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On Extrinsic Symmetric CR-Structures on the Manifolds of Complete Flags

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Abstract. In the present paper we study extrinsic symmetric CR-structures on the manifolds of complete flags G_u/T relating some of them with the inner symmetric spaces G_u/K. We show that the maximum possible CR-dimension of any extrinsic symmetric CR-manifold G_u/T cannot be larger than half the maximal dimension of the inner symmetric spaces G_u/K. The CR-dimension can be this number and in this case, the holomorphic tangent space is the tangent space of G_u/K (at some point).

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1. Introduction

The study of the diverse forms in which the classical definition of a symmetric space can be generalized, is of interest in Differential Geometry. In [10] the authors introduced the definition of symmetric Cauchy-Riemann manifolds as a very natural way to generalize the notion of symmetry to the category of CR-spaces. They notice that in this case is not adequate to maintain the usual requirement that the points of the manifold are isolated fixed points of the symmetries associated to them. This has the effect of producing a large collection of spaces with many new and interesting features that deserve to be studied.

In the paper [11] it was observed that it is very natural to extend to symmetric CR-spaces the geometric notion of extrinsic symmetric submanifold given by D. Ferus in [7] and many examples from [10] were shown to have this property.

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In the present paper we intent to initiate a more systematic study of extrinsic symmetric CR-manifolds which could provide new examples and at the same time describe some interesting subfamilies.

Our previous study of normal sections on the natural embedding of the flag manifolds G_u/T (G_u a connected compact simple Lie group and T a maximal torus in G_u) led us naturally to consider the possible extrinsic symmetric CR-structures admitted by these manifolds in their natural embeddings. This turns out to be an interesting problem which seems to have important connections to other geometric features of these manifolds.

Our initial motivation for considering these simple spaces was to describe an elementary family of examples, but it turns out that the existence or not of extrinsic symmetric CR-structures on these spaces is a non-trivial question. This question is related to the existence of real projective subspaces in the variety of planar normal section associated to the natural embeddings of these submanifolds. This is a central point of the present paper giving rise to the proof of the main results (Theorem 8 and Corollary 9).

The paper is organized as follows. In Section 2 we present the basic facts required in our work. In Section 3 we study the almost CR-structures on the manifold of complete flags $M = G_u/T$ for which the natural action of G_u on M is via CR-diffeomorphisms. This seems to be the most natural way to relate quotient and CR-structures on M. In Theorems 1 and 5 we characterize these structures. This leads naturally to consider the tangent spaces to symmetric spaces G_u/K as candidates to holomorphic tangent spaces of CR-structures on M. The main result of this section, Theorem 6, shows that only interior symmetric spaces are fit to do that.

In Section 4, for a natural embedding j of G_u/T in $\mathfrak{g}_u$, we study conditions related to the existence of certain CR-structures on M. We may consider j as the inclusion and as in [11] we say that an almost Hermitian symmetric CR-structure on M is *extrinsic symmetric* if for each $y \in M$ the symmetry σ_y of M extends to an isometry φ_y of $\mathfrak{g}_u$ such that $(\varphi_y)_*|_y$ is the identity on $T_yM^{\perp}$. We relate some extrinsic symmetric CR-structures on G_u/T with the inner symmetric spaces G_u/K. Finally, Theorem 8 shows that the maximum possible CR-dimension of *any* extrinsic symmetric CR-manifold G_u/T cannot be larger than half the maximal dimension of the inner symmetric spaces G_u/K. It also shows that in those of maximal CR-dimension, the holomorphic tangent space is in fact the tangent space (at some point) of the inner G_u/K of maximal dimension.

2. Necessary facts

This section contains the definitions and facts from [10], [8] and [12] which are needed in the rest of the paper.

2.1. SCR-spaces

Let M be a connected finite dimensional real manifold (C^{∞} or analytic) and denote by $T_x(M)$ the tangent space at the point $x \in M$. An *almost Cauchy-Riemann structure* or an *almost CR-structure* is the assignation, to every $x \in M$, of a linear subspace $\mathfrak{H}_x \subset T_x(M)$ and a complex structure J_x on $\mathfrak{H}_x$ in such a way that the subspace $\mathfrak{H}_x$ and the complex structure J_x depend differentially on x. This dependence means that every point $x \in M$ has a neighborhood $U \subset M$ and for each $y \in U$ a linear endomorphism J_y of $T_y(M)$ such that $\left(-J_y^2\right)$ is a projection from $T_y(M)$ onto $\mathfrak{H}_y$ with $J_y^2 X = -X$ for every $X \in \mathfrak{H}_y$ and J_y depending smoothly on $y \in U$. Thus all the subspaces $\mathfrak{H}_x$ have the same dimension. A connected differentiable manifold with an almost CR-structure is called an *almost CR-manifold*.

A smooth map $\varphi : M \to N$ between two almost CR-manifolds is called a *CR-map* if for every $y \in M$ the derivative $\varphi_*|_y : T_y(M) \to T_{\varphi(y)}(N)$ maps the complex subspace $\mathfrak{H}_y(M)$ complex linearly into $\mathfrak{H}_{\varphi(y)}(N)$. Then it makes sense to consider *CR-diffeomorphisms* between almost CR-manifolds.

Let us assume now that we have on M a Riemannian metric and let $\langle , \rangle_y$ denote the corresponding scalar product in $T_y(M)$. We shall say that M is an *almost Hermitian CR-manifold* if for every $y \in M$ and every $X, Z \in \mathfrak{H}_y$

$$\langle J_y X, J_y Z \rangle_y = \langle X, Z \rangle_y .$$

Let M be an almost Hermitian CR-manifold and let $\sigma : M \to M$ be an isometric CR-difeomorphism. In [10], the map σ is called a *symmetry* at the point $y \in M$ if y is a (not necessarily isolated) fixed point of σ and its derivative $\sigma_*|_y$ restricted to the subspace $\mathfrak{H}_y^{-1} \oplus \mathfrak{H}_y \subset T_y(M)$ coincides with $(-Id)$. For the definition of $\mathfrak{H}_y^{-1}$ see [10, p. 151]. The almost CR-manifold M is called *minimal* if $\mathfrak{H}_y^{-1} = 0$ for all $y \in M$.

