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Constructing Non-regular Algebraic Spreads with Asymplecticly Complemented Regulization

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Abstract. We give an application of the second extension of the Thas-Walker construction and exhibit a 4-parameter family $\mathcal{F}$ of explicit examples of spreads of $\mathrm{PG}(3,\mathbb{R})$ with asymplecticly complemented regulization. In $\mathcal{F}$ there are symplectic spreads and also asymplectic algebraic spreads. A spread $\mathcal{S}$ of $\mathrm{PG}(3,\mathbb{R})$ is called rigid if, apart from the identity, there exists no collineation leaving $\mathcal{S}$ invariant; a rigid spread $\mathcal{S}$ is said to be hyperrigid if there exists no duality leaving $\mathcal{S}$ invariant. The family $\mathcal{F}$ contains hyperrigid algebraic spreads as well as rigid algebraic spreads which are not hyperrigid.

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1. Introductory survey

The present article continues a series of nine papers [20]–[28] the author wrote on constructions of spreads, hence we give a survey of the investigations done up till now in order to position the present paper and to make reading easier.

1.1 Posing the problems. All locally compact 4-dimensional translation planes which admit an at least 7-dimensional collineation group were classified by D. Betten; cf. [30, Chapter 73]. As contrast to his results D. Betten asked for an explicit example of a 4-dimensional translation plane with smallest possible collineation group, i.e., a translation plane which admits no collineation except translations and homotheties; cf. [3, p. 140]. Equivalent to Betten's problem is

Task R. *Give an explicit example of a rigid topological spread of the real projective* 3*-space* $\mathrm{PG}(3,\mathbb{R})$.

A spread $\mathcal{S}$ of a projective space Π is called *rigid*, if the only collineation leaving $\mathcal{S}$ invariant is the identity. A spread $\mathcal{S}$ of $\mathrm{PG}(3,\mathbb{R})$ is *topological*, if $\mathcal{S}$ represents a 4-dimensional translation plane. The full collineation group of a rigid topological spread of $\mathrm{PG}(3,\mathbb{R})$ is 5-dimensional; cf. [30, p.395]. When we stress the line geometric aspects of a spread $\mathcal{S}$ of a projective 3-space Π_3 we must also take dualities of Π_3 into account. A rigid spread $\mathcal{S}$ of Π_3 is called *hyperrigid*, if there exists no duality of Π_3 leaving $\mathcal{S}$ invariant.

Task HR. *Give an explicit example of a hyperrigid topological spread of* $\mathrm{PG}(3,\mathbb{R})$.

Task R is solved in [20] and [28] by tacking together partial spreads along common reguli. In [28, Theorem 3] we exhibit a rigid topological spread of $\mathrm{PG}(3,\mathbb{R})$ which is not hyperrigid; this spread is built up by parts of four different regular spreads. Task HR is solved by [28, Theorem 4] where we exhibit spreads which are built up by parts of five different regular spreads. These "patchwork" solutions of Task R and HR in [20] and [28] make us ask for more aesthetical solutions, hence

Task AR. *Give an explicit example of an algebraic rigid spread of* $\mathrm{PG}(3,\mathbb{R})$.

Task AHR. *Give an explicit example of an algebraic hyperrigid spread of* $\mathrm{PG}(3,\mathbb{R})$.

A spread of $\mathrm{PG}(3,\mathbb{R})$ is called *algebraic*, if its Klein image is an algebraic subvariety of the Klein quadric. We may omit the demand "topological" in Task AR and AHR since we show in Section 8 of the present paper that each algebraic spread of $\mathrm{PG}(3,\mathbb{R})$ is topological. Weaker than Task AR and AHR is

Task A. *Construct a non-regular algebraic spread of* $\mathrm{PG}(3,\mathbb{R})$.

Our approach to the solution of Task A, AR, and AHR follows two guidelines:

G1. *We construct spreads as compositions of reguli.*

G2. *We conjecture that "in the neighborhood" of the regular spread there exist solutions of Task A, AR, and AHR.*

1.2 Regulizations. A first attempt are [21] and [22] where we give explicit examples of spreads of $\mathrm{PG}(3,\mathbb{R})$ whose collineation groups are 6-dimensional and which admit hyperbolic resp. parabolic regulizations in the sense of N. Knarr (cf. [23, Def. 1.1] or [16, p. 35]). The immediate addition of an elliptic supplement to [21] and [22] fails because of two obstacles: The applied constructions could not be modified to an elliptic case without using the complex extension of $\mathrm{PG}(3,\mathbb{R})$ and the same holds for Knarr's definition. Hence we give in [23] an equivalent definition which also comprises the elliptic case:

Definition 1. *Let* $\Pi_3 = \mathrm{PG}(3,\mathbb{K})$ *be a projective* 3*-space with commutative coordinating field* $\mathbb{K}$. *A proper regulus* $\mathcal{R}$ *of* Π_3 *is a set of lines meeting three given mutually skew lines, by* $\mathcal{R}^c$ *we denote the complementary regulus. A single line is improper regulus and defined to be self-complementary. By a regulization of a spread* $\mathcal{S}$ *of* Π_3 *we mean a collection* Σ *of reguli contained in* $\mathcal{S}$ *such that* Σ *contains at most two improper reguli and such that each element* *of $\mathcal{S}$ is member of exactly one regulus of Σ or of all reguli of Σ;* cf. [23, Def. 1.2]. *The set of all lines obtained by taking the union of complementary reguli to the reguli of Σ is called the complementary congruence $\mathcal{S}^c_\Sigma$ of $\mathcal{S}$ with respect to Σ; in symbols $\mathcal{S}^c_\Sigma := \cup(\mathcal{R}^c \mid \mathcal{R} \in \Sigma)$. If $\mathcal{S}^c_\Sigma$ happens to be a non-degenerate linear congruence of lines (hyperbolic, parabolic, or elliptic), then Σ is called net generating regulization (hyperbolic, parabolic, or elliptic);* cf. [23, Def. 1.3]. *If $\mathcal{S}^c_\Sigma$ belongs to a single linear complex of lines, then we say that Σ is a unisymplecticly complemented regulization of $\mathcal{S}$;* cf. [25, Def. 1]. *If $\mathcal{S}^c_\Sigma$ belongs to no linear complex of lines, then Σ is named asymplecticly complemented regulization of $\mathcal{S}$;* cf. [27, Def. 1].

