphism group. As a first step, the even lattices of covering radius $< \sqrt{2}$ are classified in this note.

If L is an even lattice with covering radius $< \sqrt{2}$, then for every $v \in V$, there is a vector $l \in L$ with $(v - l, v - l) < 2$, where (v, w) denotes the scalar product of two vectors $v, w \in V$. In particular if $v = \frac{1}{2}w$ with $w \in L$, then $(w - 2l, w - 2l) < 8$. Since L is even, this means that every coset in $L/2L$ contains a vector of square length ≤ 6. Let $\mu(L)$ denote the minimal m such that every coset in $L/2L$ contains a vector of norm $\leq m$. The easy but crucial observation is stated in Lemma 1: If L is an even lattice with $\mu(L) \leq 6$, then every norm 8 vector in L gives rise to a norm 2 vector in L which enables to classify these lattices according to the sublattices spanned by the vectors of norm 2 in L. 32 of the lattices L with $\mu(L) \leq 6$ are root lattices (Theorem 6), where the largest dimension is 10, achieved by E_8A_2. For the other 51 lattices (given in Theorem 7) the root sublattice is not of full rank. This list

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of 83 lattices includes all even lattices with covering radius strictly smaller than $\sqrt{2}$. With MAGMA ([4]), one checks that all 83 lattices have covering radius $\leq \sqrt{2}$. 69 of these lattices have covering radius $< \sqrt{2}$. The 14 lattices with covering radius $= \sqrt{2}$ and $\mu(L) \leq 6$ are listed in Remark 8. I thank R. Parker for inspiring and helpful discussions.

2. The lattices L with $\mu(L) \leq 6$

Throughout the whole note let L be an even lattice, such that each coset of $L/2L$ contains a vector of norm ≤ 6. In particular all non zero isotropic cosets of $L/2L$ contain vectors of norm 4. For even nonnegative integers i let

$$L_i := \{x \in L \mid (x, x) = i\}$$

be the set of norm i vectors in L.

The first lemma is the crucial observation, since it constructs from a vector of norm 8 in L a norm 2 vector in L.

Lemma 1. *Let $w \in L_8$. Then either $w \in 2L$ and $r := \frac{1}{2}w \in L_2$ or there is a vector $v \in L_4$ such that $(v, w) = -2$ and $r := \frac{1}{2}(v + w) \in L_2$. In the first case $(r, w) = 4$ and in the second case $(r, w) = 3$.*

Proof. Assume that $w \notin 2L$. Then the coset $w + 2L \in L/2L$ is isotropic and hence there is a vector $v \in L_4$ such that $v + w \in 2L$. Replacing v by $-v$ if necessary, one may assume that $(v, w) \leq 0$. Since $(v, w) \geq -4$ one gets

$$4 \leq (v + w, v + w) = (v, v) + (w, w) + 2(v, w) = 12 + 2(v, w) \leq 12$$

Now $(v + w, v + w)$ is divisible by 8 and therefore $(v + w, v + w) = 8$, $\frac{1}{2}(v + w) \in L_2$, and $(v, w) = -2$. □

Corollary 2. *Let $v_1, v_2 \in L_4$ with $(v_1, v_2) = 0$. Then either*
a) *$r := \frac{1}{2}(v_1 + v_2) \in L_2$ or*
b) *there is $v \in L_4$ such that $r := \frac{1}{2}(v + v_1 + v_2) \in L_2$.*
In case a) *one has $(r, v_1) = (r, v_2) = 2$.*
In case b) *after interchanging v_1 and v_2 if necessary, it holds that $(v, v_1) = -2$, $(v, v_2) = 0$ and hence $(r, v_1) = 1$ and $(r, v_2) = 2$.*

Proof. That only these two cases occur follows from Lemma 1 applied to $w := v_1 + v_2$. It remains to calculate the scalar products in case b)). Since $v_1 + 2L$ and $v_2 + 2L$ generate an isotropic subspace of $L/2L$ and $v \in v_1 + v_2 + 2L$ by assumption, (v, v_1) and (v, v_2) are even. By Lemma 1, $(v, v_1 + v_2) = -2$ and hence, after interchanging v_1 and v_2 if necessary, $(v, v_1) = -2$ and $(v, v_2) = 0$. □

Let $R := \langle L_2 \rangle$ be the sublattice spanned by the vectors of norm 2 in L. Then R is a root lattice and therefore an orthogonal sum of irreducible root lattices of type $A_n (n \geq 1), D_n (n \geq 4), E_6, E_7, E_8$. Define the *orthogonal rank* $\mathrm{OR}(M)$ of a root lattice M to be the maximal number of pairwise orthogonal norm 2 vectors in M. One has $\mathrm{OR}(A_n) = \lceil \frac{n}{2} \rceil$, $\mathrm{OR}(D_n) = 2\lfloor \frac{n}{2} \rfloor$, $\mathrm{OR}(E_6) = 4$, $\mathrm{OR}(E_7) = 7$ and $\mathrm{OR}(E_8) = 8$.