A connected almost Hermitian CR-manifold M is called a *symmetric almost Hermitian CR-manifold* if there is a symmetry at each point $y \in M$. It can be proved that there is at most one symmetry at each point of M and also that the group $I(M)$, *of all isometric CR-diffeomorphisms of* M, is a Lie group ([10, p. 152]).

2.2. The manifold of complete flags

Let G be a simply connected, complex, simple Lie group and let $\mathfrak{g}$ be its Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. We may write

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\gamma \in \Delta^+} (\mathfrak{g}_\gamma \oplus \mathfrak{g}_{-\gamma})$$

where Δ^+ indicates the set of positive roots with respect to some order. Let us consider in $\mathfrak{g}$ the Borel subalgebra

$$\mathfrak{b} = \mathfrak{h} \oplus \sum_{\gamma \in \Delta^+} \mathfrak{g}_{-\gamma}.$$

Let B be the analytic subgroup of G corresponding to the subalgebra $\mathfrak{b}$. B is closed and its own normalizer in G. The quotient space $M = G/B$ is a complex homogeneous space called the manifold of complete flags of G.

Let $\pi = \{\alpha_1, ..., \alpha_n\} \subset \Delta^+$ be the system of simple roots. We may take in $\mathfrak{g}$ a Weyl basis [8, III, 5] $\{X_\gamma : \gamma \in \Delta\}$ and $\{H_\beta : \beta \in \pi\}$. The following set of vectors provides a basis of a compact real form $\mathfrak{g}_u$ of $\mathfrak{g}$.

$$\begin{cases} U_\gamma & = \frac{1}{\sqrt{2}}\left(X_\gamma - X_{-\gamma}\right) & \gamma \in \Delta^+ \\ U_{-\gamma} & = \frac{i}{\sqrt{2}}\left(X_\gamma + X_{-\gamma}\right) & \gamma \in \Delta^+ \\ iH_\beta & & \beta \in \pi. \end{cases} \tag{1}$$

We shall denote by $\mathfrak{h}_u$ the real vector space generated by $\{iH_\beta : \beta \in \pi\}$ and by $\mathfrak{m}_\gamma$ that of $\{U_\gamma, U_{-\gamma}\}$. Then we may write

$$\mathfrak{g}_u = \mathfrak{h}_u \oplus \sum_{\gamma \in \Delta^+} \mathfrak{m}_\gamma = \mathfrak{h}_u \oplus \mathfrak{m}. \tag{2}$$

Let G_u be the analytic subgroup of G corresponding to $\mathfrak{g}_u$. G_u is compact simply connected and acts transitively on M which can be written as $M = G_u/\left(G_u \cap B\right)$. The subgroup $T = G_u \cap B = \exp \mathfrak{h}_u$ is a maximal torus in G_u. The manifold M is then a compact simply connected complex manifold.

Let $E \in \mathfrak{g}_u$ be a regular element [8]. We want to consider the orbit of E by the adjoint action of G_u on $\mathfrak{g}_u$, i.e. $Ad\left(G_u\right) E = \{Ad\left(g\right) E : g \in G_u\}$. It is clear that the isotropy subgroup of the point E is precisely T and we have a natural embedding of M in $\mathfrak{g}_u$. We may take in $\mathfrak{g}_u$ the inner product given by the opposite of the Killing form and therefore, the induced Riemannian metric on M by the embedding $j : M \to \mathfrak{g}_u$ is invariant by the action of G_u. This is the *manifold of complete flags* for the given Lie group G_u. Then

$$M = G_u/T$$

is the orbit of the regular element $E \in \mathfrak{g}_u$ whose tangent and normal spaces at E are

$$T_E(M) = [\mathfrak{g}_u, E] = [\mathfrak{m}, E] = \mathfrak{m} \quad \text{and} \quad T_E\left(M\right)^\perp = \mathfrak{h}_u.$$

3. G_u-invariant CR-structures

We shall say that an *almost CR-structure* on the manifold of complete flags $M = G_u/T$ is G_u*-invariant* if the natural action of G_u on M is via isometric CR-diffeomorphisms of M, i.e. $G_u \subset I(M)$.

Our purpose now is to study G_u-invariant almost CR-structures on M. With similar methods to those used in [10] one can obtain the following characterization of those subspaces of the tangent space $T_E(M) = \mathfrak{m}$ which are *holomorphic tangent spaces* at E of some G_u-invariant almost CR-structures on M.

Theorem 1. *Set $M = G_u/T = Ad(G_u)E$.*

(i) *If M has a G_u-invariant almost CR-structure then the holomorphic tangent space $\mathfrak{H}_E \subset T_E(M)$ and the almost complex structure J_E in $\mathfrak{H}_E$, are $Ad(T)$-invariant.*

(ii) *If $\mathfrak{H}$ is an $Ad(T)$-invariant subspace of $T_E(M)$ and J is an endomorphism of $\mathfrak{H}$ such that $J^2 = -Id$ and $Ad\left(g\right) JZ = JAd\left(g\right) Z$ for every $Z \in \mathfrak{H}$ and $g \in T$, then M has a G_u-invariant almost CR-structure where $\mathfrak{H}_E(M) = \mathfrak{H}$ and $J_E = J$.* □

This fact motivates the study of the subspaces of $\mathfrak{m} = T_E(M)$ which are $Ad(T)$-invariant.

If $A \in \mathfrak{h}_u$ it is easy to see that, for each $\gamma \in \Delta^+$,

$$[A, U_\gamma] = -\gamma(iA)U_{-\gamma} \ ; \ \ [A, U_{-\gamma}] = \gamma(iA)U_\gamma \tag{3}$$

and therefore $\mathfrak{m}_\gamma$ is an $ad(\mathfrak{h}_u)$-invariant subspace of $\mathfrak{m}$. It follows that $\mathfrak{m}_\gamma$ is also $Ad(T)$-invariant and so, if

$$\mathfrak{H} = \sum_{\gamma \in \Delta^*} \mathfrak{m}_\gamma \ , \ \text{ with } \Delta^* \subset \Delta^+$$

then $\mathfrak{H}$ is $Ad(T)$-invariant.

Our next objective is to show that every $Ad(T)$-invariant subspace of $\mathfrak{m}$ is precisely of this form. This important fact may be known but we do not know any adequate reference. Our proof will be divided in the following steps.