1.3 The Thas-Walker construction and its extensions. The concepts of Definition 1 together with Klein's correspondence λ of line geometry lead without constraint to the Thas-Walker construction and its two extensions. The Klein image of a proper (improper) regulus is called *proper (improper) conic.* In Π_3 we start from a spread $\mathcal{S}$ with regulization Σ and study following collection of conics: $\{\lambda(\mathcal{R}^c) \mid \mathcal{R} \in \Sigma\} =: \mathbf{F}$.

Case 0: Σ is net generating. By [23, Proposition 3.1], $\mathbf{F}$ is a flock of the quadric $\lambda(\mathcal{S}^c_\Sigma)$ which is elliptic or hyperbolic or a cone, if Σ is elliptic or hyperbolic or parabolic, respectively. A *flock* of a quadric Q of $\mathrm{PG}(3,\mathbb{K})$, $\mathbb{K}$ commutative, is a collection of disjoint conics which partitions Q and which contains no improper conic, if Q is hyperbolic, exactly one improper conic, if Q is a cone, and at most two improper conics, if Q is elliptic; cf. [23, Def. 3.1]. Conversely, let $\mathbf{F}$ be a flock of a quadric Q embedded into the Klein quadric H_5 and put

$$\bigcup_{k\in\mathbf{F}}\Big(\lambda^{-1}(k)\Big)^c =: \mathcal{S}(\mathbf{F}) \quad\text{and}\quad \Big\{\Big(\lambda^{-1}(k)\Big)^c \mid k\in\mathbf{F}\Big\} =: \Sigma(\mathbf{F}); \tag{1}$$

then $\mathcal{S}(\mathbf{F})$ is a spread of $\mathrm{PG}(3,\mathbb{K})$ with the net generating regulization $\Sigma(\mathbf{F})$ (cf. [23, Proposition 3.3]) and $\mathcal{S}(\mathbf{F})$ is also a dual spread (cf. [23, Theorem 2.8]).

Remark 1. The procedure of winning a spread from a flock via (1) is known from finite geometry as *Thas-Walker construction*; cf. [12, pp. 7–8], [32, p. 95], [35], [36]. In [23] we show that the Thas-Walker construction is valid in the infinite (commutative) case, too. In the finite case, i.e., in $\mathrm{PG}(3,q)$, a flock of a quadric Q is defined as a set of $q-1$ or $q+1$ or q conics of Q according Q is elliptic, hyperbolic, or a cone. Apart from two exceptional points an elliptic flock uniquely covers the carrier quadric. Note that in the infinite elliptic case we impose a weaker condition; cf. [23, Def. 3.1 and Remark 3.1].

Case 1: Σ is unisymplecticly complemented. By [25, p. 239 (S3)], the complementary congruence $\mathcal{S}^c_\Sigma$ is contained in a single linear complex $\mathcal{G}$ of lines which must be general. By [25, Proposition 1], $\mathbf{F}$ is a flockoid of the Lie quadric $\lambda(\mathcal{G})$. A *flockoid* $\mathbf{F}$ of a Lie quadric L_4 of $\mathrm{PG}(4,\mathbb{K})$, $\mathbb{K}$ commutative, is a collection of (proper or improper) conics of L_4 such that $\mathbf{F}$ contains at most two improper conics and such that for each 1-dimensional subspace ℓ of L_4 there exists exactly one conic $k \in \mathbf{F}$ with $\ell \cap k \neq \emptyset$; cf. [25, Def. 3]. If conversely $\mathbf{F}$ is a flockoid of a Lie quadric L_4 embedded into the Klein quadric H_5, then the line set $\mathcal{S}(\mathbf{F})$ from (1) is a spread of $\mathrm{PG}(3,\mathbb{K})$ and $\Sigma(\mathbf{F})$ from (1) is either a unisymplecticly complemented or an elliptic regulization of $\mathcal{S}(\mathbf{F})$ (cf. [25, Proposition 2]) and $\mathcal{S}(\mathbf{F})$ is also a dual spread (cf. [25, Corollary 1]).

Remark 2. The author calls the procedure of winning a spread from a flockoid of a Lie quadric via (1) in the subsequent *first extension of the Thas-Walker construction.* Note the difference between symplectic spreads and spreads with unisymplecticly complemented regulization; in [26, Section 5, Type 1 and 2] we give explicit examples of asymplectic spreads with unisymplecticly complemented regulization and in Section 7 of the present paper we give explicit examples of symplectic spreads with asymplecticly complemented regulization. Nevertheless there is following connection:

Lemma 1. *Let $\mathcal{S}$ be a spread of* $\mathrm{PG}(3,\mathbb{K})$*,* $\mathbb{K}$ *commutative, with a unisymplecticly complemented regulization* Σ*. Then the complementary congruence* $\mathcal{S}^{\mathrm{c}}_{\Sigma}$ *is a symplectic spread.*

Proof. Put $i(\Sigma) := \# \cap (\mathcal{R} \mid \mathcal{R} \in \Sigma)$; cf. [27, Def. 3]. By [23, Remark 2.4], $i(\Sigma) \in \{0,1,2\}$. If $i(\Sigma) \in \{1,2\}$, then Σ is parabolic or hyperbolic according to [23, Remark 2.5] and [23, Remark 2.6] and this contradicts the assumption that Σ is unisymplecticly complemented. Hence $i(\Sigma) = 0$ and, by [23, Remark 2.9], $\mathcal{S}^{\mathrm{c}}_{\Sigma}$ is a spread contained by definition in a linear congruence which by [25, p. 239 (S3)] must be general, i.e., $\mathcal{S}^{\mathrm{c}}_{\Sigma}$ is a symplectic spread. □

Remark 3. From finite geometry is known: Symplectic spreads of $\mathrm{PG}(3,q)$ and ovoids of the Lie quadric $Q(4,q)$ are equivalent objects; cf. [34], [18]. The definition of an ovoid can be taken over unchanged from the finite to infinite case: An *ovoid* of a Lie quadric L_4 of $\mathrm{PG}(4,\mathbb{K})$, $\mathbb{K}$ commutative, is a point set which has exactly one point in common with each line of L_4. Immediately we get:

If $\mathbf{F}$ *is a flockoid of the Lie quadric* L_4*, then* $\cup(k \mid k \in \mathbf{F})$ *is an ovoid of* L_4*.*

Only a few classes of ovoids of $Q(4,q)$ are known:

(1) the classical ovoids,

(2) for q even ovoids of $Q(4,q) \subset \mathrm{PG}(4,q)$ which can be projected into Tits ovoids of $\mathrm{PG}(3,q)$,

(3) for q odd: (3a) the semifield Kantor ovoid $\mathcal{K}_1$,

(3b) the non-semifield Kantor ovoid $\mathcal{K}_2$,

(3c) the Thas-Payne ovoids, and,

(3d) the Penttila-Williams ovoid of $Q(4,3^5)$; cf. [34], [19].