Corollary 3. *The number of irreducible components of R is ≤ 3.*

Proof. Let $R_1 \perp R_2 \perp R_3 \perp R_4 \leq R$ be the orthogonal sum of 4 components of R and choose norm 2 vectors $r_i \in R_i$ $(i = 1, \ldots, 4)$. Then $v_1 := r_1 + r_2$ and $v_2 := r_3 + r_4$ are orthogonal vectors in L_4. Hence by Corollary 2 there is $r \in L_2$ such that $(r, v_1) > 0$ and $(r, v_2) > 0$. This contradicts the fact that the r_i are in different components of R. □

Corollary 4. *If* $\mathrm{OR}(R) \geq 4$, *then R contains a sublattice D_4. More precisely let r_i $(i = 1, \ldots, 4)$ be pairwise orthogonal norm 2 vectors in R. Then either $r := \frac{1}{2}(r_1+r_2+r_3+r_4) \in R$ and $\langle r_1, r_2, r_3, r\rangle \cong D_4$ or there is $r \in R$ and $j \in \{1, \ldots, 4\}$ with $(r, r_i) = 1$ for $i \neq j$ and $(r, r_j) = 0$ such that $\langle r, r_i \mid i \neq j\rangle \perp \langle r_j\rangle \cong D_4 \perp A_1$.*

From this corollary one concludes that, if $\mathrm{OR}(R) \geq 4$, then R has at most two irreducible components, and if it has two components, then one of them has orthogonal rank 1, hence is A_1 or A_2.

Corollary 5. *R has no component D_m with $m \geq 8$, A_m with $m \geq 7$ and no orthogonal summand $X \perp A_1$ or $X \perp A_2$, where X is one of A_6, A_5, D_7 or D_6.*

Proof. Assume that R has an orthogonal component D_m with $m \geq 8$. View $D_m := \{(x_1, \ldots, x_m) \in \mathbb{Z}^m \mid \sum_{i=1}^m x_i \equiv 0 \pmod 2\}$. Then $v = (v_1, \ldots, v_m)$ with $v_i = 1$ for $i = 1, \ldots, 8$ and $v_i = 0$ for $i \geq 9$ is a vector of norm 8 in D_m. Hence by Lemma 1 there is a norm 2 vector $r \in D_m$ with $(r, v) \geq 3$. But there is no such vector.

The other cases are dealt with similarly: For $A_m = \{(x_1, \ldots, x_{m+1}) \in \mathbb{Z}^{m+1} \mid \sum_{i=1}^{m+1} x_i = 0\}$ $(m \geq 7)$ one takes $v = (1^4, (-1)^4, 0^{m-7})$, for $A_5 \perp A_j$ and $A_6 \perp A_j$ $(j = 1, 2)$, one takes $v = (1^3, (-1)^3(, 0)) \perp r$ where r is a norm 2 vector in A_j and for $D_6 \perp A_j$ and $D_7 \perp A_j$ $(j = 1, 2)$, one takes $v = (1^6(, 0)) \perp r$ where r is a norm 2 vector in A_j. □

Theorem 6. *If R has full rank in L then $L = R$ is one of A_1, A_2, A_3, A_4, A_5, A_6, D_4, D_5, D_6, D_7, E_6, E_7, E_8, A_1^2, A_1^3, A_1A_2, $A_1^2A_2$, $A_1A_2^2$, A_2^2, A_2^3, A_1A_3, A_1A_4, A_1D_4, A_1D_5, A_1E_6, A_1E_8, A_2A_3, A_2A_4, A_2D_4, A_2D_5, A_2E_6, or A_2E_8.*

Proof. For the irreducible root lattices M one calculates

M	A_1	A_2	A_3	A_4	A_5	A_6	E_6	E_7	E_8	D_4	D_5	D_6	D_7
$\mu(M)$	2	2	4	4	6	6	4	6	4	4	4	6	6

For orthogonal sums, one clearly has $\mu(M_1 \perp M_2) = \mu(M_1) + \mu(M_2)$. From this observation one finds that the root lattices M with $\mu(M) \leq 6$ are the ones listed in the theorem. This proves the theorem in the case $L = R$.

