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Lines of Curvature, Ridges and Conformal Invariants of Hypersurfaces

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Abstract. We define some conformally invariant differential 1-forms along the curvature lines of a hypersurface M and we observe that the ridges of M can be viewed as their zeros. We characterize the highest order ridges, which are isolated points generically, as zeros of these conformally invariant differential 1-forms along special curves of ridges. We also prove that the highest order ridges are vertices of the curvature lines when they are considered as curves in n-space.

Introduction

Conformal maps of $\mathbb{R}^n$ are defined as those preserving the angles. For $n \geq 3$ they are characterized by the fact that they transform k-spheres of $\mathbb{R}^n$ into k-spheres (here the k-planes are considered as a special case of k-sphere with infinite radius). Several conformal invariants for submanifolds in $\mathbb{R}^n$ have been defined by different authors ([5], [9], [10]). We are interested here in the study of hypersurfaces from the viewpoint of their contacts with hyperspheres and we follow an alternative approach, based on the fact that the conformal maps preserve these contacts. A straightforward consequence of this is that they preserve the contact directions of hypersurfaces with their focal hyperspheres, classically known as principal directions, and therefore the curvature lines. We use this fact in order to obtain some differential 1-forms defined along the curvature lines (considered as curves in n-space) which are preserved by conformal maps (Theorems 1, 2 and 3).

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For surfaces in 3-space and locally conformally flat and no quasi-umbilical 3-manifolds in 4-space, we show how extend these 1-forms over the whole surface so that their exterior products define conformally invariant volume forms (Theorem 4 and Corollary 1). We obtain in this way the expressions for the conformal principal curvatures of surfaces introduced by Tresse in [21] and include a generalization of these results to locally conformally flat and no quasi-umbilical 3-manifolds in 4-space (Corollary 2).

We also apply this procedure to the study of ridges. These are conformally invariant subsets of the hypersurfaces arising from the analysis of their contacts with the family of hyperspheres in the ambient space. These subsets happen to be relevant from the Image Analysis viewpoint ([11]). Their introduction from the viewpoint of generic contacts with hyperspheres is due to I. R. Porteous ([18]). In fact, an exhaustive study of ridges in the case of surfaces in 3-space can be found in his book [19]. They can be viewed, roughly speaking, as sets made of points at which the hypersurface has a contact of higher order with some of its focal hyperspheres. An interesting fact is that the ridges can be characterized as the zeros of some of the previously mentioned conformally invariant 1-forms.

We can define ridges of different orders, according to the order of contact of the hypersurface with the corresponding focal hypersphere at the given point. The ridge points of order $\geq n$ of a generic hypersurface in $\mathbb{R}^n$ form (conformally invariant) immersed curves containing the ridges of order $n+1$ as isolated points. The last are characterized here as the zeros of certain conformally invariant 1-forms defined along these curves (Theorem 7).

On the other hand, the ridge points can be characterized through the contacts of focal hyperspheres with the curvature lines of the hypersurface. This fact can be deduced from the work of I. R. Porteous for surfaces in $\mathbb{R}^3$ [19]. Its proof for the general case of hypersurfaces in $\mathbb{R}^n$ requires cumbersome technical manipulations and has not been published anywhere. We have included here a proof (Theorems 5 and 6), which is based on the handling of the expressions of the focal centers in terms of certain coefficients related to the Frenet paraphernalia of the curvature lines of the hypersurface considered as a curve in $\mathbb{R}^n$. Such coefficients were introduced in [20] in order to study the conformal invariants of curves in $\mathbb{R}^n$ and provide an important simplification to the computations associated to the problem that concerns us here. A nice consequence of this is that the highest order ridges are vertices if the curvature lines are considered as curves in n-space (Corollary 3).

1. Distance squared functions, focal sets, ridges and curvature lines

Since the conformal maps of $\mathbb{R}^n$ are defined as those that transform k-spheres of $\mathbb{R}^n$ into k-spheres, we have that given a hypersurface $M \subset \mathbb{R}^n$, any conformal map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ preserves the contacts of M with the hyperspheres of $\mathbb{R}^n$. This means that if a hypersphere S has contact of a given type with M at a point m, then the hypersphere $\phi(S)$ has the same type of contact with $\phi(M)$ at the point $\phi(m)$. The contact of M with the set of hyperspheres of $\mathbb{R}^n$ can be described through the analysis of the singularities of the distance squared functions on M. If M is viewed as the image of some embedding $g : \mathbb{R}^{n-1} \to \mathbb{R}^n$, then the family of distance squared functions on M is given by

$$\begin{array}{rccl} d: & \mathbb{R}^{n-1} \times \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & (x, a) & \longmapsto & d_a(x) = \|g(x) - a\|^2. \end{array}$$

A consequence of the work of J. Montaldi ([16]) is that the contact of M with a hypersphere of center $a \in \mathbb{R}^n$ and radius $r = \|g(x) - a\|$ at the point $g(x)$ is completely characterized by the $\mathcal{K}$-equivalence class of the germ of the function d_a at the point x. More precisely:

Defininition 1. *Let X_i and Y_i, $i = 1,\ 2$ be submanifolds of $\mathbb{R}^n$, with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. The contact of X_1 and Y_1 at a point y_1 is said to be of the same type of contact as X_2 and Y_2 at a point y_2 if there is a diffeomorphism-germ $H : (\mathbb{R}^n, y_1) \to (\mathbb{R}^n, y_2)$, such that $H(X_1) = X_2$ and $H(Y_1) = Y_2$. In this case we shall write $\mathcal{K}(X_1, Y_1; y_1) = \mathcal{K}(X_2, Y_2; y_2)$.*

J. Montaldi ([16]) proved that given immersion-germs $g_i : (X_i, x_i) \to (\mathbb{R}^n, y_i)$ and maps $f_i : (\mathbb{R}^n, y_i) \to (\mathbb{R}^p, 0)$ such that $Y_i = f_i^{-1}(0),\ i = 1, 2$, we have that

$$\mathcal{K}(X_1, Y_1; y_1) = \mathcal{K}(X_2, Y_2; y_2) \Leftrightarrow f_1 \circ g_1 \sim_{\mathcal{K}} f_2 \circ g_2,$$

where $\mathcal{K}$ is the Mather's contact group. (We refer to [14] for the definition and details on $\mathcal{K}$-equivalence). The map $\phi_i = f_i \circ g_i$ is called the *contact map* for X_i and Y_i, $i = 1, 2$.

Suppose now that $p = 1$, so Y_i is a hypersurface and ϕ_i is a function on $\mathbb{R}^n$ which has a degenerate singularity at the point x_i, $i = 1, 2$. This means that the Hessian, $\mathcal{H}(\phi_i)$, defines a degenerate quadratic form, i.e. there is some unit vector $u_i \in T_{x_i}X_i$, such that $\mathcal{H}(\phi_i)(u_i, v) = 0,\ \forall v \in T_{x_i}X_i,\ \ i = 1, 2$. We call such a vector, a *contact direction* for X_i and Y_i at $y_i = g_i(x_i)$. In fact, the contact of some curve through x_i in X_i with tangent direction u_i with the submanifold Y_i at the point x_i is of higher order (i.e. the corresponding contact map has a degenerate singularity at x_i) than that of any other curve through x_i in X_i (whose corresponding contact map has a Morse singularity at x_i).

In the case that M is a hypersurface immersed by $g : \mathbb{R}^{n-1} \to \mathbb{R}^n$ in n-space and $S(a, r)$ is a hypersphere with center a and radius r, that is $S(a, r) = f_{a,r}^{-1}(0)$, where

$$\begin{array}{rccl} f_r: & \mathbb{R}^n \times \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & (x, a) & \longmapsto & f_{a,r}(x) = \|x - a\|^2 - r^2. \end{array}$$

The contact map for M and $S(a, r)$ is given by the function $f_{a,r}(g(x)) = \|g(x) - a\|^2 - r^2 = d_a(x) - r^2$. Clearly, $f_{a,r}(g(x))$ and $d_a(x)$ have the same singularities. So, as we pointed out above, we have that the contacts of the hypersurface M with all the hyperspheres of $\mathbb{R}^n$ can be described through the analysis of the singularities of the family of all the distance squared functions on M.

It follows from the work of Looijenga [12] that for a generic $M = g(\mathbb{R}^{n-1}) \subset \mathbb{R}^n$ (in the sense that it belongs to a dense subset of submanifolds embedded in $\mathbb{R}^n$ with the Whitney topology), the family d is a generic family of functions on $\mathbb{R}^{n-1}$. For a detailed description of the term "generic family of functions" we refer to [12] or [22]. This means, in particular, that these families are topologically stable, and for $n \leq 5$, smoothly stable too.