Which of these ovoids of $Q(4,q)$ carries a flockoid? An elliptic flock of a classical ovoid is also a flockoid of $Q(4,q)$; cf. [25, Remark 9]. By [5], a Tits ovoid carries no conic. By [33, p. 230], the semifield Kantor ovoid $\mathcal{K}_1$ can be decomposed in just one way into a set of conics having a common point, but this set is no flockoid since any two different conics of a flockoid are disjoint; cf. [25, Lemma 3(i)]. For the ovoids from (3b), (3c), and (3d) no decomposition into conics is known to the author.

Case 2: Σ is asymplecticly complemented. By [27, Proposition 1], $\mathbf{F}$ is a flocklet of the Klein quadric H_5. A *flocklet* $\mathbf{F}$ of the Klein quadric H_5 of $\mathrm{PG}(5,\mathbb{K})$, $\mathbb{K}$ commutative, is a collection of (proper or improper) conics of H_5 such that $\mathbf{F}$ contains at most two improper conics and such that for each Latin plane γ of H_5 there exists exactly one conic $k \in \mathbf{F}$ with $\gamma \cap k \neq \emptyset$; cf. [27, Def. 2]. If conversely $\mathbf{F}$ is a flocklet of the Klein quadric H_5, then the line set $\mathcal{S}(\mathbf{F})$ from (1) is a spread of $\mathrm{PG}(3,\mathbb{K})$ and $\Sigma(\mathbf{F})$ from (1) is either an asymplecticly complemented or a unisymplecticly complemented or an elliptic regulization of $\mathcal{S}(\mathbf{F})$ (cf. [27, Proposition 2 and Remark 1]).

Remark 4. The author calls the procedure of winning a spread from a flocklet of the Klein quadric via (1) the *second extension of the Thas-Walker construction*. It is an open question whether a spread with asymplecticly complemented regulization must be a dual spread. In the finite and topological case each spread is also a dual spread (cf. [6] and [7], respectively), note however following fact: In $\mathrm{PG}(2t+1, \mathbb{K})$ with $t \geq 1$ and infinite field $\mathbb{K}$ there exists a spread which is not a dual spread; cf. [1, Teorema 2.2][1]. Therefore (and in contrast to Case 0 and 1) we have to consider in Case 2 also the dual of the second extension of the Thas-Walker construction. By a *flockling* $\mathbf{F}$ of the Klein quadric H_5 of $\mathrm{PG}(5, \mathbb{K})$, $\mathbb{K}$ commutative, we mean a collection of (proper or improper) conics of H_5 such that $\mathbf{F}$ contains at most two improper conics and such that for each Greek plane δ of H_5 there exists exactly one conic $k \in \mathbf{F}$ with $\delta \cap k \neq \emptyset$; cf. [27, Def. 2]. If $\mathbf{F}$ is a flockling of H_5, then $\bigcup_{k\in\mathbf{F}} \left(\lambda^{-1}(k)\right)^c$ is a dual spread.

Lemma 2. *Let $\mathcal{S}$ be a spread of $\mathrm{PG}(3, \mathbb{K})$, $\mathbb{K}$ commutative, with an asymplecticly complemented regulization Σ. Then the complementary congruence $\mathcal{S}^c_\Sigma$ is an asymplectic spread.*

Proof. Take over the proof of Lemma 1, mutatis mutandis. □

Remark 5. In finite geometry one means by an *ovoid of the Klein quadric $Q^+(5, q)$* a point set of $Q^+(5, q)$ meeting each plane of $Q^+(5, q)$ in just one point; cf. [4, p. 31]. In the infinite case it is advisable to use two concepts: An *ovoilet of the Klein quadric H_5* is a point set of H_5 meeting each Latin plane of H_5 in just one point and an *ovoiling of H_5* is a point set of H_5 meeting each Greek plane of H_5 in just one point. From [1, Teorema 2.2] follows that there exist ovoilets which are not ovoilings. Immediately we get:

If $\mathbf{F}$ is a flocklet of the Klein quadric H_5, then $\cup(k \mid k \in \mathbf{F})$ is an ovoilet of H_5. If $\mathbf{F}$ is a flockling of the Klein quadric H_5, then $\cup(k \mid k \in \mathbf{F})$ is an ovoiling of H_5.

In the finite case the concepts ovoilet and ovoiling coincide. Examples of non-classical ovoids of $Q^+(5, q)$ can be found in [4] and [9], it seems to be unknown which of these ovoids carries a flocklet.

Remark 6. Each elliptic flock can be interpreted as flockoid of a suitable Lie quadric (cf. [25, Remark 9]), this is not valid for hyperbolic or parabolic flocks (cf. [25, Remark 8]). Each flockoid can be interpreted as well as flocklet and flockling (cf. [27, Remark 3]), but it is an open question whether each flocklet must be flockling.

1.4 Thas-Walker sets. Assume $\mathrm{Char}\,\mathbb{K} \neq 2$ ($\mathbb{K}$ commutative), let $E =: Q_3$ be an elliptic quadric of $\mathrm{PG}(3, \mathbb{K})$, $L_4 =: Q_4$ be a Lie quadric of $\mathrm{PG}(4, \mathbb{K})$, $H_5 =: Q_5$ be the Klein quadric of $\mathrm{PG}(5, \mathbb{K})$, and denote the polarity of Q_j by π_j ($j = 3, 4, 5$). A proper conic of Q_j is uniquely determined by the $(j-3)$-dimensional subspace $\pi_j(\mathrm{span}\,k)$, a collection $\mathbf{C}$ of proper conics of Q_j is uniquely determined by the set $\{\pi_j(\mathrm{span}\,k) \mid k \in \mathbf{C}\}$, $j = 3, 4, 5$.