Now assume that $R < L$ is a proper sublattice of finite index in L. Then
1) L is an even overlattice of R and hence contained in the dual lattice R^* of R.
From the above corollaries it follows that:
2) The number of irreducible components of R is ≤ 3.
3) If R contains a sublattice A_1^4 then it contains D_4. In particular R has no component A_n with $n \geq 7$ or D_n with $n \geq 8$.
4) If the orthogonal rank of R is ≥ 4, then R has at most 2 components and one of them is

A_1 or A_2.
The conditions 2), and 3) result in a finite list of possible root lattices R which can be shortened with 4) and Corollary 5. For all entries R in this list, there are either no even proper overlattices of R or they contain new norm 2 vectors. □

It remains to consider the case, that R has not full rank in L. Here the following strategy is used:

Since $\mathrm{rank}(R) < n := \dim(L)$, there is $v \in L - (2L + R)$. Choose v to be of minimal norm in its coset modulo $2L + R$. Then $(v, v) = 4$ or 6 and $|(v, r)| \leq 1$ for all norm 2 vectors r. Let $L' := \langle R, v \rangle$. If F is a Gram matrix of R with respect to a basis consisting of norm 2 vectors, then

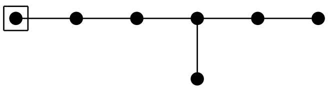
$$\left(\begin{array}{c|c} F & 0/\pm 1 \\ \hline 0/\pm 1 & 4/6 \end{array} \right)$$

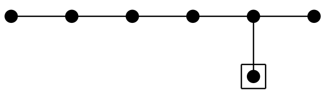
is a Gram matrix of L'.

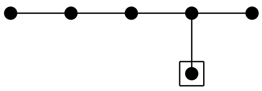
With MAGMA ([4]) one constructs all such symmetric positive definite matrices (up to isometry) and checks whether R is the sublattice of L' spanned by the norm 2 vectors in L' and for all $w \in L'$ with $(w, w) = 8$, there is a norm 2 vector $r \in R$ with $|(r, w)| \geq 3$, which is a property of any sublattice of L that contains R according to Lemma 1. To continue, one takes $v' \in L - (L' + 2L)$ of minimal norm in its coset modulo $(L' + 2L)$ and constructs all the possible Gram matrices of $L'' := \langle L', v' \rangle$ etc. Note that L is not necessarily equal to one of the lattices L', L'', ... constructed like this but might be an overlattice of odd index.

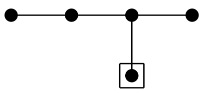
With this procedure one arrives at the following theorem:

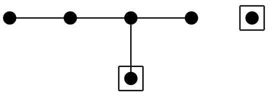
Theorem 7. *Let L be an even lattice with $\mu(L) \leq 6$. Let R be its root sublattice and assume that R has not full rank in L. If the corank of R is 1 then $L = L_j(R)$ is represented by one of the following* 27 *decorated Dynkin diagrams:*

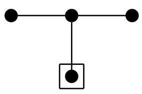
$L_1(E_7)$: 

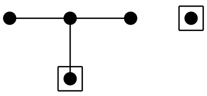
$L_1(D_7)$: 

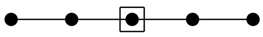
$L_1(D_6)$: 

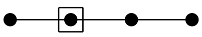
$L_1(D_5)$: 

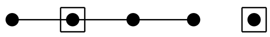
$L_1(D_5A_1)$: 

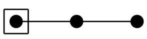
$L_1(D_4)$: 

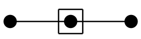
$L_1(D_4A_1)$: 

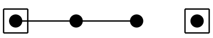
$L_1(A_5)$: 

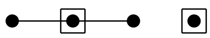
$L_1(A_4)$: 

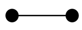
$L_1(A_4A_1)$: 

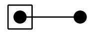
$L_1(A_3)$: 

$L_2(A_3)$: 

$L_1(A_3A_1)$: 

$L_2(A_3A_1)$: 

$L_1(A_2)$: 

$L_2(A_2)$: 

$L_1(A_2A_1)$: 

$L_2(A_2A_1)$: 

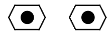
$L_1(A_1^3)$: 