The generic singularities of d were initially studied by Porteous [18], who observed that its singular set,

$$\Sigma(d) = \{(g(x), a) \in M \times \mathbb{R}^n | \frac{\partial d_a}{\partial x} = 0\}$$

is precisely the normal bundle, NM, of M in $\mathbb{R}^n$.

Defininition 2. *The restriction of the projection* $\pi : M \times \mathbb{R}^n \to \mathbb{R}^n$ *to the singular set* $\Sigma(d) = NM \subset M \times \mathbb{R}^n : \pi|_{\Sigma(d)}$*, is the catastrophe map associated to the family d. In this particular case we have that it coincides with the normal exponential map of M,* $\exp_N$*. The bifurcation set*

$$\mathcal{B}(d) = \{a \in \mathbb{R}^n | \exists\, x \in R^{n-1} \text{ where } d_a \text{ has a degenerate singularity } \}$$

is made of all the centers of hyperspheres having contact of higher order at least 2 with M in the sense that the contact function-germ d_a *at x has codimension at least 1, i.e. it is not a Morse function. This subset is classically known as focal set of M and the hyperspheres tangent to M whose centers lie in* $\mathcal{B}(d)$ *are called focal hyperspheres of M.*

We remind that if M is a hypersurface in $\mathbb{R}^n$ (locally embedded through g) and $\Gamma : M \to S^{n-1}$ represents its normal Gauss map, then the eigenvectors of $D\Gamma(g(x))$ are the *principal directions* of curvature of M at the point $g(x)$ and the corresponding eigenvalues, $\{K_i(x)\}_{i=1}^{n-1}$, are the *principal curvatures*. A curve all of whose tangents are in principal directions is a *curvature line*. We shall say that a point $g(x) \in M$ is *umbilic* if at least two of the principal curvatures coincide at this point. It can be seen that the principal directions coincide with the contact directions of the hypersurfaces with its focal hyperspheres at each point (see [13]). Moreover, we have that these directions fill up at least a whole tangent plane at the umbilics of M, in other words, the umbilics are singularities of corank at least two of distance squared functions on M. We shall denote by $U(M)$ the subset of the umbilics of M. For a generic M, the subset $M - U(M)$ is an open and dense submanifold of M.

Provided $g(x) \in M - U(M)$, we can find exactly $n-1$ focal hyperspheres at $g(x)$, whose centres are given by $a_i(x) = g(x) + r_i(x)N(g(x))$, where $N(g(x))$ is the normal vector of the hypersurface in the point $g(x)$, and whose radii are $r_i(x) = 1/K_i(x)$. If some of the principal curvatures vanishes, i.e. $g(x)$ is a *parabolic point* of M, then the corresponding focal hypersphere becomes a tangent hyperplane. We shall denote by $P(M)$ the subset of the parabolics of M. For a generic M, the subset $P(M)$ is a $(n-2)$-submanifold immersed in M.

Consider the deformation associated to the family d,

$$\begin{array}{rcl} \Psi : M \times \mathbb{R}^n & \longrightarrow & \mathbb{R} \times \mathbb{R}^n \\ (g(x), a) & \longmapsto & (d_a(x), a), \end{array}$$

and its different singularities, labelled by their corresponding Boardman symbols, $\Sigma^{k_1,\dots,k_r}\Psi$. It is not difficult to check ([18]) that

$$\Sigma^{n-1,i_1,\dots,i_r}\Psi = \Sigma^{i_1,\dots,i_r} \exp_N .$$

For a generic embedding in the sense of Looijenga ([12]), Ψ is a Boardman map and hence the subspace $NM = \Sigma(d) = \Sigma^{n-1}\Psi$ of $M \times \mathbb{R}^n$ is stratified by the subsets $\Sigma^{n-1,i_1,\dots,i_r}\Psi$, $n-1 \geq i_1 \geq \dots \geq i_r$. Moreover, this induces in turn a stratification on the lifting of the focal set

$$L\mathcal{B}(d) = \{(g(x), a) \in NM : d_a \text{ has a degenerate singularity at } x\}.$$

We shall pay special attention to the strata of type $\Sigma^{n-1,1,\ldots,1}\Psi = \Sigma^{1,\ldots,1}\exp_N$. An interesting feature, being $\exp_N$ the catastrophe map of the family d, is that $(g(x), a) \in \Sigma^{1,\stackrel{k}{\ldots},1}\exp_N$ if and only if x is a singularity of type A_{k+1} of the function d_a (i.e., the germ of d_a at x is $\mathcal{A}$-equivalent to one of the normal forms $x_1^{k+2} \pm x_2^2 \pm \cdots \pm x_{n-1}^2$). Thus the part of $L\mathcal{B}(d)$ included in $N(\bar{M})$, where $\bar{M} = M - (U(M) \cup P(M))$, is given by the $(n-1)$-submanifold $\Sigma^1\exp_N$, which is an $(n-1)$-fold covering of $\bar{M}$. So, if we denote by $p : NM \to M$ the natural projection, we have that $p|_{\Sigma^1\exp_N} : \Sigma^1\exp_N \to \bar{M}$ is a local diffeomorphism. And hence each subset $p(\Sigma^{1,\stackrel{k}{\ldots},1}\exp_N)$ is a regular submanifold of dimension $(n-k)$ immersed with normal crossings in M.

On the other hand, the restrictions of the map $\exp_N$ to the submanifolds $\Sigma^{1,\stackrel{k}{\ldots},1,0}\exp_N$ are also local diffeomorphisms onto their images. Therefore, $\exp_N(\Sigma^{1,\stackrel{k}{\ldots},1,0}\exp_N)$ is an immersed regular $(n-k+1)$-submanifold of $\mathbb{R}^n$ (contained in the focal set of M). We remark that $\exp_N(\Sigma^{1,\stackrel{k}{\ldots},1}\exp_N)$ is not a regular submanifold, its singular set being $\exp_N(\Sigma^{1,\stackrel{k+1}{\ldots},1}\exp_N)$.

$$
\begin{array}{ccccc}
\Sigma^1\exp_N & \hookrightarrow & NM & \xrightarrow{\exp_N} & \mathbb{R}^n \\
 & & (g(x), a_i(x)) & \longmapsto & a_i(x) = g(x) + 1/K_i(x)N(g(x)) \\
 & & p\downarrow & \nearrow & \\
 & & \bar{M} & & \\
 & & g(x) & &
\end{array}
$$

Defininition 3. *The different connected components of*

$$\exp_N(\Sigma^{1,\stackrel{k}{\ldots},1,0}\exp_N), \quad k > 2$$

are called ribs of order k of M, whereas those of

$$p(\Sigma^{1,\stackrel{k}{\ldots},1,0}\exp_N), \quad k > 2$$

are the ridges of order k of M.

These subsets, as mentioned in the Introduction, have been introduced by I. Porteous in [18], who has explored them with great details in the case of surfaces in $\mathbb{R}^3$ (see [19] for instance). Nevertheless, their properties are not so well stablished in the higher dimensional cases. We study them in the next two sections, providing some characterizations in terms of the conformal geometry of the hypersurface, as well as in terms of the analysis of the Euclidean geometry of the curvature lines of the hypersurface.

Remark 1. We observe that:

a) In the parabolic points at least one of the focal centers lies in the infinity. Some parabolic points can be seen as ridge points. They are characterized by the fact of being singularities of type $\Sigma^{1,\stackrel{k}{\ldots},1}$, $k > 2$ of Γ the normal Gauss map of M ([1]) and belong to the clausure of the subset $\exp_N(\Sigma^{1,\stackrel{k}{\ldots},1}\exp_N)$. In fact, by considering

$$CM = \{(g(x), v) \in M \times T_xM : v \perp T_{g(x)}M\}$$

and $\bar{\Gamma} : CM \to S^{n-1}$ where $\bar{\Gamma}(g(x), v) = v = \Gamma(g(x))$ the set $\Sigma^{1,..^k..,1}\,\bar{\Gamma}$ can be seen as a part of $\Sigma^{1,..^k..,1} \exp_N$ throng the family

$$\begin{array}{llll} G: & M \times S^n & \longrightarrow & \mathbb{R} \\ & (g(x),(a,t)) & \longmapsto & t\|g(x)\|^2 - 2a \cdot g(x) - r \end{array}$$

which measures the contacts of M with all the hyperspheres and hyperplanes (considered as degenerate hyperspheres) of $\mathbb{R}^n$.