Let T be a set of $(j-3)$-dimensional subspaces of $\mathrm{PG}(j, \mathbb{K})$ and put

$$T' := \{X \in T \mid \pi_j(X) \cap Q_j \neq \emptyset\}, \quad \text{and} \quad \mathbf{F}_j(T) := \{\pi_j(X) \cap Q_j \mid X \in T'\}. \qquad (2)$$

[1] For spreads which are not dual spreads see also [6] and [14].

Case $j = 3$: We say T is a *Thas-Walker point set with respect to the elliptic quadric Q_3*, if $\mathbf{F}_3(T)$ is a flock of Q_3.

Case $j = 4$: We say T is a *Thas-Walker line set with respect to the Lie quadric Q_4*, if $\mathbf{F}_4(T)$ is a flockoid of Q_4.

Case $j = 5$: We say T is a *Thas-Walker plane set of Latin type with respect to the Klein quadric Q_5*, if $\mathbf{F}_5(T)$ is a flocklet of Q_5.

If T is a Thas-Walker set with respect to the quadric Q_j, then the spread $\mathcal{S}(\mathbf{F}_j(T))$ constructed from $\mathbf{F}_j(T)$ via (1) has a Klein image $\lambda(\mathcal{S}(\mathbf{F}_j(T)))$ which is on H_5 and on the cone having the $(4-j)$-dimensional vertex $\pi_5(\operatorname{span} Q_j)$ and the directrix T $(j = 3, 4, 5)$, in other words, we get $\lambda(\mathcal{S}(\mathbf{F}_j(T)))$ by projecting T from $\pi_5(\operatorname{span} Q_j)$ onto H_5.

Initial examples. The latitudinal circles of a sphere Q_3 of $\mathrm{PG}(3, \mathbb{R})$ together with North pole N and South pole S form a flock $\mathbf{F}_{lat3}$ of Q_3. The range T_{03} of points on $N \vee S =: c$ is a Thas-Walker point set with respect to Q_3 satisfying $\mathbf{F}_3(T_{03}) = \mathbf{F}_{lat3}$. We embed the sphere Q_3 together with $\mathbf{F}_{lat3}$ into a Lie quadric Q_4, then $\mathbf{F}_{lat3}$ is a flockoid $\mathbf{F}_{lat4}$ of Q_4. All lines incident with the point $\pi_4(\operatorname{span} Q_3) =: V$ and meeting c form a pencil T_{04} of lines such that T_{04} is a Thas-Walker line set with respect to Q_4 satisfying $\mathbf{F}_4(T_{04}) = \mathbf{F}_{lat4}$. We embed the Lie quadric Q_4 together with $\mathbf{F}_{lat4}$ into the Klein quadric $H_5 = Q_5$, then $\mathbf{F}_{lat4}$ is a flocklet $\mathbf{F}_{lat5}$ of Q_5. All planes incident with the line $\pi_5(\operatorname{span} Q_3) =: d$ and meeting c form a pencil T_{05} of planes such that T_{05} is a Thas-Walker plane set of Latin type with respect to Q_5 satisfying $\mathbf{F}_5(T_{05}) = \mathbf{F}_{lat5}$.

$j = 3$. Following the guideline G2 we show in [24, Section 3.1] that in the neighborhood of the range T_{03} of points there exist rational cubics $w_{\varepsilon,\varphi}$ ($\varepsilon, \varphi \in \mathbb{R}$ are deviations and $w_{0,0} = T_{03}$) such that $w_{\varepsilon,\varphi}$ are Thas-Walker point sets with respect to Q_3. The spreads $\mathcal{S}(\mathbf{F}_3(w_{\varepsilon,\varphi}))$ are algebraic and for $(\varepsilon, \varphi) \neq (0, 0)$ non-regular; cf. [24, Theorem 3.2.1] together with [26, Remark 16] and [24, Theorem 3.3.1]. Thus we have solutions of Task A. Because of [23, Lemma 1.1], a spread of $\mathrm{PG}(3, \mathbb{R})$ with net generating, especially elliptic regulization is never rigid. By the way, the determination of all collineations of $\mathrm{PG}(3, \mathbb{R})$ leaving a spread $\mathcal{S}(\mathbf{F}_3(w_{\varepsilon,\varphi}))$ with $\varepsilon\varphi \neq 0$ invariant is equivalent to the determination of all collineations which leave invariant the elliptic quadric $Q_3 = E$, a distinguished point pair p on Q_3, and the skew cubic $w_{\varepsilon,\varphi}$; see [24, p. 140–141].

$j = 4$. In [26, Section 4] we replace the vertex V of T_{04} with a conic $c_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ in the neighborhood of V ($\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R}$ are deviations and $c_{0,0,0} = V$) and generate a line set $A_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ by a projectivity from c $(= g_1)$ onto $c_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ such that $A_{0,0,0} = T_{04}$; appropriate bounds for $\varepsilon_1, \varepsilon_2, \varepsilon_3$ guarantee that $A_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ is a Thas-Walker line set with respect to Q_4; cf. [26, Lemma 3]. The spreads $\mathcal{S}(\mathbf{F}_4(A_{\varepsilon_1,\varepsilon_2,\varepsilon_3})) =: \mathcal{A}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ with $\varepsilon_1\varepsilon_2 \neq 0$ are algebraic (see [26, Theorem 3]) and rigid, if $\varepsilon_3 \neq 0$ (see [26, Theorem 5]). Thus we have solutions of Task AR, but the spreads $\mathcal{A}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ with $\varepsilon_1\varepsilon_2\varepsilon_3 \neq 0$ are not hyperrigid; cf. [26, Remark 22 and 23]. For the spreads $\mathcal{A}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ with $\varepsilon_1\varepsilon_2 \neq 0$ it is possible to give beside the algebraic also a rational parametric description; see [26, Theorem 3 and 1]. This fact and properties of the normal ruled surface corresponding to the line set $A_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ enable us to determine all automorphic collineations of $\mathcal{A}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ by synthetic considerations and by comparing coefficients[2]; see [26, p. 330–335].