Lemma 2. *There exists $E^* \in \mathfrak{h}_u$ such that $\gamma(iE^*) > 0$ for each $\gamma \in \Delta^+$ and $\gamma(iE^*) \neq \beta(iE^*)$ for every pair of different roots γ, β in Δ^+.*

Proof. Let $\mu = \sum n_j \alpha_j$ be the highest root of $\mathfrak{g}$ where $\pi = \{\alpha_1, ..., \alpha_n\} \subset \Delta^+$ is the set of simple roots. Let $\{v_1, \dots, v_n\}$ be the Wolf basis of $\mathfrak{h}_u$ [14], defined by $\alpha_j(v_k) = \frac{\delta_{j,k}}{n_k}$, $\quad 1 \leq j, k \leq n$.

Each $\gamma \in \Delta^+$ can be written as $\gamma = \sum h_{\alpha_j}(\gamma)\alpha_j$ where $0 \leq h_{\alpha_j}(\gamma) \leq n_j$. Let $s > 6$ be a fixed natural number and set

$$iE^* = \sum_{j=1}^{n} s^j n_j v_j.$$

Then, if $\gamma \in \Delta^+$,

$$\gamma(iE^*) = \sum_{j=1}^{n} s^j h_{\alpha_j}(\gamma) > 0.$$

Therefore $\gamma(iE^*) = \beta(iE^*)$ is equivalent to

$$\sum_{j=1}^{n} s^j h_{\alpha_j}(\gamma) = \sum_{j=1}^{n} s^j h_{\alpha_j}(\beta).$$

By our choice of the number s, both sides of this equality represent the s-adic expression of the same integer. Consequently $h_{\alpha_j}(\gamma) = h_{\alpha_j}(\beta)$, $1 \leq j \leq n$, and therefore $\gamma = \beta$. $\square$

Lemma 3. *Let $\mathfrak{V}$ be a real subspace of $\mathfrak{m}$ supporting an irreducible representation of the abelian Lie algebra $\mathfrak{h}_u$ via the adjoint representation. For each $A \in \mathfrak{h}_u$, set $\varphi_A = ad(A)|_{\mathfrak{V}}$.*

(i) *If 0 is an eigenvalue of φ_A then $\varphi_A = 0$.*

(ii) $\dim_R(\mathfrak{V}) = 2$.

Proof. (i): Since $\mathfrak{h}_u$ is abelian, $\varphi_A \varphi_B = \varphi_B \varphi_A$ for every $B \in \mathfrak{h}_u$. Then $\{0\} \neq Ker\varphi_A$ is $ad(\mathfrak{h}_u)$-invariant and so $Ker\varphi_A = \mathfrak{V}$.

(ii): To simplify our notation we denote by $r = |\Delta^+|$, $\Delta^+ = \{\gamma_1, ..., \gamma_r\}$, $\mathfrak{m}_j = \mathfrak{m}_{\gamma_j}$, and $\mathcal{B}_j = \left\{U_{\gamma_j}, U_{-\gamma_j}\right\}$. Then $\mathfrak{m} = \sum_{j=1}^{r} \mathfrak{m}_j$.

Let E^* be an element given by Lemma 2. The real numbers $b_j = \gamma_j(iE^*)$ are non-zero and pairwise different and if

$$R_j = \begin{bmatrix} 0 & b_j \\ -b_j & 0 \end{bmatrix}$$

then, by (3), the matrix of the transformation $ad(E^*)$ on the basis $\bigcup_{j=1}^r \mathcal{B}_j$ is given by $diag\,\{R_1, ..., R_r\}$ and so its characteristic polynomial, which is a product of *different* irreducible factors, is

$$p(x) = \prod_{j=1}^{r} (x^2 + b_j^2).$$

Since $\mathfrak{V}$ is $ad(E^*)$-invariant so is $\mathfrak{V}^\perp$ (orthogonal complement with respect to the Killing form) and then, the characteristic polynomial $p^*(x)$ of φ_{E^*} divides $p(x)$. By reordering the factors, if necessary, we may write for some $1 \le s \le r$,

$$p^*(x) = \prod_{j=1}^{s} (x^2 + b_j^2).$$

Since φ_{E^*} is a skew-symmetric transformation of $\mathfrak{V}$ (with respect to de Killing form), there exists some basis of $\mathfrak{V}$, say $\bigcup_{j=1}^s \mathcal{Q}_j$, where $\mathcal{Q}_j = \{v_{1,j}, v_{2,j}\}$ generates a two dimensional φ_{E^*}-invariant subspace $\mathfrak{V}_j$ such that the matrix of the restriction of φ_{E^*} to $\mathfrak{V}_j$ in the basis $\mathcal{Q}_j$ is R_j.

Let us consider in particular the subspace $\mathfrak{V}_1$. For each $A \in \mathfrak{h}_u$, we have

$$\begin{aligned} \varphi_{E^*}\varphi_A v_{1,1} &= \varphi_A \varphi_{E^*} v_{1,1} = -b_1 \varphi_A v_{2,1}, \\ \varphi_{E^*}\varphi_A v_{2,1} &= \varphi_A \varphi_{E^*} v_{2,1} = b_1 \varphi_A v_{1,1} \end{aligned}$$

and therefore the subspace

$$\mathfrak{W}_A = Span_R\,\{\varphi_A v_{1,1}, \varphi_A v_{2,1}\} = \varphi_A(\mathfrak{V}_1)$$

is φ_{E^*}-invariant. We have then two possibilities either $\dim \mathfrak{W}_A < 2$ or $\dim \mathfrak{W}_A = 2$.

In the first case, it is easy to see that there exist real numbers a_1 and a_2 such that $v = a_1 v_{1,1} + a_2 v_{2,1} \neq 0$ and $\varphi_A v = 0$. From (i), $\varphi_A = 0$ and in this case $\mathfrak{V}_1$ is φ_A-invariant.

On the other hand, when $\dim \mathfrak{W}_A = 2$, we consider the φ_{E^*}-invariant subspace $\mathfrak{U}_A = \mathfrak{V}_1 \cap \mathfrak{W}_A$ whose dimension is $0 \le u \le 2$.

If $u = 0$ there is a subspace $\mathfrak{Z}$ and a basis in $\mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{V}_1 \oplus \mathfrak{W}_A \oplus \mathfrak{Z}$ and φ_{E^*} is represented in this basis by the matrix

$$\begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_1 & 0 \\ 0 & 0 & * \end{bmatrix}.$$

Therefore $(x^2 + b_1^2)^2$ divides $p(x)$ which is impossible.