b) We shall denote $\mathcal{R}_k = p(\Sigma^{1,..^k..,1} \exp_N) \cup \bar{p}(\Sigma^{1,..^k..,1}\,\bar{\Gamma})$, $k \geq 2$, where $\bar{p} : M \times S^{n-1} \to M$ is the natural projection. These are submanifolds of codimension k in M, made of points $g(x) \in M - U(M)$ for which there exists some $(a,t) \in \mathbb{R}^{n+1}$ such that the germ $G_{(a,t)}$ has some singularity equivalent to some A_j, $j \geq k+1$. We notice that each connected component of $\mathcal{R}_k$ will in general be a union of several ridges of order at least k.

c) There may be self-intersections in both the ribs and the ridges, and also transversal intersections between different ribs or different ridges. So, a given point a of the focal set may belong at the same time to several ribs, which means that it is the center of some hypersphere osculating with contacts of order higher than 2 (in the sense that $G_{(a,t)}$ has a singularity of type $A_{k>2}$) at more than one point of M. On the other hand, a point $g(x) \in M$ belonging to a ridge-intersection occurs whenever more than one of the focal hyperspheres at $g(x)$ has contact of type $A_{k>2}$ with M at this point.

d) The subset $\mathcal{R}_{n-1}$ is a union of non-necessarily closed curves immersed in M whose end points lie in $U(M)$. On the other hand, $\mathcal{R}_n$ is made of isolated points in M lying inside those curves.

2. Invariant 1-forms along curvature lines

We shall show first that the curvature lines grid is preserved by conformal maps.

Proposition 1. *Conformal maps preserve curvature lines of hypersurfaces.*

Proof. Suppose a hypersphere tangent to the hypersurface M (embedded through g in $\mathbb{R}^n$) at some point $g(x) = p$. The corresponding contact map is given by the function $G_{(a,t)}$. Furthermore, suppose that $S(a,t)$ is a focal hypersphere of M at p. So the contact direction of $G_{(a,t)}(x)$ is one principal direction of curvature (see [13], Lemma 2). Conformal maps transform hyperspheres into hyperspheres, and since there are diffeomorphisms, they must preserve their corresponding contacts with the hypersurface. Therefore they take focal hyperspheres into focal hyperspheres, preserving the contact directions. Consequently they take principal curvature directions into principal curvature directions and hence curvature lines into curvature lines. □

Coxeter defined in [6] the *inversive distance* between couples of circles in $\mathbb{R}^2$. This is preserved under conformal maps. The generalized expression of this formula for two hyperspheres $S_i(a_i, r_i)$, $i = 1, 2$ in $\mathbb{R}^n$, is given by

$$d(S_1, S_2) = \left| \frac{{r_1}^2 + {r_2}^2 - \|a_1 - a_2\|^2}{2 r_1 r_2} \right|,$$

where $a_i, r_i,\ i = 1, 2$ denote their centers and radii, respectively, [2].

Let us denote by $\varphi_{i,m_0}(t)$ the i-th curvature line of M passing through a point $m_0 = g(x_0) \in M - U(M)$. By considering two nearby focal hyperspheres of the hypersurface M along the curve φ_{i,m_0}, and applying the fact that the generalized inversive distance is a conformal invariant, we obtain below several invariant 1-forms on each one of the curvature lines of M considered as curves in $\mathbb{R}^n$, in the sense that any conformal map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ takes the 1-forms associated to a given curvature line of M to the corresponding ones on its image curve, which is itself a curvature line in the hypersurface $\phi(M)$.

Theorem 1. *The differential 1-form defined by*

$$\omega_{i,m_0}(t) = \sqrt{| \ K_i{}'(t) \ |} dt,\ 1 \leq i \leq n-1$$

is a conformal invariant along the curvature line $\varphi_{i,m_0}(t)$, where $K_i{}'$ represents the derivative of the principal curvature K_i of M restricted to the curve φ_{i,m_0}.

Remark 2. We observe that:

a) The 1-form ω_{i,m_0} depends only on the considered curvature line φ_{i,m_0} and not on the point m_0 chosen to determine it. Clearly, varying the point m_0 in a convenient manner (for instance along a curve transversal to the i-th curvature lines), we obtain a differentiable family of differential 1-forms, one on each i-th curvature line of M. To simplify notation we shall drop the suffix m_0 in what follows, understanding that $\varphi_i(t)$ represents someone of the i-th curvature lines of M.

b) We shall proof the conformal invariance of the 1-forms proposed by Theorems $1, 2, 3$ and 6 only at non-parabolic points. The result extends easily by continuity to the parabolic ones.

Proof. Let us consider $S_i(t)$, $S_i(t+h)$ two nearby focal hyperspheres of M with centers in the i-th focal sheet and radii $r_i(t) = 1/K_i(t)$, $r_i(t+h) = 1/K_i(t+h)$, respectively, lying along the i-th curvature line. The square of the inversive distance between the centers $a_i(t) = \varphi_i(t) + r_i(t)N(\varphi_i(t))$ of $S_i(t)$ and $a_i(t+h) = \varphi_i(t+h) + r_i(t+h)N(\varphi_i(t+h))$ of $S_i(t+h)$ is given by

$$d^2(S_i(t), S_i(t+h)) = \left| \frac{r_i(t+h)^2 + r_i(t)^2 - \|a_i(t+h) - a_i(t)\|^2}{2r_i(t+h)r_i(t)} \right|^2 .$$

We denote $d(S_i(t), S_i(t+h)) = d_i(h)$ and by expanding in Taylor series, we get:

$$d_i^2(h) = 1 - \frac{\|a_i{}'\|^2 - r_i{}'^2}{r_i{}^2} h^2 + \frac{\|a_i{}'\|^2 r_i{}' - r_i{}'^3 - a_i{}' a_i{}'' r_i + r_i r_i{}' r_i{}''}{r_i{}^3} h^3 + O(h^4).$$

Now, the Olinde Rodrigues theorem for hypersurfaces tells us that along all the curvature lines the equality $N'(\varphi_i) = -K_i \varphi_i'$ holds. By applying this formula we simplify the above Taylor series:

$$d_i^2(h) = 1 + \frac{1}{4!} K_i{}'^2 h^4 + O(h^5).$$

As the inversive distance $d_i(h)$ is invariant under the action of the conformal group, so is $\sqrt[4]{d_i^2(h)-1}$. And we get that the 1-form

$$\omega_{i,m_0}(t)=\sqrt{| K_i'(t) |}dt,\ 1\leq i\leq n-1$$

is a conformal invariant along the given corresponding curvature line passing through m_0. □

In an analogous way, we can consider the focal hyperspheres corresponding to the j-th principal direction of M, along the curve φ_{i,m_0}. The same principles as above lead us to:

Theorem 2. *Given any curvature line φ_{i,m_0}, $1\leq i\leq n-1$ of M, the 1-forms defined by*

$$\hat{\omega}_{i,j,m_0}(t)=(K_j(t)-K_i(t))dt,\ 1\leq j\neq i\leq n-1$$

are conformal invariants along φ_{i,m_0}.

Proof. The above argument for two nearby focal hyperspheres along the i-th curvature line: $S_j(t)$ and $S_j(t+h)$, with centers in the j-th focal sheet and radii $r_j(t)=1/K_j(t)$, $r_j(t+h)=1/K_j(t+h)$, respectively, leads to

$$d_j^2(h)=1-\frac{\|a_j'\|^2-{r_j'}^2}{{r_j}^2}h^2+O(h^3),\ 1\leq j\neq i\leq n-1.$$

And by applying again the generalized Olinde Rodrigues theorem $N'(\varphi_i)=-K_i{\varphi_i}'$, we obtain this time

$$d_j^2(h)=1-(K_j-K_i)^2h^2+O(h^3),\ 1\leq j\neq i\leq n-1.$$

Now, by taking into account, as above, that the inversive distance is invariant under the action of the conformal group and considering the variation of $\sqrt{1-d_j^2(h)}$ with respect to the parameter of the given curve φ_{i,m_0}, we get that

$$\hat{\omega}_{i,j,m_0}(t)=(K_j(t)-K_i(t))dt,\ 1\leq j\neq i\leq n$$

are conformal invariants along this curvature line. □

Theorem 3. *The following differential 1-form*

$$\bar{\omega}_{i,m_0}(t)=\sqrt{\frac{(n-2)\sum_{j=1}^{n-1}K_j(t)^2-2\sum_{1\leq j<k\leq n-1}K_j(t)K_k(t)}{(n-1)^2(n-2)}}dt,$$

is a conformal invariant along each curvature line φ_{i,m_0}, $1\leq i\leq n-1$ of M.