[2]If we wanted full analogy with the cases $j = 3$ and $j = 5$, we would have to alter the proceeding and

$j = 5$. This case is dealt with in the present paper. To the lines c and d we add a line $e_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4} =: e$ which belongs to the neighborhood of d ($\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \mathbb{R}$ are deviations). We generate a plane set $B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ by projectivities between c, d, and e such that $B_{0,0,0,0} = T_{05}$; appropriate bounds for $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ guarantee that $B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ is a Thas-Walker plane set of Latin type with respect to Q_5; cf. Lemma 4. If $\varepsilon_1\varepsilon_2 \neq 0$, then c, d, e are mutually skew lines of $\mathrm{PG}(5,\mathbb{R})$, i.e., $B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ is a 2-*regulus*, cf. [15, p. 199], and the corresponding point set is a Segre variety $S_{2;1}$, cf. [8, p. 116], [15, p. 190]. Each spread $\mathcal{S}(\mathbf{F}_5(B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4})) =: \mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ with $\varepsilon_1\varepsilon_2 \neq 0$ is an algebraic asymplectic spread with asymplecticly complemented regulization which is the only regulization of $\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$; see Theorem 1[3] and 2. Synthetic considerations show that the determination of all collineations and dualities of $\mathrm{PG}(3,\mathbb{R})$ leaving $\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ invariant is equivalent to the determination of all collineations of $\mathrm{PG}(5,\mathbb{R})$ which leave invariant the Klein quadric H_5, a distinguished point pair p on H_5, and the Segre variety $S_{2;1}$ corresponding to $B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$; see Corollary 1 and Lemma 6. We get the common automorphic collineations of H_5, p, and $S_{2;1}$ by comparing coefficients which involves longer computer aided calculations with numerous ramifications. Result: For $\varepsilon_1\varepsilon_2 \neq 0$, $\varepsilon_2 \neq \pm\,\varepsilon_1$, and $\varepsilon_4 \neq -\varepsilon_3$ the spread $\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ is hyperrigid; see Theorem 3. Thus we have solutions of Task AHR.

In Section 7 we shortly discuss the special case with $(\varepsilon_2, \varepsilon_4) = (0, 0)$ and $\varepsilon_1\varepsilon_3 \neq 0$. Each spread $\mathcal{B}_{\varepsilon_1,0,\varepsilon_3,0}$ is symplectic and admits an asymplecticly complemented regulization (see Theorem 5), but symplectic spreads are never hyperrigid (see Lemma 7). Each spread $\mathcal{B}_{\varepsilon_1,0,\varepsilon_3,0}$ is properly contained in the complete intersection of a general linear complex and a cubic complex of lines.

1.5 Table of solutions. The subsequent table shows where solutions of the tasks from Subsection 1.1 can be found.

Reference	Task R	Task HR	Task A
[28, Theorem 3]	yes	no	no
[28, Theorem 4]	yes	yes	no
[24, Theorem 3.2.1 and 3.3.1]	no	no	yes
[26]	yes	no	yes
present paper	yes	yes	yes

Table 1

2. Thas-Walker plane sets of Latin type in terms of coordinates

Let λ be the Klein mapping of the lines of $\Pi = \mathrm{PG}(3,\mathbb{K})$ onto the points of the Klein quadric H_5 which is embedded into a projective 5-space $\Pi_5 = \mathrm{PG}(5,\mathbb{K})$ with point set $\mathcal{P}_5$. For the

show: The determination of all collineations of $\mathrm{PG}(3,\mathbb{R})$ leaving a spread $\mathcal{A}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ with $\varepsilon_1\varepsilon_2 \neq 0$ invariant is equivalent to the determination of all collineations which leave invariant the Lie quadric $Q_4 = L_4$, a distinguished point pair p on Q_4, and the normal ruled surface corresponding to the line set $A_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$.

[3]Theorem 1 answers the question posed in [27, p. 487] for explicit examples of asymplectic algebraic spreads with asymplecticly complemented regulization.

rest of this paper, we assume that Π and Π_5 are the projective spaces on $\mathbb{K}^4$ and $\mathbb{K}^4 \wedge \mathbb{K}^4$, respectively, and that λ maps the line $\mathbf{c}\mathbb{K} \vee \mathbf{d}\mathbb{K}$ of Π onto $(\mathbf{c} \wedge \mathbf{d})\mathbb{K} \in \mathcal{P}_5$. The standard basis $\mathbf{B}$ of $\mathbb{K}^4$ yields the ordered basis $(\mathbf{p}_0, \ldots, \mathbf{p}_5) =: \mathbf{B}_5$ of $\mathbb{K}^4 \wedge \mathbb{K}^4$ with

$$\mathbf{p}_0 := \mathbf{b}_0 \wedge \mathbf{b}_1,\ \mathbf{p}_1 := \mathbf{b}_0 \wedge \mathbf{b}_2,\ \mathbf{p}_2 := \mathbf{b}_0 \wedge \mathbf{b}_3,\ \mathbf{p}_3 := \mathbf{b}_2 \wedge \mathbf{b}_3,\ \mathbf{p}_4 := \mathbf{b}_3 \wedge \mathbf{b}_1,\ \mathbf{p}_5 := \mathbf{b}_1 \wedge \mathbf{b}_2.$$

Thus

$$H_5 = \{\mathbf{p}\mathbb{K} \in \mathcal{P}_5 \mid \mathbf{p} = \sum_{k=0}^{5} \mathbf{p}_k p_k \text{ and } h_5(\mathbf{p}) := p_0p_3 + p_1p_4 + p_2p_5 = 0\}. \tag{3}$$

To the quadratic form h_5 there belongs the symmetric bilinear form Ω with

$$\Omega(\mathbf{p}, \mathbf{q}) := h_5(\mathbf{p} + \mathbf{q}) - h_5(\mathbf{p}) - h_5(\mathbf{q}) = p_0q_3 + p_3q_0 + p_1q_4 + p_4q_1 + p_2q_5 + p_5q_2 \tag{4}$$

for $\mathbf{p} = \sum_{k=0}^{5} \mathbf{p}_k p_k,\ \ \mathbf{q} = \sum_{k=0}^{5} \mathbf{p}_k q_k$. Now Ω describes the polarity π_5 of the Klein quadric H_5 [31, p. 9]; note that we do not assume $\operatorname{Char}\mathbb{K} \neq 2$.

