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When is the Algebra of Multisymmetric Polynomials Generated by the Elementary Multisymmetric Polynomials?

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Abstract. Multisymmetric polynomials are the r-fold diagonal invariants of the symmetric group $\mathfrak{S}_n$. Elementary multisymmetric polynomials are analogues of the elementary symmetric polynomials, in the multisymmetric setting. In this paper, we give a necessary and sufficient condition on a ring A for the algebra of multisymmetric polynomials with coefficients in A to be generated by the elementary multisymmetric polynomials.

Introduction

It is well-known that the ring of symmetric polynomials with integer coefficients is generated by the elementary symmetric polynomials. This result (with the algebraic independence of the elementary symmetric polynomials) is often referred as "the fundamental theorem of symmetric polynomials".

The *multisymmetric polynomials* are the r-fold diagonal invariants of the symmetric group $\mathfrak{S}_n$. When taken with coefficients in a ring A, they constitute an A-algebra we denote by $\mathfrak{I}_n^r(A)$. Multisymmetric polynomials are generalizations of symmetric polynomials (recovered with $r = 1$). The elementary symmetric polynomials have analogues in the multisymmetric setting, called *elementary multisymmetric polynomials*.

It is quite an old theorem that $\mathfrak{I}_n^r(\mathbb{Q})$, the algebra of multisymmetric polynomials with rational coefficients, is generated by the elementary multisymmetric polynomials. This was established first by Schläfli [20] and also by MacMahon [15], and later other proofs were given by Emmy Noether [17] and Hermann Weyl [24]. On the other hand, various authors [16, 2, 4, 3, 5] gave counter-examples showing that the elementary multisymmetric polynomials may not generate $\mathfrak{J}_n^r(A)$, depending on the coefficient ring A. In particular, John Dalbec proved in [4, 3] that $\mathfrak{J}_n^r(\mathbb{Z})$, the ring of multisymmetric polynomials with integer coefficients, is generated by the elementary multisymmetric polynomials only in the trivial cases $n = 1$ or $r = 1$ and in the special case $(n, r) = (2, 2)$.

For which rings A do the elementary multisymmetric polynomials generate $\mathfrak{J}_n^r(A)$? The purpose of this paper is to demonstrate that we have the following answer:

Theorem 1. *Let A be a ring, and n and r be positive integers. The elementary multisymmetric polynomials generate the A-algebra $\mathfrak{J}_n^r(A)$ if and only if $n!$ is invertible in A, except in the following special cases:*

(i) $(n, r) = (2, 2)$.

(ii) $(n, r) = (3, 2)$ *and* 3 *is invertible in* A.

(iii) $r = 1$.

In these special cases, the elementary multisymmetric polynomials generate $\mathfrak{J}_n^r(A)$, even if the condition $n!$ invertible does not hold.

Let us give a geometric motivation. Let $\mathbb{K}$ be an algebraically closed field, and V be a $(r+1)$-dimensional $\mathbb{K}$-vector space. Let us consider the mapping *formal product* with values in S^nV, the n-th symmetric power of V:

$$\begin{array}{ccc} V^n & \longrightarrow & S^nV \\ (v_1, \dots, v_n) & \mapsto & v_1 \cdots v_n \ . \end{array}$$

It induces an injective morphism φ of algebraic varieties from the n-th symmetric product $(\mathbb{P}V)^n/\mathfrak{S}_n$ of the projective space $\mathbb{P}V$ of lines in V, to the projective space $\mathbb{P}(S^nV)$ of lines in S^nV. Its image is a closed algebraic subvariety, called the *Chow variety of multi-sets of n points in $\mathbb{P}V$*. We denote it by *Chow*$(0, n, \mathbb{P}V)$. So we have two algebraic varieties parameterizing the length n multi-sets of points in $\mathbb{P}V$: the symmetric product $(\mathbb{P}V)^n/\mathfrak{S}_n$ and the Chow variety *Chow*$(0, n, \mathbb{P}V)$.

Neeman [16] showed that a necessary and sufficient condition for these two algebraic varieties to be isomorphic is that the elementary multisymmetric polynomials generate the $\mathbb{K}$-algebra $\mathfrak{J}_n^r(\mathbb{K})$ (and if it is so, then φ is an isomorphism).

We deduce from Neeman's result and Theorem 1 that $(\mathbb{P}V)^n/\mathfrak{S}_n$ and *Chow*$(0, n, \mathbb{P}V)$ are isomorphic if and only if at least one of the following conditions holds:

- the coefficient field $\mathbb{K}$ has characteristic zero,
- the coefficient field $\mathbb{K}$ has positive characteristic greater than n,
- $(n, r) = (2, 2)$ (two points in the plane),
- $(n, r) = (3, 2)$ (three points in the plane) with $\mathbb{K}$ having characteristic two,
- $r = 1$ (any number of points on the line).

This paper is organized as follows.

In the first section, multisymmetric polynomials are introduced.

The second section deals with the non-modular case (following the terminology of invariant theoretists), that is the case when $n!$ is invertible in the coefficient ring A. It is proved that in this case, the elementary multisymmetric polynomials generate $\mathfrak{J}_n^r(A)$.

In the third section, it is demonstrated that when $n!$ is not invertible, the elementary multisymmetric polynomials don't generate $\mathfrak{J}_n^r(A)$, provided $r > 1$ and $(n,r) \neq (2,2)$, $(n,r) \neq (3,2)$.

The last section is devoted to the two special cases $(n,r) = (2,2)$ and $(n,r) = (3,2)$.

In the sequel, the set of natural numbers is denoted by $\mathbb{N}$; and for $\alpha \in \mathbb{N}^r$, we set $|\alpha| = \alpha_1 + \cdots + \alpha_r$. The zero vector of $\mathbb{N}^r$ is denoted with a bold zero: $\mathbf{0}$.