Proof. By using the above argument for two nearby focal hyperspheres $S_k(t)$ and $S_k(t+h)$ along a curvature line φ_i, with centers in the k-th focal sheet $k=1,\dots,n-1$, applying again the generalized O. Rodrigues theorem, the fact that the inversive distance is invariant under the action of the conformal group and considering the variation of the function:

$$\sqrt{\frac{2}{(n-1)^2(n-2)}\left(1-\prod_{j=1}^{n-1}d_j(h)+\sum_{1\leq j<k\leq n-1}\left(\sqrt{1-d_j(h)}-\sqrt{1-d_k(h)}\right)^2\right)}$$

$$= \left(\sqrt{\frac{(n-2)\sum_{j=1}^{n-1} {K_j}^2 - 2\sum_{1\le j<k\le n-1} K_j K_k}{(n-1)^2(n-2)}} \right) h + O(h^2),$$

we get that:

$$\bar{\omega}_{i,m_0}(t) = \sqrt{\frac{(n-2)\sum_{j=1}^{n-1} K_j(t)^2 - 2\sum_{1\le j<k\le n-1} K_j(t)K_k(t)}{(n-1)^2(n-2)}} dt,$$

is a conformal invariant along each i-th curvature line φ_{i,m_0}. □

The next result tells us how all the 1-forms of the families $\{\omega_{i,m_0}\}_{m_0 \in M-U(M)}$, $\{\hat{\omega}_{i,j,m_0}\}_{m_0\in M-U(M)}$ and $\{\bar{\omega}_{i,m_0}\}_{m_0\in M-U(M)}$ along each i-th curvature line φ_{i,m_0}, can be respectively glued in order to define conformally invariant 1-forms ω_i, $\hat{\omega}_{i,j}$ and $\bar{\omega}_i$ $1 \le i \ne j \le n-1$ on the open and dense submanifold $M - U(M)$ when M is a locally conformally flat and no quasi-umbilical hypersurface. This happens to be the case of any surface in 3-space and a large class of 3-manifolds in 4-space. Unfortunately, hypersurfaces of higher dimensions cannot be included in our analysis for local conformal flatness is equivalent to quasi-umbilicity ($M = U(M)$) in this case (Cartan's Theorem, [8]).

Theorem 4. *Suppose that M is a locally conformally flat and no quasi-umbilical hypersurface in $\mathbb{R}^n$, $n = 3, 4$. The following differential 1-forms for all $1 \le i \le n-1$:*

$$\omega_i(x) = \sqrt{|\ {K_i}'(x)\ |} dx_i,$$

$$\hat{\omega}_{ij}(x) = (K_j(x) - K_i(x))dx_i,\ 1 \le j \ne i \le n-1,$$

and

$$\bar{\omega}_i(x) = \sqrt{\frac{(n-2)\sum_{j=1}^{n-1} K_j(x)^2 - 2\sum_{1\le j<k\le n-1} K_j(x)K_k(x)}{(n-1)^2(n-2)}} dx_i,$$

are conformally invariant on the open submanifold $M - U(M)$.

Proof. We consider the parametrization of $M - U(M)$ given by the curvature lines of M in a neighborhood of $m_0 = g(x_0)$ [7]. Let $\{X_i\}_{i=1}^{n-1}$ be the principal direction fields of M. We observe that this is the dual basis of the one given by the differential 1-forms $\{dx_i\}_{i=1}^{n-1}$ in the chosen coordinates, so we have:

$$\begin{aligned} \omega_i(X_i) =&\ \sqrt{{K_i}'}, \\ \omega_i(X_j) =&\ 0,\ 1 \le j \ne i \le n-1, \end{aligned}$$

$$\begin{aligned} \hat{\omega}_{i,j}(X_i) =&\ K_j - K_i, \\ \hat{\omega}_{i,j}(X_j) =&\ 0,\ 1 \le j \ne i \le n-1, \end{aligned}$$

$$\begin{aligned} \bar{\omega}_i(X_i) =&\ \sqrt{\frac{(n-2)\sum_{p=1}^{n-1} {K_j}^2 - 2\sum_{1\le p<k\le n-1} K_p K_k}{(n-1)^2(n-2)}}, \\ \bar{\omega}_i(X_j) =&\ 0,\ 1 \le j \ne i \le n-1. \end{aligned}$$

Now, since the basis $\{X_i\}_{i=1}^{n-1}$ is a conformal invariant, the fact that the $\{\omega_i\}_{i=1}^{n-1}$, $\{\hat{\omega}_{i,j}\}_{1\leq j\neq i\leq n-1}$ and $\{\bar{\omega}_i\}_{i=1}^{n-1}$ are conformally invariants along the curvature lines implies that so they are on the whole manifold M [17]. $\square$

We now see how to obtain some of the well-known conformal invariants on surfaces and locally conformally flat and no quasi-umbilical 3-manifolds in 4-space:

Corollary 1. *If M is a surface in 3-space, the following differential 2-form defined on $M - U(M)$:*

$$\sqrt{(H_1^2 - H_2)^2}\, dx_1 \wedge dx_2$$

is a conformal invariant, where $2\,H_1 = K_1 + K_2$ and $H_2 = K_1 K_2$. If M is locally conformally flat and no quasi-umbilical 3-manifold in 4-space, the following differential 3-form defined on $M - U(M)$:

$$\sqrt{(H_1^2 - H_2)^3}\, dx_1 \wedge dx_2 \wedge dx_3$$

is a conformal invariant, where:

$$3\,H_1 = K_1 + K_2 + K_3, \quad 3\,H_2 = K_1K_2 + K_1K_3 + K_2K_3.$$

Proof. We know that $H_r = \begin{pmatrix} n \\ r \end{pmatrix}^{-1} \sum_{1\leq i_1<\cdots<i_r\leq n} K_{i_1}\cdots K_{i_1}$, then we get that

$$H_1^2 - H_2 = \frac{\sum_{i=1}^{n-1} {K_i}^2 + 2\sum_{1\leq i<j\leq n-1} K_iK_j}{(n-1)^2} - \frac{2\sum_{1\leq i<j\leq n-1} K_iK_j}{(n-1)(n-2)}$$

$$= \frac{(n-2)(\sum_{i=1}^{n-1} {K_i}^2 + 2\sum_{1\leq i<j\leq n-1} K_iK_j) - 2(n-1)\sum_{1\leq i<j\leq n-1} K_iK_j}{(n-1)^2(n-2)}$$

$$= \frac{(n-2)\sum_{i=1}^{n-1} {K_i}^2 - 2\sum_{1\leq i<j\leq n-1} K_iK_j}{(n-1)^2(n-2)}.$$ $\square$

Remark 3. The above differential $(n-1)$-form is known as the *conformal volume.* This conformal invariant was first obtained W. J. Blasche for surfaces in $\mathbb{R}^3$, [4]. A generalization for surfaces in $\mathbb{R}^n$ was later given by B.-Y. Chen in [9]. The general case of a m-submanifold in $\mathbb{R}^n$ has been traited by Ch.-Ch. Hsiung and L. R. Mugridge in [10]. We point out that the approach followed in all these cases is essentially different from ours.

A further consequence of Theorem 4 is the obtention of the following conformal invariants, that can be seen as a generalization of the conformal principal curvatures of surfaces in 3-space defined by Tresse ([21]) to locally conformally flat and no quasi-umbilical 3-submanifolds in $\mathbb{R}^4$.

Corollary 2. *The functions*

$$\frac{\partial K_i}{\partial x_i}\left(\frac{(n-2)\sum_{j=1}^{n-1} K_i{}^2 - 2\sum_{1\le j<k\le n-1} K_j K_k}{(n-1)^2(n-2)}\right)^{-1},$$

are conformal invariant of a locally conformally flat and no quasi-umbilical hypersurface in $\mathbb{R}^n$, $n = 3, 4$.

The above results tell us that the conformal geometry of the hypersurface can be recognized from its conformal geometry along its curvature lines. This idea leads us, in the following section, to detect the points at which the hypersurface has the highest possible contact with hyperspheres through the geometry of these "special" curves.

3. On the existence and detection of higher order ridges

Let $\alpha : \mathbb{R} \to \mathbb{R}^n$ be a curve parametrized by arc-length and consider its associated family of squared functions

$$\begin{array}{rcl} d^\alpha : \mathbb{R}\times\mathbb{R}^n & \longrightarrow & \mathbb{R} \\ (t,a) & \longmapsto & d^\alpha_a(t) = \|\alpha(t)-a\|^2. \end{array}$$

The *focal set* F_α of α is made by all the centers if hyperspheres of $\mathbb{R}^n$, having contact of order at least 2 with the curve, i.e. the focal hyperspheres of α. In other words F_α is composed of all the points $a \in \mathbb{R}^n$ such that the distance squared function on α from a, d^α_a, has some singularity of type $A_k, k \ge 2$ at some point $\alpha(t)$ (in which case we say that the hypersphere of center a passing through $\alpha(t)$ has contact of order k with the curve). For $k \ge n$ we have the *osculating hypersphere* of α at $\alpha(t)$.