We generate a set B of planes of Π_5 by joining points of equal parameter of three "directing curves" c, d, and e given by parametric representations. Thus

$$c = \{\mathbf{c}_u\mathbb{K} \mid \mathbf{c}_u = \sum_{k=0}^{5} \mathbf{p}_k c_k(u) \ \text{ and } \ u \in \mathbb{U} \subseteq \mathbb{K} \cup \{\infty\}\}, \tag{5}$$

$$d = \{\mathbf{d}_u\mathbb{K} \mid \mathbf{d}_u = \sum_{k=0}^{5} \mathbf{p}_k d_k(u) \ \text{ and } \ u \in \mathbb{U} \subseteq \mathbb{K} \cup \{\infty\}\}, \tag{6}$$

$$e = \{\mathbf{e}_u\mathbb{K} \mid \mathbf{e}_u = \sum_{k=0}^{5} \mathbf{p}_k e_k(u) \ \text{ and } \ u \in \mathbb{U} \subseteq \mathbb{K} \cup \{\infty\}\}, \tag{7}$$

where c_k, d_k, and e_k are mappings from $\mathbb{U}$ into $\mathbb{K}$ such that

$$\{\mathbf{c}_u, \mathbf{d}_u, \mathbf{e}_u\} \ \text{ is a triangle for each } u \in \mathbb{U}; \tag{8}$$

$$B = \{\beta_u := \mathbf{c}_u\mathbb{K} \vee \mathbf{d}_u\mathbb{K} \vee \mathbf{e}_u\mathbb{K} \mid u \in \mathbb{U}\}. \tag{9}$$

In order to have a clearly arranged description of the set B, we define (3×6)-matrices

$$M_B(u) := \begin{pmatrix} c_0(u) & \cdots & c_5(u) \\ d_0(u) & \cdots & d_5(u) \\ e_0(u) & \cdots & e_5(u) \end{pmatrix} \ \text{ for } u \in \mathbb{U}. \tag{10}$$

The subsequent Lemma 3 sums up the conditions which guarantee that B is a Thas-Walker plane set of Latin type with respect to the Klein quadric (3). In spite of its length, Lemma 3 is nearly trivial, since it is only the translation of (TWLa2)–(TWLa4) from [27, Lemma 3][4] into coordinates; compare also [26, Lemma 1].

[4] Note that in [27, Lemma 3] we had to assume $\operatorname{Char}\mathbb{K} \neq 2$.

Lemma 3. *Assume* $\operatorname{Char}\mathbb{K} \neq 2$. *The set B of planes described by* (5) *up to* (9) *is a Thas-Walker plane set of Latin type with respect to the Klein quadric* (3) *if, and only if, the following six conditions hold*[5]*:*

(C2) $\#(\mathbb{U}_e) \leq 2$ *with* $\mathbb{U}_e := \{u \in \mathbb{U} \mid F_3(u) = 0\}$ *wherein*

$$F_3(u) := \begin{vmatrix} \Omega(\mathbf{c}_u, \mathbf{c}_u) & \Omega(\mathbf{c}_u, \mathbf{d}_u) & \Omega(\mathbf{c}_u, \mathbf{e}_u) \\ \Omega(\mathbf{d}_u, \mathbf{c}_u) & \Omega(\mathbf{d}_u, \mathbf{d}_u) & \Omega(\mathbf{d}_u, \mathbf{e}_u) \\ \Omega(\mathbf{e}_u, \mathbf{c}_u) & \Omega(\mathbf{e}_u, \mathbf{d}_u) & \Omega(\mathbf{e}_u, \mathbf{e}_u) \end{vmatrix}.$$

(C3) *If* $b \in \mathbb{U}_e$, *then there exists exactly one point* $\mathbf{s}\mathbb{K} \in H_5$ *with* $\Omega(\mathbf{s}, \mathbf{c}_b) = \Omega(\mathbf{s}, \mathbf{d}_b) = \Omega(\mathbf{s}, \mathbf{e}_b) = 0$.

(C4) *Put* $C_4(\xi, \eta, \zeta, u) :=$

$$\begin{vmatrix} (-c_3 - \zeta\, c_1 + \eta\, c_2)(u) & (-c_4 + \zeta\, c_0 - \xi\, c_2)(u) & (-c_5 + \xi\, c_1 - \eta\, c_0)(u) \\ (-d_3 - \zeta\, d_1 + \eta\, d_2)(u) & (-d_4 + \zeta\, d_0 - \xi\, d_2)(u) & (-d_5 + \xi\, d_1 - \eta\, d_0)(u) \\ (-e_3 - \zeta\, e_1 + \eta\, e_2)(u) & (-e_4 + \zeta\, e_0 - \xi\, e_2)(u) & (-e_5 + \xi\, e_1 - \eta\, e_0)(u) \end{vmatrix}.$$

For each $(\xi, \eta, \zeta) \in \mathbb{K}^3$ *the equation* $C_4(\xi, \eta, \zeta, u) = 0$ *in the unknown* u *has exactly one solution in* $\mathbb{U}$.

(C5) *Put* $C_5(\xi, \eta, u) :=$

$$\begin{vmatrix} (c_3 + \xi\, c_4 + \eta\, c_5)(u) & (\eta\, c_0 - c_2)(u) & (-\xi\, c_0 + c_1)(u) \\ (d_3 + \xi\, d_4 + \eta\, d_5)(u) & (\eta\, d_0 - d_2)(u) & (-\xi\, d_0 + d_1)(u) \\ (e_3 + \xi\, e_4 + \eta\, e_5)(u) & (\eta\, e_0 - e_2)(u) & (-\xi\, e_0 + e_1)(u) \end{vmatrix}.$$

For each $(\xi, \eta) \in \mathbb{K}^2$ *the equation* $C_5(\xi, \eta, u) = 0$ *in the unknown* u *has exactly one solution in* $\mathbb{U}$.

(C6) *Put* $C_6(\xi, u) :=$

$$\begin{vmatrix} (c_4 + \xi\, c_5)(u) & (-\xi\, c_1 + c_2)(u) & -c_0(u) \\ (d_4 + \xi\, d_5)(u) & (-\xi\, d_1 + d_2)(u) & -d_0(u) \\ (e_4 + \xi\, e_5)(u) & (-\xi\, e_1 + e_2)(u) & -e_0(u) \end{vmatrix}.$$

For each $\xi \in \mathbb{K}$ *the equation* $C_6(\xi, u) = 0$ *in the unknown* u *has exactly one solution in* $\mathbb{U}$.