1. Multisymmetric polynomials

In this section we present the multisymmetric polynomials. We introduce only the necessary material. More can be found about these objects in the original works of MacMahon [15] and Junker [8, 9, 10, 11, 12, 13], or in modern accounts by Dalbec [3, 4], by Rosas [19] and by the author [1].

Let $a, b, \ldots, z$ be a finite alphabet with n letters. To each of these letters we associate r variables indexed with the integers from 1 to r:

$$\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_r \\ \vdots \\ z_1, \ldots, z_r \end{matrix} .$$

We will use the following notations: the bold symbol $\boldsymbol{a}$ will denote the family of variables $(a_1, \ldots, a_r)$, and $\boldsymbol{a}^\alpha$, where $\alpha \in \mathbb{N}^r$, will denote the monomial $a_1^{\alpha_1} \cdots a_r^{\alpha_r}$.

Definition 1. *The multisymmetric polynomials in the n families of r variables $\boldsymbol{a}, \ldots, \boldsymbol{z}$ are the polynomials in the $n \times r$ variables $a_1, a_2, \ldots, z_r$ that remain unchanged under every permutation of the n letters $a, b, \ldots, z$.*

Let A be a ring. The algebra of the multisymmetric polynomials in n families of r variables with coefficients in A is denoted $\mathfrak{J}_n^r(A)$.

Example. When $n = r = 2$, the multisymmetric polynomials are those polynomials P in a_1, a_2, b_1, b_2 such that:

$$P(a_1, a_2, b_1, b_2) = P(b_1, b_2, a_1, a_2).$$

We provide $\mathfrak{J}_n^r(A)$ with a grading with values in $\mathbb{N}^r$: the one obtained by giving to every variable x_i (for x any letter in the alphabet $a, \ldots, z$ and i any index between 1 and r) as its multidegree the i-th vector of the canonical basis of $\mathbb{Z}^r$ (denoted with ξ_i in the sequel).

The multidegree of an homogeneous multisymmetric polynomial P will be denoted with $\operatorname{mdeg} P$.

As in the case of symmetric polynomials, the algebra of multisymmetric polynomials admits an obvious module basis: the one made of the symmetrizations of the monomials, called *monomial (multisymmetric) functions.* Their symmetric analogues are the monomial symmetric functions, indexed by the integer partitions. Integer partitions are the finished decreasing sequences of positive integers, that we may think as representing the finished multi-sets of integers. Let λ be an integer partition with length at most n. It is obtained by forgetting the ordering and the possible occurrences of 0 in a length n integer sequence $\boldsymbol{s} = (s_a, \ldots, s_z)$. The monomial symmetric function in the variables $a, b, \ldots, z$ indexed with λ is the symmetrization of the monomial with exponent $\boldsymbol{s}$, that is:

$$m_\lambda = \sum_{\boldsymbol{t}} a^{t_a} b^{t_b} \cdots z^{t_z}$$

where $\boldsymbol{t}$ runs over the orbit of $\boldsymbol{s}$ under the permutations of its n terms.

In the same way, and mimicking the vocabulary associated to integer partitions, we set the following definitions:

Definition 2. *A vector partition of $\mathbb{N}^r$ is a finished multi-set of vectors of $\mathbb{N}^r \setminus \{\mathbf{0}\}$. If the vector partition $\mathfrak{p}$ is obtained from the sequence*

$$\left(\alpha^{(1)}, \ldots, \alpha^{(k)}\right)$$

of vectors of $\mathbb{N}^r$, by forgetting the ordering, and the possible occurrences of the zero vector, we will denote

$$\mathfrak{p} = \left[\alpha^{(1)}, \ldots, \alpha^{(k)}\right].$$

If moreover all of the $\alpha^{(i)}$ are non-zero, we will call them the parts of $\mathfrak{p}$, and say that k is the length of $\mathfrak{p}$, denoted $\ell_{\mathfrak{p}}$.

The vector $\beta = \alpha^{(1)} + \cdots + \alpha^{(k)}$ is called the sum of $\mathfrak{p}$, and denoted $s(\mathfrak{p})$. It will also be written $\mathfrak{p} \vdash \beta$, and read that $\mathfrak{p}$ is a partition of β.

Let α in $\mathbb{N}^r \setminus \{\mathbf{0}\}$. Its multiplicity in $\mathfrak{p}$ is denoted $\mu_{\mathfrak{p}}(\alpha)$. It is often useful to consider the multi-set of multiplicities of $\mathfrak{p}$, that is the multi-set obtained from the sequence of the multiplicities of the parts of $\mathfrak{p}$. It is denoted $\mu_{\mathfrak{p}}$. Last, the notation $\mu_{\mathfrak{p}}!$ stands for the product of the $\mu_{\mathfrak{p}}(\alpha)!$, for α running in the parts of $\mathfrak{p}$.

Definition 3. *Let $\mathfrak{p}$ be a vector partition of $\mathbb{N}^r$ with length at most n. There exists a sequence of n vectors $\boldsymbol{\alpha} = (\alpha^{(a)}, \ldots, \alpha^{(z)})$ (with possibly some of them being zero) such that*

$$\mathfrak{p} = [\alpha^{(a)}, \ldots, \alpha^{(z)}].$$

The monomial symmetric function indexed by $\mathfrak{p}$ in $\mathfrak{I}_n^r(A)$ is

$$m_{\mathfrak{p}} = \sum \boldsymbol{a}^{\beta^{(a)}} \cdots \boldsymbol{z}^{\beta^{(z)}}$$

where the sum is carried over the sequences $\left(\beta^{(a)}, \ldots, \beta^{(z)}\right)$ in the orbit of $\boldsymbol{\alpha}$ under the permutations of its n terms.