Consider the Frenet frame $\{T(t), N_1(t), \ldots, N_{n-1}(t)\}$ and the corresponding curvature functions $\{k_i(t)\}_{i=1}^{n-1}$ at the point $\alpha(t)$ of a generic curve α. The centers of the osculating hyperspheres form a smooth curve in $\mathbb{R}^n$, given by (see [20])

$$c_\alpha(t) = \alpha(t) + \sum_{i=1}^{n-1} \mu_i(t) N_i(t),$$

where $\{\mu_i(t)\}_{i=1}^{n-1}$ are rational functions of the curvatures $\{k_i(t)\}_{i=1}^{n-1}$ and their derivatives and satisfy the following relation (as shown in [20]):

$$\begin{aligned} &\mu_1(t)k_1(t) = 1, \\ &\mu_2(t)k_2(t) = \mu_1'(t), \\ &\mu_i(t)k_i(t) = \mu_{i-1}'(t) + \mu_{i-2}(t)k_{i-1}(t),\ i = 3, ..., n-1. \end{aligned}$$

We call c_α the *generalized evolute* of α. The singular points of c_α, called *vertices*, are precisely the points at which the curve has contact of order higher than n with its osculating hyperspheres and we characterize its in [20] by the formula $\mu_{n-1}'(t) + \mu_{n-2}(t)k_{n-1}(t) = 0$.

A curve in the n-space with $k_i(t) \neq 0$ and free of i-vertices $i = 1, \ldots, n-2$ [15] is a *generic curve.*

Let α be a generic curve and take coordinates $\{\gamma_1, ..., \gamma_{n-1}\}$ in the normal plane $N_{\alpha(t_0)}\alpha = \alpha(t_0) + < N_1(t_0), \dots, N_{n-1}(t_0) >$ of α at the point $\alpha(t_0)$.

Suppose that $S(a, r)$ is a hypersphere tangent to α at $\alpha(t_0)$, it is not difficult to verify that $S(a, r)$ has contact of order $\geq k$ (for $k \leq n$) with the curve if and only if the point a belongs to the $(n-k)$-subspace of $N_{\alpha(t_0)}\alpha$ defined by the linear equations

$$\begin{aligned} \gamma_1 &= \mu_1(t_0), \\ &\vdots \\ \gamma_{k-1} &= \mu_{k-1}(t_0). \end{aligned}$$

We observe that, in general, the i-th focal hypersphere $S_i(a_i, r_i)$ of M at a given point $m_0 \in M - U(M)$ and the osculating hypersphere on the i-th curvature line φ_i at this point do not need to coincide. Moreover, the last one does not need to be tangent to M. Nevertheless, we have:

Proposition 2. *The focal hypersphere $S_i(a_i, r_i)$ of M at a non umbilic point m_0 has contact of order at least 2 with the corresponding curvature line φ_{i,m_0} considered as curves in the n-space.*

Proof. We know that the focal hyperspheres of the hypersurface M, along the curvature lines, are given by $S_i(a_i, r_i)$, where $r_i(t) = 1/K_i(t)$, with $K_i \neq 0$ the i-th principal curvature of M, and $a_i(t) = \varphi_i(t) + r_i(t)N(\varphi_i(t))$, $1 \leq i \leq n-1$. The derivative of φ_i respect its arc-length is the tangent of the curvature line considered as a curve in the n-space, i.e. $\varphi_i'(t) = T(t)$. So $< N(\varphi_i(t)), T(t) > = 0$. By deriving in the above expression, with respect the arc-length, we obtain

$$< N(\varphi_i(t)), T(t) >' = < N'(\varphi_i(t)), T(t) > + < N(\varphi_i(t)), T'(t) > = 0.$$

And then, by applying the Frenet's formulas for the curvature line and the Olinde Rodrigues theorem, $N'(\varphi_i(t)) = -K_i(t)\varphi_i'(t)$, we get

$$< N(\varphi_i(t)), N_1(t) > = \frac{K_i(t)}{k_1(t)}, \ 1 \leq i \leq n-1,$$

where $N_1(t)$ and $k_1(t)$ are the first normal vector and the first Euclidean curvature of the curve φ_i in the n-space, respectively. Then, we observe that the center of the focal hypersphere of the hypersurface $S_i(a_i, r_i)$, along the curvature line, can be rewritten as

$$\begin{aligned} a_i(t) = & \ \varphi_i(t) + r_i(t)N(\varphi_i(t)) \\ = & \ \varphi_i(t) + \mu_1(t)N_1(t) + \textstyle\sum_{i=2}^{n} \gamma_i N_i(t), \ 1 \leq i \leq n-1, \end{aligned}$$

where $N_i(t)$ are the i-th normal vector of the curve φ_i at the point $\varphi_i(t)$ and $\mu_1(t) = 1/k_1(t)$. Hence the focal hypersphere of the hypersurface $S_i(a_i, r_i)$ has contact at least 2 with the curvature line in the n-space.

In a parabolic point $\varphi_i(t_0)$ the focal hypersphere becomes to a tangent hyperplane. In this case, by using the below formula, we know that

$$< N(\varphi_i(t_0)), T(t_0) > = 0, \ < N(\varphi_i(t_0)), N_1(t_0) > = 0$$

and this implies that the tangent hyperplane has contact at least of order 2 with the curvature line in the n-space. □

The ridges points of a surface in $\mathbb{R}^3$ can be recognized as critical points of the principal curvatures along the principal curvature lines (see [3] and [19]). This is naturally generalized to the case of hypersurfaces in $\mathbb{R}^n$ by using the methods of [19], as follows:

Lemma 1. *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with a degenerate singularity at the origin and suppose that $\theta \in Ker(\mathcal{H}(h)(0))$. Then we have that θ is a singularity of type A_k of h if and only if the vector θ belongs to the kernel of the k-linear form, $D^k h(0)$, given by the k-th differential of h, $k \geq 2$.*

Proof. By taking an appropriate change of coordinates in $\mathbb{R}^n$ we can write

$$h(x_1, \dots, x_n) = \pm x_1^{k+1} \pm x_2^2 \pm \cdots \pm x_n^2.$$

Then the result follows from a straightforward verification for this function and the fact that if Φ is a change of coordinates in $\mathbb{R}^n$, the isomorphism $D\Phi$ transforms the kernel of the differential $D^k h(0)$ into the kernel of the differential $D^k(h \cdot \Phi)(0)$. □

Proposition 3. *A point $m_0 \in M - U(M)$ belongs to a k-th order ridge ($k \geq 2$) if and only if there is some curvature line $\varphi_{i,m_0}(t)$ on M, with $m_0 = \varphi_i(t_0) = g(x_0)$ and such that the corresponding principal curvature K_i restricted to it (as a function of t) satisfies: $K_i'(t_0) = ... = K_i^{(k-1)}(t_0) = 0$.*

Proof. We know that the point $m_0 = g(x_0) \in M - (U(M) \cup P(M))$ belongs to a second order ridge (i.e. $(m_0, a) \in \Sigma^{1,1,0}(\exp_N)$) if and only if $Dd_a(x_0) = 0$ and there exists a tangent vector $X \in T_x\mathbb{R}^n$, such that $D^2 d_a(x_0)(X) = 0$, i.e. $Dg(x_0)(X)$ is a contact direction for M and the focal hypersphere at $m_0 = g(x_0)$ and $D^3 d_a(x_0)(X, X, X) = 0$, where:

$$D^2 d_a(x) = 2(a - g(x)) \cdot D^2 g(x) - 2Dg(x) \cdot Dg(x) \in L_S(\mathbb{R}^n(L(\mathbb{R}^n, \mathbb{R}))),$$

$$D^3 d_a(x) = 2(a - g(x)) \cdot D^3 g(x) - 6Dg(x) \cdot D^2 g(x) \in L_S(\mathbb{R}^n(L_S(\mathbb{R}^n(L(\mathbb{R}^n, \mathbb{R})))).$$