If $u = 1$ there exists $v \neq 0$ in $\mathfrak{U}_A$ such that $\varphi_{E^*}(v) = cv$ which is also impossible since φ_{E^*} has no eigenvalues in $\mathfrak{V}_1$. Then $u = 2$ yielding $\mathfrak{V}_1 = \mathfrak{W}_A$ and so $\mathfrak{V}_1$ is also in this case φ_A-invariant.

We conclude that $\mathfrak{V}_1$ is $ad(\mathfrak{h}_u)$-invariant and therefore $\mathfrak{V}_1 = \mathfrak{V}$. □

Lemma 4. *If $\mathfrak{V}$ is a real bidimensional $ad(\mathfrak{h}_u)$-invariant subspace of $\mathfrak{m}$ then there exists $\gamma_o \in \Delta^+$ such that $\mathfrak{V} = \mathfrak{m}_{\gamma_o}$.*

Proof. Let us take $X = \sum_{\gamma \in \Delta_X^+} X_\gamma \neq 0$ in $\mathfrak{V}$ where $X_\gamma = a_\gamma U_\gamma + b_\gamma U_{-\gamma} \neq 0$ in $\mathfrak{m}_\gamma$ for each $\gamma \in \Delta_X^+$. Let γ_o be a fixed element in Δ_X^+ and we consider $E \in \mathfrak{h}_u$ such that $M = Ad(G)E$.

Since $\mathfrak{V}$ is an $ad(\mathfrak{h}_u)$-invariant subspace of $\mathfrak{m}$ and $Ker(ad(E)) = T_E(M)^\perp = \mathfrak{h}_u$, it follows that $Y = [E, X] \in \mathfrak{V}$ is different from zero. Due to the skew-symmetry of the transformation $ad(E)$, the vectors Y and X are orthogonal with respect to the Killing form and therefore

$$\mathfrak{V} = Span\{X, Y\}.$$

By (3), for each $A \in \mathfrak{h}_u$ we have the following vectors of $\mathfrak{V}$, $Y_A = [A, X] = \sum_{\gamma \in \Delta_X^+} \gamma(iA)(b_\gamma U_\gamma - a_\gamma U_{-\gamma})$ and

$$Z_A = [A, Y_A] = \sum_{\gamma \in \Delta_X^+} (\gamma(iA))^2(-a_\gamma U_\gamma - b_\gamma U_{-\gamma}).$$

It is easy to see that Z_A and Y are orthogonal, then $Z_A = c_A X$ with $c_A \in R$ and therefore $(\gamma(iA))^2 = -c_A$ for every $\gamma \in \Delta_X^+$. Then

$$(\gamma(iA))^2 = (\gamma_o(iA))^2, \quad \forall\, \gamma \in \Delta_X^+.$$

The last identity holds for every $A \in \mathfrak{h}_u$, in particular for the element E^* given by Lemma 2 and consequently $\Delta_X^+ = \{\gamma_o\}$ and $X, Y \in \mathfrak{m}_{\gamma_o}$. Then $\mathfrak{V} = \mathfrak{m}_{\gamma_o}$. □

Theorem 5. *If $\mathfrak{V}$ is an $Ad(T)$-invariant real subspace of $\mathfrak{m}$ then there exists $\Delta^* \subset \Delta^+$ such that $\mathfrak{V} = \sum_{\gamma \in \Delta^*} \mathfrak{m}_\gamma$.*

Proof. Let us consider $\mathfrak{V} = \oplus \mathfrak{V}_j$ where each $\mathfrak{V}_j$ is supporting an irreducible representation of T via the adjoint one. So, each $\mathfrak{V}_j$ is supporting an irreducible representation of the abelian Lie algebra $\mathfrak{h}_u$ via the adjoint one. By Lemma 3 each $\mathfrak{V}_j$ has real-dimension 2 and Lemma 4 guaranties the existence of $\gamma_j \in \Delta^+$ such that $\mathfrak{V}_j = \mathfrak{m}_{\gamma_j}$. □

This theorem gives a complete characterization of those subspaces of $\mathfrak{m} = T_E(G_u/T)$ which are $Ad(T)$-invariant. Each one of these subspaces admits $Ad(T)$-invariant complex structures. Since here $\mathfrak{V} = \sum_{\gamma \in \Delta^*} \mathfrak{m}_\gamma$ we may take on each $\mathfrak{m}_\gamma$ ($\gamma \in \Delta^*$) the *canonical complex structure* given by $J(U_\gamma) = U_{-\gamma}$ and $J(U_{-\gamma}) = -U_\gamma$ which is $Ad(T)$-invariant by the identities in (3).

Our next objective is to relate the subspaces of $\mathfrak{m}$ which may give rise to G_u-invariant almost CR-structures on $M = G_u/T$ with those subspaces that are tangent spaces of symmetric spaces of type G_u/K.

Associated to our simple group G_u we have its family of symmetric spaces of type I [8, p. 518] and among them, those which are *inner,* i.e. the spaces in which the symmetry at each point belongs to the group G_u. These are, among all symmetric spaces, the only ones that are related (in a way to be described bellow) with our purpose. It is well-known that each one of the simple groups gives rise to at least one of these symmetric spaces. They are those of the form G_u/K, where K is a subgroup of maximal rank in G_u. The ones which are not inner in the list in [8, p. 518] are AI, AII, BDI ($p+q = 2n$, p *odd*, $1 \leq p \leq n$), EI and EIV.

By conjugating K, if necessary, we may assume that K contains T.

It is known, (see [5, p.226]) that if G_u/K is an inner symmetric space, then $\mathfrak{V} = T_{[K]}G_u/K$ is the form $\mathfrak{V} = \sum_{\gamma\in\Delta^*} \mathfrak{m}_\gamma$ for some $\Delta^* \subset \Delta^+$. Moreover, there exists a root $\gamma^* \in \pi$ such that

$$\mathfrak{V} \ = \textstyle\sum_{\gamma\in\Delta^*} \mathfrak{m}_\gamma, \quad \textit{where} \quad \Delta^* = \{\gamma \in \Delta^+ : h_{\gamma^*}(\gamma) = 1\}\,. \tag{4}$$

In fact, when G_u/K is an Hermitian symmetric space γ^* is one of the simple roots such that $h_{\gamma^*}(\mu) = 1$, where μ is the highest root of $\mathfrak{g}$. For each one of the non-Hermitian inner symmetric spaces G_u/K of type I , the root γ^* is indicated in the following table according to the notation in [8, p. 477 and 518].