Remark. The monomial multisymmetric function $m_{\mathfrak{p}}$ has multidegree $s(\mathfrak{p})$.

Two special cases of monomial functions are, on the first hand, when the indexing vector partition has all its parts members of the canonical basis $(\xi_1, \dots, \xi_r)$, then we obtain an *elementary multisymmetric polynomial*[1]; and on the second hand, when the indexing vector partition has only one part, then we get a *multisymmetric power sum.* Precise definitions follow. Note that when $r = 1$ the elementary symmetric polynomials and the symmetric power sums are recovered this way.

Definition 4. *Let $\alpha \in \mathbb{N}^r$ with $1 \leq |\alpha| \leq n$. The elementary multisymmetric polynomial with multidegree α is*

$$e_\alpha = m_{[\xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_r^{\alpha_r}]} \,.$$

Otherwise stated, an elementary multisymmetric polynomial is the symmetrization of a monomial in which every letter appears at most once. It is coherent to set:

$$e_{\mathbf{0}} = 1$$

and for $\alpha \in \mathbb{N}^r$ with $|\alpha| > n$

$$e_\alpha = 0.$$

Definition 5. *Let $\alpha \in \mathbb{N}^r \setminus \{\mathbf{0}\}$. The multisymmetric power sum with multidegree α is*

$$p_\alpha = m_{[\alpha]} = \boldsymbol{a}^\alpha + \boldsymbol{b}^\alpha + \ldots + \boldsymbol{z}^\alpha.$$

Among them, we will distinguish the first power sums: those p_α with $|\alpha| \leq n$.

We will also index the products of elementary multisymmetric polynomials and multisymmetric power sums by vector partitions. So if

$$\mathfrak{p} = \left[\alpha^{(1)}, \dots, \alpha^{(k)}\right]$$

then

$$e_{\mathfrak{p}} = e_{\alpha^{(1)}} \cdots e_{\alpha^{(k)}}$$

and

$$p_{\mathfrak{p}} = p_{\alpha^{(1)}} \cdots p_{\alpha^{(k)}}.$$

Elementary multisymmetric polynomials and multisymmetric power sums are not only generalizations of elementary symmetric polynomials and symmetric power sums. Actually, they are obtained through a systematic process, called *polarization*, from their symmetric analogues. We present now this process, following the presentation given by Larry Smith [21].

[1] In [23], a different natural multisymmetric generalization of the elementary symmetric polynomials appears.

Let $\alpha \in \mathbb{N}^r$ with $|\alpha| = k$. The *polarization operator* Δ^r_α is the function from the homogeneous component of degree k of $\mathbb{Z}[a, b, \dots, z]$ to the homogeneous component of multidegree α of $\mathbb{Z}[a_1, a_2, \dots, z_r]$ obtained first by making the substitutions:

$$\begin{array}{ccc} a & \mapsto & a_1 + \ldots + a_r \\ b & \mapsto & b_1 + \ldots + b_r \\ \vdots & & \\ z & \mapsto & z_1 + \ldots + z_r \end{array}$$

and next by selecting the homogeneous component of multidegree α in the result.

For instance, for a power of the variable a, one has:

$$\Delta^r_\alpha(a^k) = \binom{k}{\alpha} \boldsymbol{a}^\alpha$$

where $\binom{k}{\alpha}$ is the multinomial coefficient:

$$\frac{k!}{\alpha_1!\alpha_2! \cdots \alpha_r!}.$$

From the definitions it is readily seen that polarization sends symmetric polynomials to multisymmetric polynomials. Better,

Proposition 2. *Polarization sends elementary symmetric polynomials to elementary multisymmetric polynomials, and symmetric power sums to integer multiples of multisymmetric power sums. Precisely, for $\alpha \in \mathbb{N}^r$ with $|\alpha| = k$,*

$$\Delta^r_\alpha(e_k) = e_\alpha$$

and if $\alpha \neq \mathbf{0}$,

$$\Delta^r_\alpha(p_k) = \binom{k}{\alpha} p_\alpha.$$

We will denote the subalgebra of $\mathfrak{I}^r_n(A)$ generated by the polarized symmetric polynomials by $\mathfrak{E}^r_n(A)$.

Remark. Let f be any homogeneous symmetric polynomial with degree k. Then there is a polynomial with integer coefficients P such that

$$f = P(e_1, \dots, e_k).$$

Let $\alpha \in \mathbb{N}^r$ with $|\alpha| = k$. Apply the polarization operator Δ^r_α, this yields an expression of $\Delta^r_\alpha(f)$ as a polynomial with integer coefficients in the elementary multisymmetric polynomials. This shows that $\mathfrak{E}^r_n(A)$ is generated by the elementary multisymmetric polynomials, and the problem raised in the present paper is also the problem of knowing when all the polarized symmetric polynomials generate $\mathfrak{I}^r_n(A)$.

2. Non-modular case

In this section, we prove that if $n!$ is invertible in A then the elementary multisymmetric polynomials generate $\mathfrak{J}_n^r(A)$. Our proof is essentially the proof already given by Richman [18], but we think our presentation is clearer.

The proof has three steps.

First, it is shown that if $n!$ is invertible, then $\mathfrak{J}_n^r(A)$ is generated by the power sums.

Next, an inductive formula (the *reduction formula for multisymmetric power sums*, Proposition 4) is established; this formula proves that any power sum is in the $\mathfrak{E}_n^r(A)$-module generated by the first power sums, under the condition $n!$ invertible (Corollary 5).

The last step demonstrates that the first power sums are in $\mathfrak{E}_n^r(A)$, under the condition $n!$ invertible (Lemma 6).