We remind that the principal directions coincide with the contact direction, therefore, we consider the contact map d_a, with $a(t) = \varphi_i(t) + 1/K_i(t)N_{\varphi_i(t)}$ along the curvature line $\varphi_i(t) = g(\alpha_i(t))$, where $\alpha_i(t) \subset \mathbb{R}^n$, corresponding to the principal direction $Dg(\alpha_i(t))(X)$ i.e. $\alpha_i(t) = x$ and $\alpha_i'(t) = X$. In this case $D^2 d_a(\alpha_i(t))(\alpha_i'(t)) = 0$ along φ_i. By deriving the function $D^2 d_a(\alpha_i(t))(\alpha_i'(t))$ along the curvature line and applying the generalized O. Rodrigues theorem, we get:

$$\begin{aligned} 0 = (D^2 d_a(\alpha_i(t))(\alpha_i'(t)))' &= D^3 d_a(\alpha_i(t))(\alpha_i'(t), \alpha_i'(t)) + D^2 d_a(\alpha_i(t))(\alpha_i''(t)) \\ &\quad + 2(\tfrac{1}{K_i(t)})' N_{g(\alpha_i(t))} \cdot D^2 g(\alpha_i(t))(\alpha_i'(t)) \\ &= (Dd_a(\alpha_i(t)))'' + 2(\tfrac{1}{K_i(t)})' N_{g(\alpha_i(t))} \cdot D^2 g(\alpha_i(t))(\alpha_i'(t)). \end{aligned}$$

As the point $m_0 = g(x_0) \in M - (U(M) \cup P(M))$ belongs to a second order ridge, by the Lemma 1 we say that

$$0 = (Dd_a(\alpha_i(t_0)))''(\alpha_i'(t_0)) = D^3 d_a(\alpha_i(t_0))(\alpha_i'(t_0), \alpha_i'(t_0), \alpha_i'(t_0)) + D^2 d_a(\alpha_i(t_0))(\alpha_i''(t_0), \alpha_i'(t_0)),$$

then

$$\left(\frac{1}{K_i(t_0)}\right)' = \frac{K_i'(t_0)}{K_i^2(t_0)} = 0,$$

and we obtain $m_0 = g(x_0) = \varphi_i(t_0)$ belongs to a second order ridge if and only if $K_i'(t_0) = 0$, i.e., t_0 is a critical point of K_i along φ_i.

By deriving again, we get:

$$\begin{aligned} 0 &= (D^2 d_a(\alpha_i(t))(\alpha_i'(t)))'' = D^4 d_a(\alpha_i(t))(\alpha_i'(t), \alpha_i'(t), \alpha_i'(t)) \\ &+ 3D^3 d_a(\alpha_i(t))(\alpha_i''(t), \alpha_i'(t)) + D^2 d_a(\alpha_i(t))(\alpha_i'''(t)) \\ &+ 2\left((\tfrac{1}{K_i(t)})' N_{g(\alpha_i(t))} \cdot D^2 g(\alpha_i(t))(\alpha_i'(t))\right)' \\ &= (Dd_a(\alpha_i(t)))^{(3)} + 2\left(\tfrac{1}{K_i(t)}\right)' (N_{g(\alpha_i(t))} \cdot D^2 g(\alpha_i(t))(\alpha_i'(t)))' \\ &+ 2\left(\tfrac{1}{K_i(t)}\right)'' (N_{g(\alpha_i(t))} \cdot D^2 g(\alpha_i(t))(\alpha_i'(t))). \end{aligned}$$

If the point $m_0 = g(x_0) \in M - (U(M) \cup P(M))$ belongs to a ridge of order 3, by using the Lemma 1 we say that

$$0 = (Dd_a(\alpha_i(t_0)))^{(3)}(\alpha_i'(t_0)) = D^4 d_a(\alpha_i(t_0))(\alpha_i'(t_0), \alpha_i'(t_0), \alpha_i'(t_0), \alpha_i'(t_0)) + 3D^3 d_a(\alpha_i(t_0))(\alpha_i''(t_0), \alpha_i'(t_0), \alpha_i'(t_0)) + D^2 d_a(\alpha_i(t_0))(\alpha_i'''(t_0), \alpha_i'(t_0)),$$

then $g(x_0) = \varphi_i(t_0)$ belongs to a ridge of order 3, if and only if $K_i'(t_0) = 0$ and $K_i''(t_0) = 0$ along φ_i.

By using an induction argument we obtain that

$$(D^2 d_a(\alpha_i(t))(\alpha_i'(t)))^{(k)} = (Dd_a(\alpha_i(t)))^{(k+1)} + 2\left((\tfrac{1}{K_i(t)})(N_{g(\alpha_i(t))} \cdot D^2 g(\alpha_i(t))(\alpha_i'(t))\right)^{(k-1)}.$$

By the Lemma 1 we get that a point $m_0 \in M - (U(M) \cup P(M))$ belongs to a k-th order ridge if and only if there is the corresponding principal curvature K_i restricted to φ_i satisfies: $K_i'(t_0) = \cdots = K_i^{(k-1)}(t_0) = 0$.

By applying the generalized O. Rodrigues theorem

$$N'_{g(\alpha_i(t))}(Dg(\alpha_i(t))(\alpha_i'(t)) = -K_i(t) Dg(\alpha_i(t))(\alpha_i'(t),$$

we get that $m_0 \in P(M) - U(M)$ if and only if m_0 is a singular point of the normal Gauss map.

By deriving this expression along the curvature line

$$\left(N'_{g(\alpha_i(t))}(Dg(\alpha_i(t))(\alpha'_i(t)))\right)^{(k)} = -\sum_{j=1}^{k} \begin{pmatrix} k \\ j \end{pmatrix} K_i^{(k-j)}(t)(Dg(\alpha_i(t))(\alpha'_i(t))^{(j)},$$

we obtain m_0 is a singular point of order at least k of the normal Gauss map if and only $K_i(t_0) = K'_i(t_0) = \cdots = K_i^{(k-1)}(t_0) = 0$ along the curvature line φ_i. □

We shall see now how to obtain the order of the ridge from the kind of contact that the focal hyperspheres have with the curvature lines.

Remark 4. Let be a hypersurface M locally given by some embedding $g : \mathbb{R}^{n-1} \to \mathbb{R}^n$. We observe, as a consequence of Thom's Transversality Theorem [14], that points determined by more than $n-1$ conditions on the derivatives of g do not appear generically on M. Since we are considering generic hypersurfaces, we have that its curvature lines φ_i, $i = 1, \ldots, n-1$ are generic curves.

Theorem 5. *Let m_0 be a non umbilic point of a generic hypersurface M. The point m_0 belongs to some ridge of M if and only if a focal hypersphere of M at m_0 has contact of order al least 3 with the corresponding curvature line.*

Proof. By deriving with respect to the arc-length of the curvature line φ_i the expression $< N(\varphi_i(t)), N_1(t) > = K_i(t)/k_1(t)$, obtained in the proof of Proposition 2, we get

$$< N'(\varphi_i(t)), N_1(t) > + < N(\varphi_i(t)), N'_1(t) > = \left(\frac{K_i(t)}{k_1(t)}\right)'.$$

If we are considering hypersurfaces of dimension $n \geq 3$, where the curvature lines are generic curves, then we have that $k_i(t) \neq 0$ and free of i-vertices, $i = 1, \ldots, n-2$. From the Frenet's formulas for the curvature line φ_i considering as a curve in the n-space and the O. Rodrigues theorem we obtain

$$k_2(t) < N(\varphi_i(t)), N_2(t) > = \frac{-k'_1(t)K_i(t)}{k_1(t)^2} + \frac{K'_i(t)}{k_1(t)}. \tag{1}$$

Therefore, the point $m_0 = \varphi_i(t_0) \in \bar{M}$ belongs to a second order ridge point, i.e. $K'_i(t_0) = 0$, if and only if $< N(\varphi_i(t_0)), N_2(t_0) > = K_i(t_0)\mu_2(t_0)$, where

$$\mu_2(t) = \frac{1}{k_2(t)} \left(\frac{-k'_1(t)}{k_1(t)^2}\right).$$

So, the center of the focal hypersphere of the hypersurface, at the point $\varphi_i(t_0)$ of the curvature line is given by

$$\begin{aligned} a_i(t_0) &= \varphi_i(t_0) + 1/K_i(t_0)N(\varphi_i(t_0) \\ &= \varphi_i(t_0) + \mu_1(t_0)N_1(t_0) + \mu_2(t_0)N_2(t_0) + \textstyle\sum_{i=3}^{n}\gamma_i N_i(t_0), \end{aligned}$$

where $N_i(t_0)$ are the i-th normal vectors of the curve φ_i at the point $\varphi_i(t_0)$. Hence, the point m_0 belongs to a ridge of M if and only if $S_i(a_i, r_i)$ has contact of order at least 3 with the curve φ_i in the n-space.