(C7) *Put* $C_7(u) :=$

$$\begin{vmatrix} c_0(u) & c_1(u) & c_5(u) \\ d_0(u) & d_1(u) & d_5(u) \\ e_0(u) & e_1(u) & e_5(u) \end{vmatrix}.$$

The equation $C_7(u) = 0$ *in the unknown* u *has exactly one solution in* $\mathbb{U}$.

[5] In order to have full correspondence with [26, Lemma 1] the conditions start with (C2).

Proof. We use the characterization of a Thas-Walker plane set of Latin type by the properties (TWLa2)–(TWLa4) given in [27, Lemma 3].

If $\beta_u \in B$, then

$$\delta_u := \pi_5(\beta_u) = \{\mathbf{p}\mathbb{K} \in \mathcal{P}_5 \mid \Omega(\mathbf{p}, \mathbf{c}_u) = \Omega(\mathbf{p}, \mathbf{d}_u) = \Omega(\mathbf{p}, \mathbf{e}_u) = 0\}. \tag{11}$$

An arbitrary point $(\mathbf{c}_u\xi + \mathbf{d}_u\eta + \mathbf{e}_u\zeta)\mathbb{K}$, $(\xi, \eta, \zeta) \in \mathbb{K} \setminus \{(0,0,0)\}$, of $\mathbf{c}_u\mathbb{K} \vee \mathbf{d}_u\mathbb{K} \vee \mathbf{e}_u\mathbb{K}$ is incident with the plane δ_u if, and only if, (ξ, η, ζ) is a solution of the system of linear equations $\Omega(\mathbf{c}_u\xi + \mathbf{d}_u\eta + \mathbf{e}_u\zeta, \mathbf{c}_u) = \Omega(\mathbf{c}_u\xi + \mathbf{d}_u\eta + \mathbf{e}_u\zeta, \mathbf{d}_u) = \Omega(\mathbf{c}_u\xi + \mathbf{d}_u\eta + \mathbf{e}_u\zeta, \mathbf{e}_u) = 0$ with determinant $F_3(u)$. As $\beta_u \cap \delta_u = \emptyset \Leftrightarrow F_3(u) \neq 0$, so (C2)⇔(TWLa2) and (C3)⇔(TWLa3)[6].

By applying the antiautomorphism π_5, (TWLa4) turns into the equivalent condition

(TWLa4*) *For each Latin plane γ of H_5 there exists exactly one plane δ_u in $D_{\mathbb{U}} := \{\delta_u \mid u \in \mathbb{U}\}$ with $\gamma \vee \delta_u \neq \mathcal{P}_5$ ($\Leftrightarrow$ $\gamma \cap \delta_u \neq \emptyset$).*

Next we apply λ^{-1} in order to replace the condition $\gamma \cap \delta_u \neq \emptyset$ with an equivalent condition in the 3-space Π. By $\mathcal{L}[P]$ we denote the star of lines incident with a point P of Π. We put $\lambda^{-1}(\pi_5(\mathbf{c}_u)) =: \mathcal{N}_{u,1}$, $\lambda^{-1}(\pi_5(\mathbf{d}_u)) =: \mathcal{N}_{u,2}$, $\lambda^{-1}(\pi_5(\mathbf{e}_u)) =: \mathcal{N}_{u,3}$; the linear complexes $\mathcal{N}_{u,i}$ ($i = 1,2,3$) of lines need not be general for each $u \in \mathbb{U}$. Let X be a point of γ with $X \in \delta_u$. Now $\lambda^{-1}(\gamma)$ is a star of lines with a vertex, say $Y \in \mathcal{P}$. As X and $\mathbf{c}_u\mathbb{K}$ are conjugate with respect to H_5, so $\lambda^{-1}(X) \in \mathcal{L}[Y] \cap \mathcal{N}_{u,1}$; analogously, $\lambda^{-1}(X) \in \mathcal{L}[Y] \cap \mathcal{N}_{u,i}$ for $i = 2,3$. Thus we have: $\gamma \cap \delta_u \neq \emptyset \Leftrightarrow \#(\mathcal{L}[Y] \cap \mathcal{N}_{u,1} \cap \mathcal{N}_{u,2} \cap \mathcal{N}_{u,3}) \geq 1$.

Now it is evident that the following condition is equivalent to (TWLa4) resp. (TWLa4*):

(CONP) *For each $Y \in \mathcal{P}$ there exists exactly one $u \in \mathbb{U}$ with*

$$\#((\mathcal{L}[Y] \cap \mathcal{N}_{u,1}) \cap (\mathcal{L}[Y] \cap \mathcal{N}_{u,2}) \cap (\mathcal{L}[Y] \cap \mathcal{N}_{u,3})) \geq 1.$$

How to express (CONP) in coordinates can be taken over from [26, Proof of Lemma 1] without any changes. □

Remark 7. Let B be a set of planes described by (5)–(9). Provided that (C2) and (C3) hold for B, then $\dim(\gamma \cap \beta_u) \in \{-1, 0\}$ for all pairs (γ, u) where γ is a Latin plane of H_5 and $u \in \mathbb{U}$. Furthermore, $\#(\mathcal{L}[Y] \cap \mathcal{N}_{u,1} \cap \mathcal{N}_{u,2} \cap \mathcal{N}_{u,3}) \in \{0, 1\}$ for all $(Y, u) \in \mathcal{P} \times \mathbb{U}$.

Proof. Assume, to the contrary, $\dim(\gamma \cap \beta_u) \in \{1, 2\}$; then $\beta_u \cap H_5$ contains a line, a contradiction to footnote 6. □

Remark 8. In Section 3, we aim at cubic equations $C_4(\xi, \eta, \zeta, u) = 0, \ldots, C_7(u) = 0$ in u. Hence we will choose linear functions c_j, d_j, e_j; in other words, the directing curves c, d, e will be linearly parametrized lines.

Remark 9. Lemma 3 comprises the first extension of the Thas-Walker construction, too, namely for certain constant functions c_j; cf. [26, (11) and Lemma 1]. In [26] we also aimed at cubic equations $C_4(\xi, \eta, \zeta, u) = 0, \ldots, C_7(u) = 0$ in u and the d_j were chosen as linear, the e_j as quadratic functions; cf. [26, Remark 2].