Theorem 3. *Let A be a ring in which $n!$ is invertible. Then the multisymmetric power sums generate $\mathfrak{J}_n^r(A)$ as an A-algebra.*

This theorem appears in [21], but with a gap in the proof. This gap is closed the proof that follows. Another short proof of the theorem is given in [6].

Proof. Let $\mathfrak{p} = [\alpha^{(1)}, \dots, \alpha^{(k)}]$ be a vector partition with length $k \leq n$. We want to show that $m_{\mathfrak{p}}$ is in the subalgebra of $\mathfrak{J}_n^r(A)$ generated by the power sums.

It is known that there is a polynomial P with integer coefficients such that:

$$k!e_k = P(p_1, \dots, p_k)$$

(see [14] for formulas giving P). Apply the polarization operator $\Delta^k_{(1,1,\dots,1)}$, this yields an expression of $k!e_{1,1,\dots,1}$ as a polynomial with integer coefficients in the multisymmetric power sums. Now evaluate every variable x_i to $\boldsymbol{x}^{\alpha^{(i)}}$. The image of $e_{1,1,\dots,1}$ is readily seen to be an integer multiple of $m_{\mathfrak{p}}$. To determine the multiplicative coefficient, note that $m_{\mathfrak{p}}$ is the sum of $k!/\mu_{\mathfrak{p}}!$ monomials, while $e_{1,1,\dots,1}$ is the sum of $k!$ monomials. So the multiplicative coefficient is $\mu_{\mathfrak{p}}!$.

This map also sends multisymmetric power sums to multisymmetric power sums.

So it is established that $k!\mu_{\mathfrak{p}}!m_{\mathfrak{p}}$ is a polynomial with integer coefficients in the multisymmetric power sums. Note that $k!\mu_{\mathfrak{p}}!$ is invertible, since $\mu_{\mathfrak{p}}!$ is a product of integers less than or equal to k, and $k \leq n$. □

The following formula was suggested to me by Nicolas Thiéry [22].

Proposition 4. (Reduction formula) *Let n be a positive integer. Let $\omega \in \mathbb{N}^r$ with $|\omega| = n$. Let $\alpha \in \mathbb{N}^r$. Then, in $\mathfrak{J}_n^r(\mathbb{Z})$,*

$$\binom{n}{\omega} p_{\omega+\alpha} = \sum_{\substack{\beta,\gamma \in \mathbb{N}^r, \gamma \neq \mathbf{0} \\ \beta+\gamma=\omega}} (-1)^{|\gamma|-1} e_\gamma \binom{|\beta|}{\beta} p_{\beta+\alpha}. \tag{1}$$

Note that $\binom{n}{\omega}$ is invertible as soon as $n!$ is invertible.

Proof. In $\mathbb{Z}[a, b, \ldots, z]$ one has:

$$\sum_{i+j=n} (-1)^i \, e_i \, a^j = 0$$

where the e_i's are the elementary symmetric polynomials in the variables $a, \ldots, z$. Let us apply Δ^r_ω, this yields:

$$\sum_{\gamma+\beta=\omega} (-1)^{|\gamma|} \, e_\gamma \binom{|\beta|}{\beta} \boldsymbol{a}^\beta = 0.$$

Let us multiply by $\boldsymbol{a}^\alpha$:

$$\sum_{\gamma+\beta=\omega} (-1)^{|\gamma|} \, e_\gamma \binom{|\beta|}{\beta} \boldsymbol{a}^{\beta+\alpha} = 0.$$

A similar formula, with a replaced by any letter $b, \ldots, z$, can be obtained. Let us sum over the expressions obtained this way and isolate the term with $\gamma = \mathbf{0}$ to get formula (1). □

Corollary 5. *Let A be a ring in which $n!$ is invertible. Then any power sum is in the module over $\mathfrak{E}^r_n(A)$ generated by the first power sums.*

Proof. This is a direct consequence of the reduction formula above, by induction. □

Lemma 6. *Let A be a ring in which $n!$ is invertible. Then any of the first power sums is in $\mathfrak{E}^r_n(A)$.*

Proof. Let $\alpha \in \mathbb{N}^r$ with $|\alpha| = k$. There is a polynomial P with integer coefficients such that:

$$p_k = P(e_1, \ldots, e_k).$$

Apply the polarization operator Δ^r_α, this yields an expression of $\binom{k}{\alpha} p_\alpha$ as a polynomial with integer coefficients in the elementary multi-symmetric polynomials.

Now, since $k \le n$, the multinomial coefficient $\binom{k}{\alpha}$ is invertible in A. □

3. Modular case

We now study the modular case, that is the case when $n!$ is not invertible in A.

First, several counter-examples for which $\mathfrak{E}^r_n(A) \neq \mathfrak{I}^r_n(A)$ are presented (Lemmas 7,8,9). Next, it is shown that these counter-examples are sufficient to prove that, under the condition $n!$ non-invertible, $\mathfrak{E}^r_n(A) \neq \mathfrak{I}^r_n(A)$ for nearly every value of (n, r), except when $r = 1$ and two special cases whose study is reported to the last section.

Lemma 7. *Let $k > 2$ be an odd number. Let A be a ring in which k is not invertible. Let f_k be the monomial function:*

$$f_k = m_{[\underbrace{(1,0)(1,0)\cdots(1,0)}_{k-1}(0,2)]}.$$

Then f_k is not inside the sub-algebra of $\mathfrak{I}_k^2(A)$ generated by the elementary multisymmetric polynomials.

This lemma is inspired by a counter-example due to Fleischmann ([5], proof of Theorem 4.7).