In a parabolic point $\varphi_i(t_0)$, when $K_i(t) = 0$, and the focal hypersphere becomes to a tangent hyperplane, by using the formula (1), we know that

$$< N(\varphi_i(t_0)), T(t_0) > = 0, \quad < N(\varphi_i(t_0)), N_1(t_0) > = 0, \quad < N(\varphi_i(t_0)), N_2(t_0) > = 0.$$

This implies that the tangent hyperplane has contact at least of order 3 with the curvature line in the n-space.

In the particular case of a surface, if $m_0 = \varphi_i(t_0) \in \bar{M}$ belongs to a second order ridge (i.e. $K_i'(t_0) = 0$), then we obtain that the focal sphere is also the osculating sphere. By using the equation (1) and the fact that the surface is generic and m_0 is not a 1-vertex (i.e. $k_1'(t_0) \neq 0$), we obtain that if $\varphi_i(t_0)$ belongs to a second order ridge then it is a parabolic point (i.e. $K_i(t_0) = 0$) if and only if $k_2(t_0) = 0$, because in this particular case $< N(\varphi_i(t)), N_2(t) > \neq 0$. Hence, the degenerate focal sphere (tangent plane) has contact of order at least 3 with the curve φ_i in the space, i.e. coincides with the degenerate osculating sphere (osculating plane) of φ_i. When $< N(\varphi_i(t_0)), N_1(t_0) > = 0$ and $k_2(t_0) = 0$, we have that m_0 belongs to a ridge of at least order 2 of M. □

Theorem 6. *Let m_0 be a non umbilic point of a generic hypersurface M. The point m_0 belongs to some ridge of order k of M if and only if a focal hypersphere of M at m_0 has contact of order al least $k+1$ with the corresponding curvature line.*

Proof. We consider hypersurfaces of dimension $n \geq 4$. By deriving the expression

$$< N(\varphi_i(t)), N_2(t) > = \frac{K_i'(t)}{k_1(t)k_2(t)} + K_i(t)\mu_2(t),$$

we obtain:

$$< N'(\varphi_i(t)), N_2(t) > + < N(\varphi_i(t)), N_2'(t) > = \frac{K_i''(t)}{k_1(t)k_2(t)} +$$

$$+ \frac{-(k_1(t)k_2(t))'K_i'(t)}{k_1^2(t)k_2^2(t)} + K_i'(t)\mu_2(t) + K_i(t)\mu_2'(t).$$

By applying O. Rodrigues theorem, Frenet's formula $N_2'(t) = -k_2(t)N_1(t) + k_3(t)N_3(t)$ and $< N(\varphi_i(t)), N_1(t) > = K_i(t)\mu_1(t)$ we have

$$\begin{aligned} < N(\varphi_i(t)), k_3(t)N_3(t) > = & \frac{K_i''(t)}{k_1(t)k_2(t)} + \frac{-(k_1(t)k_2(t))'K_i'(t)}{k_1^2(t)k_2^2(t)} \\ & + K_i'(t)\mu_2(t) + K_i(t)(\mu_2'(t) + k_2(t)\mu_1(t)), \end{aligned} \quad (2)$$

and using the formula $k_3(t)\mu_3(t) = \mu_2'(t) + k_2(t)\mu_1(t)$ we obtain the coefficient of the center a_i in N_3. Therefore, $1/K_i(t_0)N(\varphi_i(t_0)) = \mu_1(t_0)N_1(t_0) + \mu_2(t_0)N_2(t_0) + \mu_3(t_0)N_3(t_0) +$

$\sum_{i=3}^{n} a_i N_i(t_0)$ if and only if $K_i'(t_0) = K_i''(t_0) = 0$. Hence, $m_0 = \varphi_i(t_0) \in \bar{M}$ belongs to a ridge of order 3 of M if and only if $S_i(a_i, r_i)$ has contact of order at least 4 with φ in $\varphi_i(t_0)$.

If the point m_0 belongs to a ridge of order 3 and is a parabolic point of M, i.e. $K_i(t_0) = K_i'(t_0) = K_i''(t_0) = 0$, and the focal hypersphere becomes to a tangent hyperplane, by using the formula (2), we know that

$$< N(\varphi_i(t_0)), T(t_0) > = 0, \quad < N(\varphi_i(t_0)), N_i(t_0) > = 0, \; i = 1, 2, 3$$

and this implies that the tangent hyperplane has contact at least of order 4 with the curvature line in the n-space.

When M is a 3-submanifold in 4-space, if $m_0 \in \bar{M}$ belongs to a ridge of order 3, the focal sphere is also the osculating 3-sphere of φ_i.

By using the equation (2) and the fact that M is generic (m_0 is not a 2-vertex i.e. $\mu_2'(t_0) + k_2(t_0)\mu_1(t_0) \neq 0$), we obtain that if $m_0 = \varphi_i(t_0)$ belongs to a ridge of at least order 3 of M (i.e. $K_i'(t_0) = K_i''(t_0) = 0$), then $K_i(t_0) = 0$ if and only if $k_3(t_0) = 0$, because in this particular case $< N(\varphi_i(t)), N_3(t) > \neq 0$. Then the degenerate focal hypersphere (tangent hyperplane) has contact of order at least 4 with the curve φ_i in the 4-space, i.e. coincides with the degenerate osculating 3-sphere (osculating hyperplane) of φ_i. When $< N(\varphi_i(t_0)), N_i(t_0) > = 0, \; i = 1, 2$ and $k_3(t_0) = 0$ we have that m_0 belongs to a ridge of at least order 3 of M.

Finally, when M is a surface in $\mathbb{R}^3$ the Frenet formula $N_2'(t) = -k_2(t)N_1(t)$, and we get

$$0 = \frac{K_i''(t)}{k_1(t)k_2(t)} + \frac{-(k_1(t)k_2(t))' K_i'(t)}{k_1^2(t)k_2^2(t)} + K_i'(t)\mu_2(t) + K_i(t)(\mu_2'(t) + k_2(t)\mu_1(t)).$$

If we suppose that $m_0 = \varphi_i(t_0)$ is not parabolic point at the curvature line $\varphi_i(t)$, then $K_i(t_0) \neq 0$. Hence if $K_i'(t_0) = K_i''(t_0) = 0$, we know by Theorem 5 that $k_2(t_0) \neq 0$, then $\mu_2'(t_0) + k_2(t_0)\mu_1(t_0) = 0$. So if m_0 belongs to a ridge of order 3 of the surface then it is a 2-vertex of φ.

By deriving the expression (1), we obtain:

$$k_2'(t) < N(\varphi_i(t)), N_2(t) > + k_2(t) < N(\varphi_i(t)), N_2(t) >' = \frac{K_i''(t)}{k_1(t)} - \frac{K_i'(t)k_1'(t)}{k_1^2(t)} + K_i'(t)\mu_1'(t) + K_i(t)\mu_1''(t).$$

Hence, we obtain that if a parabolic point m_0 belongs to a ridge of order 3 then $k_2(t_0) = k_2'(t_0) = 0$, because in this case $< N(\varphi_i(t)), N_2(t) > \neq 0$. So if the parabolic point m_0 belongs to a ridge of order 3 of the surface then it is a degenerate 2-vertex of φ.

We consider now hypersurfaces of dimension $n \geq k + 2$, where the curvature lines are generic curves. We will obtain by induction the following expression for all $1 \leq j \leq k + 1$:

$$< N(\varphi_i(t)), N_j(t) > = \frac{K_i^{(j-1)}(t)}{\prod_{m=1}^{j} k_m(t)} + K_i(t)\mu_j(t) + \sum_{m=1}^{j-2} \eta_m(t) K_i^{(m)}(t),$$

where $\{\eta_m(t)\}_{m=1}^{j-2}$ are functions of $k_m(t)$, $m = 1, \ldots, j-1$ and their derivatives. In the particular case $j = 1, 2, 3$ we are proved that this expression occurs.

Therefore, $m_0 \in \bar{M}$ belongs to a ridge at least of order $k+1$ i.e. $K_i^{(j)}(t_0) = 0$, $j = 1, \dots, k$ if and only if $< N(\varphi_i(t_0)), N_j(t_0) > = K_i(t_0)\mu_j(t_0)$, $j = 1, \dots, k+1$ and

$$1/K_i(t_0)N(\varphi_i(t_0)) = \sum\nolimits_{j=1}^{k+1} \mu_j(t_0)N_j(t_0) + \sum\nolimits_{j=k+2}^{n} a_j N_j(t_0).$$

Hence, the focal hypersphere of the hypersurface has contact of order at least $k+1$ with the curvature line φ_i at the point m_0.