$L_2(A_1^3)$: 

$L_3(A_1^3)$: 

$L_1(A_1^2)$: 

$L_2(A_1^2)$: 

$L_3(A_1^2)$: 

$L_1(A_1)$: 

$L_2(A_1)$: 

$L_3(A_1)$: 

A basis of L with a given decorated Dynkin diagram consists of the respective fundamental roots of R and an additional norm 4 vector v which has scalar product -1 with all the fundamental roots surrounded by a box and 0 with the other ones. For the three lattices $L_3(A_1^3)$, $L_3(A_1^2)$ and $L_3(A_1)$, this additional vector v has norm 6, which is indicated by changing the boxes to hexagons.

If the corank of R is bigger than 1, or $R = \{0\}$, then $L = L_j(R)$ is defined by one of the following 24 *Gram matrices* $F_j(R)$

$$F_2(D_4) = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 & -1 \\ -1 & 2 & -1 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \\ 0 & -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 4 & -1 \\ -1 & 0 & 1 & 0 & -1 & 4 \end{pmatrix}, \quad F_{11}(\{0\}) = \begin{pmatrix} 4 & -2 & -1 & 1 & -2 & -1 \\ -2 & 4 & -1 & -2 & 1 & 2 \\ -1 & -1 & 4 & -1 & -1 & 1 \\ 1 & -2 & -1 & 4 & 1 & -1 \\ -2 & 1 & -1 & 1 & 4 & -1 \\ -1 & 2 & 1 & -1 & -1 & 4 \end{pmatrix},$$

$$F_3(A_3) = \begin{pmatrix} 2 & -1 & 0 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 \\ -1 & 0 & 0 & -1 & 4 \end{pmatrix}, \quad F_4(A_1^3) = \begin{pmatrix} 2 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & -1 \\ -1 & -1 & 0 & 4 & -1 \\ -1 & 0 & -1 & -1 & 4 \end{pmatrix},$$

$$F_3(A_2) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & -2 & 4 \end{pmatrix}, \quad F_4(A_2) = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 4 & -2 \\ -1 & 0 & -2 & 4 \end{pmatrix},$$

$$F_5(A_2) = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 4 & -1 \\ -1 & 0 & -1 & 4 \end{pmatrix}, \quad F_8(A_1) = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 4 & -1 & -2 \\ -1 & -1 & 4 & -1 \\ 0 & -2 & -1 & 4 \end{pmatrix},$$

$$F_4(A_1^2) = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}, \qquad F_5(A_1^2) = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}$$

$$F_9(\{0\}) = \begin{pmatrix} 4 & -2 & -2 & 1 \\ -2 & 4 & 1 & -2 \\ -2 & 1 & 4 & -2 \\ 1 & -2 & -2 & 4 \end{pmatrix}, \qquad F_{10}(\{0\}) = \begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix},$$

$$F_4(A_1) = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix}, \qquad F_5(A_1) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix},$$

$$F_7(\{0\}) = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix}, \qquad F_8(\{0\}) = \begin{pmatrix} 4 & -1 & -2 \\ -1 & 4 & -1 \\ -2 & -1 & 4 \end{pmatrix},$$

$$F_6(A_1) = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 4 & -2 \\ -1 & -2 & 4 \end{pmatrix}, \qquad F_7(A_1) = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 4 & -2 \\ -1 & -2 & 6 \end{pmatrix},$$

$$F_3(\{0\}) = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}, \qquad F_4(\{0\}) = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix},$$

$$F_5(\{0\}) = \begin{pmatrix} 4 & -2 \\ -2 & 6 \end{pmatrix}, \qquad F_6(\{0\}) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix},$$

$$F_1(\{0\}) = (4), \qquad F_2(\{0\}) = (6).$$

Remark 8. *The lattices L with $\mu(L) \leq 6$ and covering radius $= \sqrt{2}$ are A_2^3, $A_2 \perp E_6$ and the 12 lattices $L_1(D_7)$, $L_1(D_6)$, $L_2(D_4)$, $L_3(A_2)$, $L_3(A_1^3)$, $L_4(A_1^3)$, $L_4(A_1^2)$, $L_7(A_1)$, $L_6(\{0\})$, $L_9(\{0\})$, $L_{10}(\{0\})$ and $L_{11}(\{0\})$ of Theorem 7.*

All the other 69 even lattices L with $\mu(L) \leq 6$ have covering radius $< \sqrt{2}$.

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