Proof. Let us suppose that f_k lies in the subalgebra generated by the elementary multisymmetric polynomials. Let us consider the projection from $\mathfrak{I}_k^2(A)$ to $\mathfrak{I}_k^1(A)$, the algebra of symmetric polynomials in $a_1, b_1, \ldots, z_1$, obtained by sending every variable $a_2, b_2, \ldots, z_2$ to 1 and leaving unchanged the variables $a_1, b_1, \ldots, z_1$. It sends f_k to e_{k-1}. It also sends every elementary multisymmetric polynomial $e_{i,j}$ to an integer multiple of e_i. Specially, the only $e_{\mathfrak{p}}$ whose image contributes to e_{k-1} are:

$$e_{k-1,1}\, e_{0,1}, \quad e_{k-1,0}\, e_{0,1}^2, \quad e_{k-1,0}\, e_{0,2}\,.$$

Their images are respectively:

$$k\, e_{k-1}, \quad k^2\, e_{k-1}, \quad \binom{k}{2}\, e_{k-1}\,.$$

All of them lie in the ideal of $\mathfrak{I}_k^1(A)$ generated by the integer k (this integer being odd, the binomial coefficient $\binom{k}{2}$ is a multiple of k). Therefore, 1 lies in the ideal of A generated by k; otherwise stated, k is invertible in A. □

Lemma 8. *Let A be a ring in which 2 is not invertible. The the power sum $p_{(1,1,1)}$ is not inside the subalgebra of $\mathfrak{I}_2^3(A)$ generated by the elementary multisymmetric polynomials.*

This counter-example was known by Campbell, Hughes and Pollack ([2], Section 6) and Dalbec ([4], Section 1.4 and [3], Section 2.1.4).

Proof. We suppose the contrary, that is that $p_{(1,1,1)}$ is a A-linear combination of:

$$e_{1,0,0}\, e_{0,1,0}\, e_{0,0,1}, \quad e_{1,0,0}\, e_{0,1,1}, \quad e_{0,1,0}\, e_{1,0,1}, \quad e_{0,0,1}\, e_{1,1,0}\,. \tag{2}$$

Let us consider the projection from $\mathfrak{I}_2^3(A)$ to the algebra of symmetric polynomials $\mathfrak{I}_2^1(A)$ obtained by substituting 1 to every one of the variables a_2, b_2, a_3, b_3. It sends $p_{(1,1,1)}$ to $p_1 = e_1$. It also sends the polynomials in (2) respectively to $4\,e_1$, $2\,e_1$, $2\,e_1$ and $2\,e_1$. As a result, we get that 2 is invertible in A. □

Lemma 9. *Let A be a ring in which 2 is not invertible. Then the power sum $p_{(3,2)}$ is not inside the subalgebra of $\mathfrak{I}_4^2(A)$ generated by the elementary multisymmetric polynomials.*

Proof. Let us consider the algebra morphism from $\mathfrak{I}_4^2(A)$ to $\mathfrak{I}_4^1(A)$, the algebra of symmetric polynomials in the variables a_1, b_1, c_1, d_1, obtained by leaving a_1, b_1, c_1, d_1 unchanged and sending a_2, b_2, c_2, d_2 to 1. The elementary multisymmetric polynomials have their images given by the following table:

$$\begin{array}{lllll} & e_{0,1} \mapsto 4 & e_{0,2} \mapsto 6 & e_{0,3} \mapsto 4 & e_{0,4} \mapsto 1 \\ e_{1,0} \mapsto e_1 & e_{1,1} \mapsto 3e_1 & e_{1,2} \mapsto 3e_1 & e_{1,3} \mapsto e_1 & \\ e_{2,0} \mapsto e_2 & e_{2,1} \mapsto 2e_2 & e_{2,2} \mapsto \ e_2 & & \\ e_{3,0} \mapsto e_3 & e_{3,1} \mapsto \ e_3 & & & \\ e_{4,0} \mapsto e_4 & & & & \end{array}$$

The image of the multisymmetric power sum $p_{(3,2)}$ is the symmetric power sum $p_3 = a_1^3 + b_1^3 + c_1^3 + d_1^3$. This power sum has a unique decomposition in terms of elementary symmetric polynomials:

$$p_3 = 3\,e_3 - 3\,e_2\,e_1 + e_1^3.$$

Let us suppose that $p_{(3,2)}$ lies in the subalgebra generated by the elementary multisymmetric polynomials. The only monomials in the e_α with projection contributing e_3 are

$$e_{3,0}\,e_{0,1}^2, \quad e_{3,0}\,e_{0,2}, \quad e_{3,1}\,e_{0,1}.$$

But their images are actually

$$16\,e_3, \quad 6\,e_3, \quad 4\,e_3.$$

This implies that the coefficient, 3, of e_3, lies inside the ideal generated by 2 in A, and so that 2 is invertible in A. □

Now these counter-examples are used to produce counter-examples in nearly every $\mathfrak{I}_n^r(A)$, thanks to the following lemma:

Lemma 10. *Let A be a ring. Let r, n be integers. If the algebra $\mathfrak{I}_n^r(A)$ is not generated by the elementary multisymmetric polynomials, then neither is $\mathfrak{I}_{n+1}^r(A)$, nor $\mathfrak{I}_n^{r+1}(A)$.*

Proof. We prove the contrapositive.

There is an algebra epimorphism from $\mathfrak{I}_{n+1}^r(A)$ to $\mathfrak{I}_n^r(A)$ that sends the elementary multisymmetric polynomials of $\mathfrak{I}_{n+1}^r(A)$ to elementary multisymmetric polynomials of $\mathfrak{I}_n^r(A)$. It is the one obtained by sending the variables associated to the last letter to 0. So if $\mathfrak{I}_{n+1}^r(A)$ is generated by its elementary multisymmetric polynomials, so is $\mathfrak{I}_n^r(A)$.