When m_0 belongs to a ridge at least of order $k+1$ and is a parabolic point, i.e. $K_i^{(j)}(t_0) = 0$, $j = 0, \dots, k+1$, the focal hypersphere becomes to a tangent hyperplane and by using the previous formula, we know that

$$< N(\varphi_i(t_0)), T(t_0) > = 0, \quad < N(\varphi_i(t_0)), N_i(t_0) > = 0, \ i = 1, \dots, k+1.$$

Hence, the tangent hyperplane has contact at least of order $k+1$ with the curvature line in the n-space.

If $m_0 = \varphi_i(t_0)$ belongs to a ridge of order $k = n-1$ and $m_0 \in \bar{M}$, then we get

$$< N(\varphi_i(t_0)), N_j(t_0) > = K_i(t_0)\mu_j(t_0), \ j = 1, \dots, n-1$$

and $a_i(x_0) = \varphi_i(t_0) + 1/K_i(t_0)N(\varphi_i(t_0)) = \varphi_i(t_0) + \sum_{j=1}^{n-1} \mu_j(t_0)N_j(t_0)$, then the focal hypersphere is also the osculating hypersphere.

When $m_0 \in P(M)$ belongs to a ridge of order $n-1$, by using the formula:

$$k_{n-1}(t) < N(\varphi_i(t)), N_{n-1}(t) > = \frac{K_i^{(n-2)}(t)}{\prod_{m=1}^{n-2} k_m(t)} + \sum_{m=1}^{n-3} \check{\eta}_m(t)K_i^{(m)}(t) + K_i(t)(\mu'_{n-2}(t) + \mu_{n-3}(t)k_{n-2}(t)), \tag{3}$$

and considering the genericity of the hypersurface M (m_0 is not a $(n-2)$-vertex, i.e. $\mu'_{n-2}(t_0) + \mu_{n-3}(t_0)k_{n-2}(t_0) \neq 0$) then $K_i(t_0) = 0$ if and only if $k_{n-1}(t_0) = 0$. Then the degenerate focal hypersphere (tangent hyperplane) has contact of order at least n with the curve φ_i in the n-space, i.e. coincides with the degenerate osculating n-sphere (osculating hyperplane) of φ_i. When $< N(\varphi_i(t_0)), N_i(t_0) > = 0$, $i = 1, \dots, n-2$ and $k_{n-1}(t_0) = 0$ we have that m_0 belongs to a ridge of at least order n of M.

Finally if m_0 belongs to a ridge of order n, from the last Frenet formula that in this case is given by $N'_{n-1}(t) = -k_{n-1}(t)N_{n-2}(t)$, thus

$$0 = \frac{K_i^{(n-1)}(t)}{\prod_{m=1}^{n-1} k_m(t)} + K_i(t)(\mu'_{n-1}(t) + \mu_{n-2}(t)k_{n-1}(t)) + \sum_{m=1}^{n-2} \bar{\eta}_m(t)K_i^{(m)}(t),$$

where $\{\check{\eta}_m(t)\}_{m=1}^{n-2}$ are functions of $k_m(t)$, $m = 1, ..., n-2$ and their derivatives. Hence if $K_i^{(j)}(t_0) = 0$, $j = 1, \dots, n-1$, we know by formula (3) that $k_{n-1}(t_0) \neq 0$, then $\mu'_{n-1}(t_0) + \mu_{n-2}(t_0)k_{n-1}(t_0) = 0$. Therefore m_0 belongs to a ridge of order n of M, then it is a $(n-1)$-vertex of φ.

When $m_0 = \varphi_i(t_0)$ is parabolic, by deriving the expression (3), we obtain:

$$k'_{n-1}(t) < N(\varphi_i(t)), N_{n-1}(t) > +k_{n-1}(t) < N(\varphi_i(t)), N_{n-1}(t) >' =$$

$$\frac{K_i^{(n-1)}(t)}{\prod_{m=1}^{n-2} k_m(t)} + \sum_{m=1}^{n-2} \check{\eta}_m(t) K_i^{(m)}(t) + K_i(t)(\mu'_{n-2}(t) + \mu_{n-3}(t)k_{n-2}(t))'.$$

Hence, if m_0 is a parabolic point that belongs to a ridge of order n, then $k_{n-1}(t_0) = k'_{n-1}(t_0) = 0$, because in this case $< N(\varphi_i(t)), N_{n-1}(t) > \neq 0$. So if the parabolic point m_0 belongs to a ridge of order n of M, then it is a degenerate $(n-1)$-vertex of φ. □

As a consequence of the Theorems 5 and 6 we get the following characterization for the curves of ridges of order higher or equal to n:

Corollary 3. *Let m_0 be a non umbilic point:*

a) *The point m_0 lies on a ridge of order higher or equal to $n-1$ for the i-th principal direction if and only if the i-th focal hypersphere of M coincides with the osculating hypersphere of the i-th curvature line at the point m_0.*

b) *The points in the ridges of order higher or equal than n of a hypersurface M are $(n-1)$-vertices of its curvature lines.*

We finally see that for generically immersed hypersurfaces the ridges of order n (isolated points) can also be detected as zeros of conformally invariant 1-forms along curves made of ridges of order $\geq n-1$. Again, by saying that ω_φ is a conformally invariant 1-form along a curve φ in $\mathcal{R}_{n-1}$ we understand that if $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is any conformal map, then $\phi \circ \varphi = \bar{\varphi}$ is a curve contained in the subset of ridges of order $\geq n$ of $\phi(M)$ and $\phi^*(\omega_{\bar{\varphi}}) = \omega_\varphi$.

Theorem 7. *Let $\varphi(t)$ be a parametrization by arc-length of any of the curves given by the connected components of the subset $\mathcal{R}_{n-1}(M)$, i.e. $\varphi(t)$ is a union of ridges of order $\geq n-1$. Then any conformal map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ transforms connected components of $\mathcal{R}_{n-1}(M)$ into connected components of $\mathcal{R}_{n-1}(h(M))$ and the differential 1-form $\omega(t) = \|K_i(t)d_{\alpha(t)}g(\alpha'(t)) + N'_{g(\alpha(t))}(d_{\alpha(t)}g(\alpha'(t)))\|dt$, is a conformal invariant on the curve $\varphi(t)$.*

Proof. Given the curve φ consider the osculating hyperspheres of the hypersurface M corresponding to two nearby points on it. By applying the generalized squared inversive distance between them and by expanding in Taylor series, we obtain

$$d^2(h) = 1 - \frac{\|a'\|^2 - r'^2}{r^2} h^2 + O(h^3).$$

A simple calculation leads to

$$\begin{aligned} \|c_i'(t)\|^2 - r_i'^2(t) &= \|d_{\alpha(t)}g(\alpha'(t)) + \tfrac{1}{K_i(t)} N'_{g(\alpha(t))}(d_{\alpha(t)}g(\alpha'(t)))\|^2 \\ &= \frac{\|K_i(t)d_{\alpha(t)}g(\alpha'(t)) + N'_{g(\alpha(t))}(d_{\alpha(t)}g(\alpha'(t)))\|^2}{K_i^2(t)}. \end{aligned}$$

Now, by using again a similar argument to the one used in the case of the curvature lines, we consider the function $\sqrt{1-d^2(h)}$ and we get that

$$\omega(t) = \|K_i(t)d_{\alpha(t)}g(\alpha'(t)) + N'_{g(\alpha(t))}(d_{\alpha(t)}g(\alpha'(t)))\|dt,$$

is a conformal invariant defined over the curve $\varphi(t)$. □

Corollary 4. *A non umbilic point m_0 over $\mathcal{R}_{n-1} \subset M - U(M)$ belongs to a ridge of order higher or equal to n if and only if it is a zero of the 1-form associated to the curve φ as in the theorem above.*

Proof. We know that

$$\|K_i(t_0)d_{\alpha(t_0)}g(\alpha'(t_0)) + N'_{g(\alpha(t_0))}(d_{\alpha(t_0)}g(\alpha'(t_0)))\| = 0$$

$$\Updownarrow$$

$$N'_{g(\alpha(t_0))}(d_{\alpha(t_0)}g(\alpha'(t_0))) = -K_i(t_0)d_{\alpha(t_0)}g(\alpha'(t_0))$$

and it follows from the theorem of O. Rodrigues for hypersurfaces that this is true if and only if the corresponding principal direction coincides with the tangent line to the ridge at the point m_0. But this is equivalent to asking that the point $m_0 = \varphi(t_0)$ belongs to a higher order ridge. □

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