$BDI\,(*)$	p,q even	$\alpha_{\frac{p}{2}}$	;	EII	α_2	;	EIX	α_8
	p odd	$\alpha_{\frac{q}{2}}$	;	EV	α_2	;	FI	α_1
	q odd	$\alpha_{\frac{p}{2}}$	;	EVI	α_1	;	FII	α_4
CII		α_p	;	$EVIII$	α_1	;	G	α_2

$(*)$ The groups $SO\,(p+q)$ and $SO\,(p) \times SO\,(q)$ have the same rank only when pq is even.

Therefore, all tangent spaces at $[K]$ of an inner symmetric space of type G_u/K are $Ad(T)$-invariant.

On the other hand, the tangent spaces at $[K]$ of the *outer* symmetric spaces cannot give rise to G_u-invariant CR-structures on $M = G_u/T$ because they are not $Ad(T)$-invariant as we show now.

Maintaining the notation in Subsection 2.2 we add the following. Let θ be the involutive automorphism of $\mathfrak{g}_u$ giving rise to the symmetric space G_u/K. This produces a decomposition $\mathfrak{g}_u = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ is the Lie algebra of K, $\mathfrak{k} = \{X \in \mathfrak{g}_u : \theta X = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g}_u : \theta X = -X\} = T_{[K]}G_u/K$.

AI. $\mathfrak{g}_u = \mathfrak{su}(n)$, $\theta(X) = \overline{X}$.

In this case $\mathfrak{k} = \mathfrak{so}(n)$ and $\mathfrak{p}$ is the space of symmetric $n \times n$ matrices of purely imaginary entries and zero trace. So

$$T_{[K]}(G_u/K) = \sum_{\gamma\in\Delta^+} RU_{-\gamma}$$

and then this tangent space is not $Ad(T)$-invariant.

AII. $\mathfrak{g}_u = \mathfrak{su}(2n)$, $\theta(X) = J_n\overline{X}J_n^{-1}$ where $J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$.

Then $\mathfrak{k} = \mathfrak{sp}(n)$ and $\mathfrak{p} = \left\{ \begin{bmatrix} A & B \\ \overline{B} & -\overline{A} \end{bmatrix} : A \in \mathfrak{su}(n),\ B \in \mathfrak{so}(n, C) \right\}$.

From this follows that $U_{\alpha_1} - U_{\alpha_{n+1}} \in \mathfrak{p}$ but $U_{\alpha_1} \notin \mathfrak{p}$ and $U_{\alpha_{n+1}} \notin \mathfrak{p}$.

BDI. In these spaces, $T_{[K]}G_u/K$ is not $Ad(T)$-invariant because its dimension is the odd number pq.

In the next two spaces, the simple roots used, correspond to the Dynkin diagram for the algebra $\mathfrak{e}_6$ indicated in [8, p. 477].

EIV. The space $(\mathfrak{e}_{6(-78)}, \mathfrak{f}_{4(-52)})$ is obtained from an automorphism θ of $\mathfrak{e}_6$ induced by the automorphism of the Dynkin diagram which interchanges the roots α_1 and α_6, α_3 and α_5 and fixes the roots α_2 and α_4.

In terms of the basis of the algebra $\mathfrak{e}_6$ defined in [8, p. 482, Prop. 4.1] the following vectors generate the subalgebra $\mathfrak{f}_4$ in $\mathfrak{e}_6$ such that $\mathfrak{f}_{4(-52)} = \mathfrak{k}$,

$$\begin{array}{llll} H_{\beta_1} = H_{\alpha_1} + H_{\alpha_6} & H_{\beta_2} = H_{\alpha_3} + H_{\alpha_5} & H_{\beta_3} = H_{\alpha_4} & H_{\beta_4} = H_{\alpha_2} \\ X_{\beta_1} = X_{\alpha_1} + X_{\alpha_6} & X_{\beta_2} = X_{\alpha_3} + X_{\alpha_5} & X_{\beta_3} = X_{\alpha_4} & X_{\beta_4} = X_{\alpha_2} \\ X_{-\beta_1} = X_{-\alpha_1} + X_{-\alpha_6} & X_{-\beta_2} = X_{-\alpha_3} + X_{-\alpha_5} & X_{-\beta_3} = X_{-\alpha_4} & X_{-\beta_4} = X_{-\alpha_2} \end{array}$$

denoting by $\{\beta_j\}$ a system of simple roots of $\mathfrak{f}_4$ such that $2\beta_1 + 3\beta_2 + 4\beta_3 + 2\beta_4$ is the highest one. (More details of this construction can be found in [8, p. 507, Ex.2, case 4]).

Since iH_{α_1} is not an element in $\mathfrak{k} = \mathfrak{f}_{4(-52)} \subset \mathfrak{e}_{6(-78)}$ and $[U_{\alpha_1}, U_{-\alpha_1}] = iH_{\alpha_1}$, then the subspace $\mathfrak{m}_{\alpha_1}$ is not contained in $\mathfrak{k}$. Similarly $\mathfrak{m}_{\alpha_6} \not\subseteq \mathfrak{k}$. On the other hand, one can see that $U_{\alpha_1} + U_{\alpha_6} \in \mathfrak{k}$. Then there is not a subset of positive roots of $\mathfrak{e}_6$, Δ^*, such that $\mathfrak{k} = \sum_{\gamma\in\Delta^*} \mathfrak{m}_\gamma$. Therefore $\mathfrak{p}$ is not $Ad(T)$-invariant either.

EI. This space is $(\mathfrak{e}_{6(-78)}, \mathfrak{sp}(4))$.

Let us call θ_o the automorphism that defines the symmetric space EIV. It is known that (compare [13, p. 287]):

(i) There exists an element H in a Cartan subalgebra of $\mathfrak{f}_{4(-52)} \subset \mathfrak{e}_{6(-78)}$ such that

$$\theta = \theta_o Ad(\exp H)$$

is the automorphism that defines the symmetric space EI.

(ii) In the compact symmetric space FI which is $(\mathfrak{f}_{4(-52)}, \mathfrak{sp}(3) \oplus \mathfrak{su}(2))$ the subalgebra $\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$ is the fixed point set of $\nu = Ad(\exp H)$ in $\mathfrak{f}_{4(-52)}$. Then $\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$ is the fixed point set of θ in $\mathfrak{f}_{4(-52)}$.