Let us consider the algebra epimorphism from $\mathfrak{I}_n^{r+1}(A)$ to $\mathfrak{I}_n^r(A)$ obtained by annihilating the variables $a_{r+1}, b_{r+1}, \dots, z_{r+1}$. It sends the elementary multisymmetric polynomials of $\mathfrak{I}_n^{r+1}(A)$ to zero, or to elementary multisymmetric polynomials of $\mathfrak{I}_n^r(A)$. Therefore if $\mathfrak{I}_n^{r+1}(A)$ is generated by its elementary multisymmetric polynomials, so is $\mathfrak{I}_n^r(A)$. □

The final product of this section is the following proposition.

Proposition 11. *If $n!$ is not invertible in A, and $r > 1$, and $(n, r) \neq (2, 2)$, and $(n, r) \neq (3, 2)$, then the elementary multisymmetric polynomials do not generate $\mathfrak{I}_n^r(A)$.*

Proof. If $n!$ is not invertible in A, then there exists a prime number $k \leq n$, not invertible in A. If k is odd and $r > 1$, then Lemma 7 and Lemma 10 show that the elementary multisymmetric polynomials do not generate $\mathfrak{I}_n^r(A)$. If $k = 2$ then the same conclusion is reached thanks to Lemma 8 and Lemma 10 when $r \geq 3$, and thanks to Lemma 9 and Lemma 10 when $r = 2$ with $n \geq 4$. □

4. Special cases

The previous reasonings gave, up to now, no necessary and sufficient condition on A for the elementary multisymmetric polynomials to generate $\mathfrak{I}_2^2(A)$ and $\mathfrak{I}_3^2(A)$.

In the case $\mathfrak{I}_2^2(A)$ we have no condition at all, and in the case $\mathfrak{I}_3^2(A)$ we just know, after Lemma 7, the necessary condition that 3 has to be invertible.

In this section we show that there are no further conditions.

The main tool for these proofs is a reduction algorithm due to Fleischmann [5].

4.1. Fleischmann's reduction algorithm

Before presenting Fleischmann's algorithm, we need a lemma about the product of monomial functions.

Lemma 12. (Product formula). *Let $\mathfrak{p}$, $\mathfrak{q}$ be vector partitions of $\mathbb{N}^r$. Then:*

$$m_{\mathfrak{p}} \cdot m_{\mathfrak{q}} = \sum_{\mathfrak{r}} c(\mathfrak{p}, \mathfrak{q}; \mathfrak{r}) m_{\mathfrak{r}}.$$

where the coefficient $c(\mathfrak{p}, \mathfrak{q}; \mathfrak{r})$ is defined as follows: a sequence of vectors, $\boldsymbol{\gamma} = (\gamma^{(1)}, \dots, \gamma^{(\ell)})$, such that $[\boldsymbol{\gamma}] = \mathfrak{r}$ and that ℓ be no less than $\max(\ell_{\mathfrak{p}}, \ell_{\mathfrak{q}})$, is arbitrarily chosen. Then this coefficient is the number of decompositions:

$$\begin{pmatrix} \gamma^{(1)} \\ \gamma^{(2)} \\ \vdots \\ \gamma^{(\ell)} \end{pmatrix} = \begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \vdots \\ \alpha^{(\ell)} \end{pmatrix} + \begin{pmatrix} \beta^{(1)} \\ \beta^{(2)} \\ \vdots \\ \beta^{(\ell)} \end{pmatrix}$$

where

$$\begin{aligned} \left[(\alpha^{(1)}, \dots, \alpha^{(\ell)})\right] &= \mathfrak{p} \\ \left[(\beta^{(1)}, \dots, \beta^{(\ell)})\right] &= \mathfrak{q}. \end{aligned}$$

Example. Let $\delta \in \mathbb{N}^r \setminus \{\mathbf{0}\}$. Then $m_{[\delta,\mathbf{0}]} m_{[\delta,\mathbf{0}]} = m_{[2\delta,\mathbf{0}]} + 2m_{[\delta,\delta]}$. Indeed:

$$\begin{pmatrix} 2\delta \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \delta \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \delta \\ \mathbf{0} \end{pmatrix},$$

(only one decomposition, thus the coefficient of $m_{[2\delta,\mathbf{0}]}$ is 1) and

$$\begin{pmatrix} \delta \\ \delta \end{pmatrix} = \begin{pmatrix} \delta \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \delta \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \delta \end{pmatrix} + \begin{pmatrix} \delta \\ \mathbf{0} \end{pmatrix}$$

(two decompositions, thus the coefficient of $m_{[\delta,\delta]}$ is 2).

Proof. Let us recall that for any vector partition $\mathfrak{p}$ one has:

$$m_{\mathfrak{p}} = \sum_{[\alpha^{(a)},\alpha^{(b)},\dots,\alpha^{(z)}]=\mathfrak{p}} a^{\alpha^{(a)}} b^{\alpha^{(b)}} \cdots z^{\alpha^{(z)}}.$$

Let $\mathfrak{p}$ and $\mathfrak{q}$ be vector partitions, then:

$$m_{\mathfrak{p}} \cdot m_{\mathfrak{q}} = \sum a^{\alpha^{(a)}+\beta^{(a)}} b^{\alpha^{(b)}+\beta^{(b)}} \cdots z^{\alpha^{(z)}+\beta^{(z)}}$$

the sum being carried over all couples of sequences

$$\begin{array}{l}(\alpha^{(a)},\dots,\alpha^{(z)})\\(\beta^{(a)},\dots,\beta^{(z)})\end{array}$$

such that $[(\alpha^{(a)},\dots,\alpha^{(z)})] = \mathfrak{p}$ and $[(\beta^{(a)},\dots,\beta^{(z)})] = \mathfrak{q}$. So we are done. □

We will say that $\alpha \in \mathbb{N}^r$ *dominates* $\beta \in \mathbb{N}^r$ if for all i, $\alpha_i \geq \beta_i$.