It is easy to see that, in terms of the simple roots $\{\beta_j\}$ mentioned above, the roots of $\mathfrak{f}_4$ that are roots of the subalgebra $\mathfrak{c}_3 \oplus \mathfrak{a}_1$ are those that are written only in terms of β_2, β_3 and β_4 and the highest one. Therefore $U_{\beta_2} = U_{\alpha_3} + U_{\alpha_5}$ and $U_{-\beta_2} = U_{-\alpha_3} + U_{-\alpha_5}$ belong to $\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$.

Let us assume now that $\mathfrak{k} = \{X \in \mathfrak{g}_u : \theta X = X\} = \sum_{\gamma\in\Delta^*} \mathfrak{m}_\gamma$ for some $\Delta^* \subset \Delta^+(\mathfrak{e}_6)$. By definition of ν and (3), $\nu(\mathfrak{m}_\gamma) = \mathfrak{m}_\gamma$ for all $\gamma \in \Delta^*$ and then $\mathfrak{m}_\gamma = \theta(\mathfrak{m}_\gamma) = \theta_o(\mathfrak{m}_\gamma)$ for all $\gamma \in \Delta^*$. Therefore

$$\mathfrak{m}_\gamma \text{ is } \theta_o\text{-invariant for all } \gamma \in \Delta^*. \tag{5}$$

Since the space $Span_R\{U_{\beta_2}, U_{-\beta_2}\} \subset \mathfrak{k}$, we have $U_{\pm\alpha_3}, U_{\pm\alpha_5} \in \mathfrak{k}$. Consequently the roots α_3 and α_5 are in Δ^*.

By the case of the space EIV we see that $\theta_o(U_{\pm\alpha_3}) = U_{\pm\alpha_5}$ and from (5) we obtain a contradiction that proves that $\mathfrak{p}$ is not invariant.

The previous development could be summarized in the following result.

Theorem 6. *The tangent space at $[K]$ of the symmetric space G_u/K is the holomorphic tangent space at the basic point E of some G_u-invariant almost CR-structures on G_u/T if and only if G_u/K is an inner symmetric space.* □

4. Extrinsic CR-structures

In this section $M = G_u/T = Ad(G_u)E$ will be always considered as a submanifold of $\mathfrak{g}_u$ because we want to study extrinsic structures.

Theorem 7. *Let G_u/K be an inner symmetric space such that $K \supset T$ and let us consider the space $\mathfrak{H} = T_{[K]}(G_u/K)$ as a subspace of $\mathfrak{m}$ (see (2)). Then M is a G_u-invariant minimal almost Hermitian extrinsic symmetric CR-manifold whose holomorphic tangent space at the basic point E is $\mathfrak{H}$ and canonical complex structure J_E in $\mathfrak{H}$ given by $ad(A_o)$ with an adequate element A_o in $\mathfrak{h}_u$.*

Proof. The correspondence $gT \longmapsto Ad(g)E$ defines an isometric embedding of G_u/T onto $M \subset \mathfrak{g}_u$ and so, by Theorem 6, M has a G_u-invariant almost CR-structure whose holomorphic tangent space at the basic point E is $\mathfrak{H}$.

To the end of describing the endomorphism that defines the canonical complex structure J_E, from (4) we notice that there exists a root $\gamma^* \in \pi$ such that $\mathfrak{H} = \sum_{\gamma \in \Delta^*} \mathfrak{m}_\gamma$ where $\Delta^* = \{\gamma \in \Delta^+ : h_{\gamma^*}(\gamma) = 1\}$. Using the notation in Lemma 2 and calling α_{j_o} to γ^* we define an element A_o in $\mathfrak{h}_u$ by

$$iA_o = -n_{j_o} v_{j_o}$$

which satisfies $\gamma(iA_o) = -1 \ \forall\ \gamma \in \Delta^*$. From this and (3) we conclude that $[A_o, U_\gamma] = U_{-\gamma}$ and $[A_o, U_{-\gamma}] = -U_\gamma$. Therefore

$$J_E = ad(A_o)|_{\mathfrak{H}}\,.$$

Naturally, at the point $x = Ad(g)E$ we have $\mathfrak{H}_x = Ad(g)\mathfrak{H}$ and $J_x = ad(Ad(g)A_o)$.

This CR-structure on M is easily seen to be Hermitian.

Since G_u/K is an inner symmetric space, the symmetry at the point $[K]$ is given by an element $s \in K$ [13, p. 255, Th. 8.6.7]. Hence s belongs to the center of K_e, the identity component of K (because K_e is the identity component of the fixed point set in G_u of the automorphism $g \longmapsto sgs$) and so s is contained also in our chosen torus $T \subset K$. Hence

$$\sigma_E = Ad(s)$$

defines a function from M to itself which is an involutive isometric CR-diffeomorphism fixing the point E.

Since $\mathfrak{g}_u = \mathfrak{k} \oplus \mathfrak{H}$, $\mathfrak{k} = Lie(K) = F(\sigma_E, \mathfrak{g}_u)$ and $\mathfrak{H} = F(-\sigma_E, \mathfrak{g}_u)$ we see that this situation is a particular case of the construction given in [10, p. 164] and σ_E is the *symmetry* of M at E if and only if $\mathfrak{H}_E^{-1} \subset \mathfrak{H}$. Let $\mathfrak{b}$ be the subalgebra of $\mathfrak{g}_u$ generated by $\mathfrak{H}$. Then by [10, p. 165, Prop. 6.2], $\mathfrak{b} = \mathfrak{H} \oplus [\mathfrak{H}, \mathfrak{H}]$ (notice that in our situation the subalgebras $\mathfrak{a}$ and $\mathfrak{b}$ in [10, p. 165, Prop. 6.2] coincide) and also M is a minimal almost Hermitian symmetric CR-manifold and the inclusion $\mathfrak{H}_E^{-1} \subset \mathfrak{H}$ holds if and only if $\mathfrak{g}_u = \mathfrak{h}_u + \mathfrak{b}$. So in order to prove this equality it is enough to show that $\mathfrak{A} = \sum_{\gamma \in (\Delta^+ - \Delta^*)} \mathfrak{m}_\gamma$, the orthogonal complement of $\mathfrak{h}_u$ in $\mathfrak{k}$, is contained in $[\mathfrak{H}, \mathfrak{H}]$.

Let ε and ρ be elements of Δ^+ such that $\varepsilon - \rho \in \Delta$. We denote by

$$sg_{\varepsilon-\rho} = \begin{cases} \ \ 1 & if\ \varepsilon - \rho \in \Delta^+ \\ -1 & if\ \rho - \varepsilon \in \Delta^+ \end{cases}$$

and

$$|\varepsilon-\rho| = \begin{cases} \varepsilon-\rho & if\ \varepsilon-\rho\in\Delta^+ \\ \rho-\varepsilon & if\ \rho-\varepsilon\in\Delta^+. \end{cases}$$

Let $\varepsilon,\rho\in\Delta^+,\varepsilon\neq\rho$. Then

$$\begin{array}{lcl} [U_\varepsilon,U_\rho] & = & \frac{1}{\sqrt{2}}\left\{N_{\varepsilon,\rho}U_{\varepsilon+\rho}+sg_{\rho-\varepsilon}N_{\varepsilon,-\rho}U_{|\varepsilon-\rho|}\right\} \\ [U_\varepsilon,U_{-\rho}] & = & \frac{1}{\sqrt{2}}\left\{N_{\varepsilon,\rho}U_{-(\varepsilon+\rho)}+N_{\varepsilon,-\rho}U_{-|\varepsilon-\rho|}\right\} \\ [U_{-\varepsilon},U_{-\rho}] & = & \frac{1}{\sqrt{2}}\left\{N_{-\varepsilon,-\rho}U_{\varepsilon+\rho}+sg_{\varepsilon-\rho}N_{-\varepsilon,\rho}U_{|\varepsilon-\rho|}\right\}. \end{array} \tag{6}$$

(Here we understand that if $\varepsilon\pm\rho$ are not roots then the terms $N_{\varepsilon,-\rho}U_{|\varepsilon-\rho|}$, $N_{\varepsilon,\rho}U_{\varepsilon+\rho}$, etc., vanish).

From [5, p. 227, 228, Remark 4.2, 4.3] we know that if $\gamma\in\Delta^+-\Delta^*$ then there exist ε and ρ in Δ^* such that $\gamma=\varepsilon+\rho$ or $\gamma=\varepsilon-\rho$.

(i) If $\gamma=\varepsilon+\rho$ and $\varepsilon-\rho\notin\Delta$, by (6), we obtain that

$$\begin{array}{lcl} U_\gamma & = & \frac{\sqrt{2}}{N_{\varepsilon,\rho}}[U_\varepsilon,U_\rho]\in[\mathfrak{H},\mathfrak{H}] \\ U_{-\gamma} & = & \frac{\sqrt{2}}{N_{\varepsilon,\rho}}[U_\varepsilon,U_{-\rho}]\in[\mathfrak{H},\mathfrak{H}]. \end{array}$$

When $\gamma=\varepsilon-\rho$ and $\varepsilon+\rho\notin\Delta$ we reach a similar conclusion.

(ii) If $\varepsilon+\rho\in\Delta$ and $\varepsilon-\rho\in\Delta$, by using again (6), we can see that there exist nonzero real constants c_1 and c_2 such that

$$\begin{array}{lcl} [U_\varepsilon,U_\rho] & = & c_1U_{\varepsilon+\rho}+c_2U_{|\varepsilon-\rho|} \\ [U_{-\varepsilon},U_{-\rho}] & = & -c_1U_{\varepsilon+\rho}+c_2U_{|\varepsilon-\rho|} \end{array}$$

and so, if $\gamma=\varepsilon+\rho$ (respectively $\gamma=\varepsilon-\rho$) we have that $U_\gamma=\frac{1}{2c_1}\{[U_\varepsilon,U_\rho]-[U_{-\varepsilon},U_{-\rho}]\}\in[\mathfrak{H},\mathfrak{H}]$ (respectively $U_\gamma=\frac{1}{2c_2}\{[U_\varepsilon,U_\rho]+[U_{-\varepsilon},U_{-\rho}]\}\in[\mathfrak{H},\mathfrak{H}]$).

In analogous way, we can prove that $U_{-\gamma}\in[\mathfrak{H},\mathfrak{H}]$ and therefore we conclude that $\mathfrak{A}\subset[\mathfrak{H},\mathfrak{H}]$. Then this structure is minimal and $M=Ad(G_u)E$ is a G_u-invariant minimal almost Hermitian symmetric CR-manifold whose holomorphic tangent space and symmetry at E are $\mathfrak{H}$ and σ_E respectively. Since the structure is G_u-invariant, the symmetries, the holomorphic tangent spaces and the almost complex operators at any other point are given by translation.

By definition the symmetry at any point is an *extrinsic symmetry* and then the proof is complete. □

Remark 1. The extrinsic symmetries of the previous theorem are automorphisms of the algebra $\mathfrak{g}_u$.

Let us suppose now that $M=Ad(G_u)E\subset\mathfrak{g}_u$ is a G_u-invariant minimal almost Hermitian extrinsic symmetric CR-manifold such that the extrinsic symmetries σ_x are automorphisms of the algebra $\mathfrak{g}_u$. Set $\mathfrak{k}=F(\sigma_E,\mathfrak{g}_u)$. If $\mathfrak{H}$ denotes, as above, the holomorphic tangent space at the basic point E then, since M is minimal, $\mathfrak{H}=F(-\sigma_E,\mathfrak{g}_u)$. Therefore $\mathfrak{g}_u=\mathfrak{k}\oplus\mathfrak{H}$ is a reductive decomposition ($[\mathfrak{k},\mathfrak{H}]\subset\mathfrak{H}$) that satisfies $[\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k}$ and $[\mathfrak{H},\mathfrak{H}]\subset\mathfrak{k}$. If K is the analytic subgroup of G_u whose Lie algebra is $\mathfrak{k}$ then K has maximal rank. Since G_u is simply connected, from [8, p. 209, Prop. 3.4] and [13, p. 255, Th. 8.6.7], G_u/K is an inner symmetric space with $\mathfrak{H} = T_{[K]}(G_u/K)$ where the G_u-invariant metric is induced by translation of the Killing form of $\mathfrak{g}_u$ restricted to $\mathfrak{H}$.