Fleischmann's algorithm is presented in the proof of the following lemma.

Lemma 13. ([5], Theorem 4.6) *The multisymmetric polynomials with multidegree dominated by $(n-1, n-1, \dots, n-1)$ generate $\mathfrak{I}_n^r(\mathbb{Z})$ as an $\mathfrak{E}_n^r(\mathbb{Z})$-module.*

Proof. The proof consists in a reduction algorithm, expressing any monomial function with multidegree not dominated by $(n-1,\dots,n-1)$ as a linear combination, with coefficients in $\mathfrak{E}_n^r(\mathbb{Z})$, in the monomial functions with multidegree strictly lower (according to the dominance ordering).

Let us choose a coordinate $i \in \{1,\dots,r\}$. To every $\mathfrak{p}$, vector partition of $\mathbb{N}^r$, we associate $\lambda(i;\mathfrak{p})$: the sequence of the i-th coordinates of the parts of $\mathfrak{p}$, in the decreasing order. For instance, to $\mathfrak{p} = [(3,1)(2,0)(1,1)]$, it is associated $\lambda(1;\mathfrak{p}) = (3,2,1)$ and $\lambda(2;\mathfrak{p}) = (1,1,0)$.

We define a partial ordering $\preceq_i$ on the set of vector partitions of $\mathbb{N}^r$ with length at most n in the following way: $\mathfrak{p} \preceq_i \mathfrak{q}$ if and only if $\lambda(i;\mathfrak{p})$ is smaller than $\lambda(i;\mathfrak{q})$ in lexicographic order.

Let $\mathfrak{p}$ be a vector partition whose sum of its i-th coordinates is not less than n.

- If $\lambda(i;\mathfrak{p}) = (t_1, t_2, \dots, t_s, k, \dots, k, 0, \dots, 0)$ with $t_1 \geq t_2 \geq \dots \geq t_s > k > 0$, then we set $\mathfrak{r}$ the vector partition obtained from $\mathfrak{p}$ by changing in the i-th coordinates of its parts t_j into $t_j - 1$.
  The product formula shows that:
  $$m_{\mathfrak{p}} = m_{\mathfrak{r}}\, e_{s\xi_i} - \sum m_{\mathfrak{q}}$$
  for vector partitions $\mathfrak{q} \prec_i \mathfrak{p}$.
- If $\lambda(i;\mathfrak{p}) = (k, k, \dots, k, 0, \dots, 0)$ with s occurrences of $k > 1$, then we set $\mathfrak{r}$ the vector partition obtained from $\mathfrak{p}$ by changing in the i-th coordinates of its parts k into $k-1$. The product formula shows that:
  $$m_{\mathfrak{p}} = m_{\mathfrak{r}}\, e_{s\xi_i} - \sum m_{\mathfrak{q}}$$
  for vector partitions $\mathfrak{q} \prec_i \mathfrak{p}$.

- If $\lambda(i;\mathfrak{p}) = (1,1,\ldots,1)$ with n occurrences of 1, then one has the factorization:

$$m_{\mathfrak{p}} = m_{\mathfrak{r}}\, e_{n\xi_i}$$

where $\mathfrak{r}$ is the vector partition obtained form $\mathfrak{p}$ by vanishing the i-th coordinates of its parts.

By applying these three types of reduction, any monomial function with multidegree α is expressed as a linear combination, with coefficients in $\mathfrak{E}_n^r(\mathbb{Z})$, in the monomial functions with multidegree β, with β dominated by α, distinct from α, and $\beta_i \leq n-1$.

By applying successively this procedure for $i = 1$, and next $i = 2, \ldots, i = r$, it is obtained an expression as a polynomial with integer coefficients, in the elementary multisymmetric polynomials and in the monomial functions with multidegree dominated by $(n-1, n-1, \ldots, n-1)$. □

4.2. Case $(n, r) = (2, 2)$

Here we want to prove that the elementary multisymmetric polynomials generate $\mathfrak{J}_2^2(\mathbb{Z})$. After Lemma 13, this is reduced to prove that every monomial function with multidegree $(1,0), (0,1)$ or $(1,1)$ can be expressed as a linear combination with integer coefficients of products of elementary multisymmetric polynomials. The multisymmetric polynomials of multidegree $(1,0)$ (or more generally $(k,0)$ for any $k \in \mathbb{N}$) are actually symmetric polynomials in the variables a_1, b_1, and thus can be expressed as polynomials with integer coefficients in the elementary symmetric polynomials in a_1, b_2. The latter are elementary multisymmetric polynomials. The same reasoning holds for the multisymmetric polynomials of multidegree $(0,k)$: they are symmetric polynomials in the variables a_2, b_2.

So we have just to deal with the multidegree $(1,1)$ case. There are two monomial functions of multidegree $(1,1)$ in $\mathfrak{J}_2^2$, that are $m_{[(1,1)]}$ and $m_{[(1,0),(0,1)]}$. It is easy to check that:

$$\begin{aligned} m_{[(1,1)]} &= e_{1,0}e_{0,1} - e_{1,1} \\ m_{[(1,0),(0,1)]} &= e_{1,1}. \end{aligned}$$

Therefore:

Proposition 14. *The elementary multisymmetric polynomials generate the ring $\mathfrak{J}_2^2(\mathbb{Z})$.*

This proposition 14 was proved by John Dalbec in [4]. Dalbec's argument is the following: the elementary multisymmetric functions and the monomial functions $m_{[(n,0)(0,1)]}$ form a SAGBI basis of $\mathfrak{J}_2^2(\mathbb{Z})$ for a certain ordering on the variables a_1, a_2, b_1, b_2.