Remark 2. The previous comment indicates that if in Theorem 7 we also ask that the extrinsic symmetries are automorphisms of the algebra $\mathfrak{g}_u$, then the sufficient condition given in that theorem, is also necessary.

This fact is not very surprising but it suggests an interesting problem which is to decide whether or not all extrinsic symmetries come from automorphisms of the Lie algebra $\mathfrak{g}_u$. This seems to be a difficult problem for which we do not have yet a complete solution, however we have obtained Theorem 8 that we feel is very interesting since, besides giving a partial answer to this problem (Corollary 9), contains an inequality giving a bound of the possible CR-dimensions of M.

To reach Theorem 8 we introduce new notation and some results needed in the proof.

The following table contains the list of the irreducible inner symmetric spaces *of maximal dimension* for each simple Lie group G_u.

$\mathfrak{g}$	name	G_u/K		$\dim G_u/K$
$\mathfrak{a}_l$	AIII	$SU(l+1)/S(U(k+1)\times U(k))$	$l=2k$	$\frac{1}{2}l(l+2)$
		$SU(l+1)/S\left(U\left(k+1\right)\times U\left(k+1\right)\right)$	$l=2k+1$	$\frac{1}{2}\left(l+1\right)^2$
$\mathfrak{b}_l$	BDI	$SO(2l+1)/SO(l+1)\times SO(l)$		$l(l+1)$
$\mathfrak{c}_l$	CI	$Sp(l)/U(l)$		$l(l+1)$
$\mathfrak{d}_l$	BDI	$SO(2l)/SO(l)\times SO(l)$	l *even*	l^2
		$SO(2l)/SO(l+1)\times SO(l-1)$	l *odd*	l^2-1
$\mathfrak{e}_6$	EII	$E_6/SU(6)Sp(1)$		40
$\mathfrak{e}_7$	EV	$E_7/(SU(8)/Z_2)$		70
$\mathfrak{e}_8$	EVIII	$E_8/(Spin(16)/Z_{2)}$		128
$\mathfrak{f}_4$	FI	$F_4/Sp(3)Sp(1)$		28
$\mathfrak{g}_2$	G	$G_2/SO(4)$		8

We shall denote by $d\left(G_u\right)$ the dimension $\dim\left(G_u/K\right)$ indicated in this table.

Let $j : M \to R^N$ be an isometric embedding and p a point in M. Let us consider, in the tangent space $T_p\left(M\right)$, a unit vector X and define an affine subspace of R^N by

$$S\left(p,X\right) = p + Span\left\{X, T_p\left(M\right)^{\perp}\right\}.$$

If U is a small enough neighborhood of p in M, then the intersection $U \cap S\left(p,X\right)$ can be considered as the image of a C^∞ regular curve $\gamma\left(s\right)$, parametrized by arc-length, such that $\gamma\left(0\right) = p$, $\gamma'\left(0\right) = X$. This curve is called a *normal section of M at p in the direction of X* and it is *pointwise planar* at p if its first three derivatives $\gamma'\left(0\right), \gamma''\left(0\right)$ and $\gamma'''\left(0\right)$ are linearly dependent.

In this paper j is a natural embedding of the flag manifold $M = G_u/T$. Let α be the second fundamental form of this embedding and D the torsion tensor of the *canonical connection* on M ([4]).

It is known that the normal section γ with $\gamma(0) = p$ and $\gamma'(0) = X$ is pointwise planar at p if and only if the unit tangent vector X at p satisfies the equation

$$\alpha (D (X, X), X) = 0.$$

It is also known that the set of those unitary vectors X defining pointwise planar normal sections at p gives rise to a real projective algebraic variety $X[M]$ of $RP^{n-1}(\dim_R M = n)$ called the *variety of directions of pointwise planar normal sections of* M. For details of these facts see [4].

Theorem 8. *Let $M = Ad(G_u)E \subset \mathfrak{g}_u$ be a G_u-invariant minimal almost Hermitian extrinsic symmetric CR-manifold whose holomorphic tangent space at the basic point E is $\mathfrak{H}$. Then*

(i) $\dim_R \mathfrak{H} \leq d(G_u)$.

(ii) *If $\dim_R \mathfrak{H} = d(G_u)$ then there exists an inner symmetric space G_u/K such that $\mathfrak{H} = T_{[K]}(G_u/K)$.*

Proof. Let φ_E be the extrinsic symmetry of M at E. Since it is an isometry of the ambient space it must satisfy, for every $X \in T_E(M)$,

$$\varphi_E \alpha_E (D (X, X), X) = \alpha_E (D (\varphi_E X, \varphi_E X), \varphi_E X).$$

If we take now $X \in \mathfrak{H}$ then $\varphi_E X = -X$ and so

$$\alpha_E (D (\varphi_E X, \varphi_E X), \varphi_E X) = \alpha_E (D (-X, -X), -X) = -\alpha_E (D (X, X), X)$$

which yields

$$\varphi_E \alpha_E (D (X, X), X) = -\alpha_E (D (X, X), X).$$

But since $\alpha_E (D (X, X), X)$ is normal and φ_E on $T_E(M)^{\perp}$ is the identity we conclude that $\alpha_E (D (X, X), X) = 0$ and therefore $RP(\mathfrak{H}) \subset X[M]$.

From Theorems 1 and 5 there exists $\Delta^* \subset \Delta^+$ such that $\mathfrak{H} = \sum_{\gamma \in \Delta^*} \mathfrak{m}_\gamma$. By applying [6, p. 416,Th. 1.1] we obtain $|\Delta^*| \leq \frac{1}{2} d(G_u)$ and so part (i) follows.

Now by [6, p. 416,Th. 1.2], the subspace $\mathfrak{H}$ is tangent to the inner symmetric space G_u/K at a fixed point of the action of the torus T where G_u/K is given in the previous table for each $\mathfrak{g}_u$ and this proves (ii). □

From this and Theorem 7 we have the following characterization.

Corollary 9. *$M = Ad(G_u)E \subset \mathfrak{g}_u$ has a G_u-invariant minimal almost Hermitian extrinsic symmetric CR-structure satisfying $\dim_R \mathfrak{H} = d(G_u)$, where $\mathfrak{H}$ is the holomorphic tangent space at the basic point E, if and only if there exists an inner symmetric space G_u/K such that $\mathfrak{H} = T_{[K]}(G_u/K)$.* □

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