4.3. Case $(n, r) = (3, 2)$

We want to prove that the elementary multisymmetric polynomials generate the algebra $\mathfrak{J}_3^2(A)$, provided 3 is invertible in A. After Proposition 13, this is reduced to prove that any monomial function with multidegree among:

$$\begin{gathered} (1,0),\ (2,0),\ (0,1),\ (0,2), \\ (1,1),\ (2,1),\ (1,2),\ (2,2) \end{gathered}$$

lies inside the subalgebra generated by the elementary multisymmetric polynomials.

The case of a monomial function with multidegree $(1,0)$,$(2,0)$,$(0,1)$ or $(0,2)$ is trivial, as explained in the case $(n,r)=(2,2)$. Moreover, any result valid for the case $(2,1)$ is also valid for the case $(1,2)$, by permutation of the coordinates. So we are done once we have established the result for multidegrees $(1,1),(2,1)$ and $(2,2)$.

There are two vector partitions indexing the monomial functions with multidegree $(1,1)$, as well as the products of elementary multisymmetric polynomials of this multidegree. These vector partitions are $[(1,1)]$ and $[(1,0),(0,1)]$, and we have:

$$\begin{aligned} m_{[(1,1)]} &= e_{1,0}e_{0,1}-e_{1,1} \\ m_{[(1,0)(0,1)]} &= e_{1,1}. \end{aligned}$$

There are four vector partitions indexing the monomial functions with multidegree $(2,1)$, as well as the products of elementary multisymmetric polynomials of this multidegree. These vector partitions are:

$$\begin{aligned} \mathfrak{p}_1 &= [(2,1)], & \mathfrak{p}_2 &= [(2,0)(0,1)], \\ \mathfrak{p}_3 &= [(1,1)(1,0)], & \mathfrak{p}_4 &= [(1,0)(1,0)(0,1)]. \end{aligned}$$

Let us introduce the *conversion matrix* from the family e to the family m in the multidegree $(2,1)$ component of $\mathfrak{J}_3^2(\mathbb{Z})$. This is the matrix whose entry in column $e_\mathfrak{p}$ and line $m_\mathfrak{q}$ is the coefficient of $m_\mathfrak{q}$ in the decomposition of $e_\mathfrak{p}$ as a linear combination of monomial functions. We computed it:

	$e_{\mathfrak{p}_1}$	$e_{\mathfrak{p}_2}$	$e_{\mathfrak{p}_3}$	$e_{\mathfrak{p}_4}$
$m_{\mathfrak{p}_1}$	0	0	0	1
$m_{\mathfrak{p}_2}$	0	0	1	1
$m_{\mathfrak{p}_3}$	0	1	1	2
$m_{\mathfrak{p}_4}$	1	1	2	2

It is invertible over $\mathbb{Z}$. Thus every monomial function with multidegree $(2,1)$ lies inside the subring generated by the elementary multisymmetric polynomials.

There are eight vector partitions indexing the monomial functions with multidegree $(2,2)$, that are:

$$\begin{aligned} \mathfrak{q}_1 &= [(2,2)], & \mathfrak{q}_2 &= [(2,1)(0,1)], \\ \mathfrak{q}_3 &= [(2,0)(0,2)], & \mathfrak{q}_4 &= [(2,0)(0,1)(0,1)], \\ \mathfrak{q}_5 &= [(1,2)(1,0)], & \mathfrak{q}_6 &= [(1,1)(1,1)], \\ \mathfrak{q}_7 &= [(1,1)(1,0)(0,1)], & \mathfrak{q}_8 &= [(1,0)(1,0)(0,2)]. \end{aligned}$$

There are also eight vector partitions indexing the products of elementary multisymmetric polynomials with multidegree $(2,2)$. These are

$$\mathfrak{q}_2,\ \mathfrak{q}_3,\ \mathfrak{q}_4,\ \mathfrak{q}_5,\ \mathfrak{q}_6,\ \mathfrak{q}_7,\ \mathfrak{q}_8$$

and $\mathfrak{q}_9=[(1,0)(1,0)(0,1)(0,1)]$.

The conversion matrix is:

	$e_{\mathfrak{q}_2}$	$e_{\mathfrak{q}_3}$	$e_{\mathfrak{q}_4}$	$e_{\mathfrak{q}_5}$	$e_{\mathfrak{q}_6}$	$e_{\mathfrak{q}_7}$	$e_{\mathfrak{q}_8}$	$e_{\mathfrak{q}_9}$
$m_{\mathfrak{q}_1}$	0	0	0	0	0	0	0	1
$m_{\mathfrak{q}_2}$	0	0	0	0	0	1	1	2
$m_{\mathfrak{q}_3}$	0	0	0	0	1	1	0	1
$m_{\mathfrak{q}_4}$	0	0	0	1	2	2	1	2
$m_{\mathfrak{q}_5}$	0	0	1	0	0	1	0	2
$m_{\mathfrak{q}_6}$	0	1	2	0	2	2	2	4
$m_{\mathfrak{q}_7}$	1	1	2	1	2	3	2	4
$m_{\mathfrak{q}_8}$	1	0	1	0	2	2	0	2

Its determinant is 3. Therefore, the elementary multisymmetric polynomials generate the multidegree $(2,2)$ component of $\mathfrak{I}_3^2(A)$ if and only if 3 is invertible in A.

We have finally proved the following proposition:

Proposition 15. *The elementary multisymmetric polynomials generate $\mathfrak{I}_3^2(A)$ if and only if* 3 *is invertible in A.*

This completes the proof of Theorem 1.

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