[6] From the proof of [27, Lemma 3] we read off: (C2) and (C3) guarantee that $\beta_u \cap H_5$ is either a (proper or improper) conic or empty for each $u \in \mathbb{U}$.

Remark 10. Lemma 3 comprises the elliptic case of the ordinary Thas-Walker construction, too, namely for certain constant functions c_j and d_j. We get cubic equations $C_4(\xi,\eta,\zeta,u) = 0,\dots, C_7(u) = 0$ in u, if the e_j are chosen as cubic functions. This idea is pursued in [24].

3. A family of Thas-Walker plane sets of Latin type

At the beginning of this Section we exhibit the setting (12) for a set $B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ of planes by using (5)–(8) and a matrix of the form (9). Subsequently we expose the geometric background of the setting (12) and finally we determine bounds for the deviations $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$ such that $B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ becomes a Thas-Walker plane set of Latin type.

For the rest of this paper we assume $\mathbb{K}=\mathbb{R}$. By $B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ we denote the set of planes described by the (3×6)-matrices:

$$M_B(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,u) := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & u \\ 1 & u & 0 & -1 & -u & 0 \\ -u(1+\varepsilon_1) & 1+\varepsilon_2 & \varepsilon_4 & u(1-\varepsilon_1) & -(1-\varepsilon_2) & -u\varepsilon_3 \end{pmatrix} \tag{12}$$

and

$$M_B(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\infty) := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ -(1+\varepsilon_1) & 0 & 0 & 1-\varepsilon_1 & 0 & -\varepsilon_3 \end{pmatrix} \tag{13}$$

for all $u\in\mathbb{R}$ and for $\varepsilon_j\in\mathbb{R}$.

In order to check (8), we form the submatrix of $M_B(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,u)$, $u\in\mathbb{R}$, consisting of the first three columns and the submatrix of $M_B(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\infty)$ consisting of the last three columns, and get for the values of the two corresponding determinants $1+\varepsilon_2+u^2(1+\varepsilon_1)$ and $1-\varepsilon_1$, respectively. Hence we have:

$$\text{If } |\varepsilon_1|<1 \text{ and } |\varepsilon_2|<1, \text{ then } \operatorname{rank}(M_B(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,u)) = 3 \text{ for all } u\in\mathbb{R}\cup\{\infty\}. \tag{14}$$

An arbitrary plane set $B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ of Π_5 yields the line set

$$\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4} := \bigcup\Big(\lambda^{-1}(\xi)\mid \xi\in B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\Big) \tag{15}$$

of Π; compare [27, Lemma 4].

First we give a short, but detailed description of the initial Thas-Walker sets T_{03}, T_{04}, and T_{05} (compare Section 1.4). For sake of convenience we use the basis $(\mathbf{p}''_0,\dots,\mathbf{p}''_5) =: \mathbf{B}''_5$ of $\mathbb{K}^4\wedge\mathbb{K}^4$ with

$$\mathbf{p}''_j = \mathbf{p}_j+\mathbf{p}_{j+3},\quad \mathbf{p}''_{j+3} = \mathbf{p}_j-\mathbf{p}_{j+3},\quad (j=0,1,2); \tag{16}$$

$$H_5 = \{\mathbf{p}\mathbb{K}\in\mathcal{P}_5 \mid \mathbf{p} = \sum_{k=0}^{5}\mathbf{p}''_k p''_k \text{ and } {p''_0}^2+{p''_1}^2+{p''_2}^2-{p''_3}^2-{p''_4}^2-{p''_5}^2 = 0\}, \tag{17}$$

compare [26, (4) and (5)]. For the elliptic quadric ("sphere") we choose

$$Q_3 = \{\mathbf{p}\mathbb{K}\in H_5 \mid p''_3 = p''_4 = 0\}, \tag{18}$$

and $N = (\mathbf{p}_2'' + \mathbf{p}_5'')\mathbb{R}$ as North pole, $S = (\mathbf{p}_2'' - \mathbf{p}_5'')\mathbb{R}$ as South pole of the latitudinal flock $\mathbf{F}_{lat3}$. Hence

$$T_{03} = c = N \vee S = \{C_u \mid u \in \mathbb{R} \cup \{\infty\}\} \text{ with}$$

$$C_u := (\mathbf{p}_2''(1+u)+\mathbf{p}_5''(1-u))\mathbb{R} = (\mathbf{p}_2+\mathbf{p}_5 u)\mathbb{R} \text{ for } u \in \mathbb{R} \text{ and } C_\infty = (\mathbf{p}_2''-\mathbf{p}_5'')\mathbb{R} = \mathbf{p}_5\mathbb{R}. \quad (19)$$

We embed Q_3 into the Lie quadric

$$L_4 = Q_4 = \{\mathbf{p}\mathbb{K} \in H_5 \mid p_3'' = 0\}, \quad (20)$$

then $V = \pi_4(\operatorname{span} Q_3) = \mathbf{p}_4''\mathbb{R}$ and $T_{04} = \{\mathbf{p}_4''\mathbb{R} \vee C_u \mid u \in \mathbb{R} \cup \{\infty\}\}$; cf. [26, (6), p. 315 Step 1]. Finally, $Q_4 \subset Q_5 = H_5$, $d = \pi_5(\operatorname{span} Q_3) = \mathbf{p}_3''\mathbb{R} \vee \mathbf{p}_4''\mathbb{R}$, and $T_{05} = \{C_u \vee \mathbf{p}_3''\mathbb{R} \vee \mathbf{p}_4''\mathbb{R} \mid u \in \mathbb{R} \cup \{\infty\}\}$. In order to describe T_{05} according to Remark 8, we choose $d = e = \mathbf{p}_3''\mathbb{R} \vee \mathbf{p}_4''\mathbb{R}$ and endow $d = e$ with two different linear parametrizations such that points with equal parameter correspond in an elliptic autoprojectivity of $d = e$ because of (8). For our examples we use on the one hand

$$d := \{D_u \mid u \in \mathbb{R} \cup \{\infty\}\} \text{ with}$$

$$D_u := (\mathbf{p}_3''+\mathbf{p}_4''u))\mathbb{R} = (\mathbf{p}_0+\mathbf{p}_1 u-\mathbf{p}_3-\mathbf{p}_4 u)\mathbb{R} \text{ for } u \in \mathbb{R} \text{ and } D_\infty = \mathbf{p}_4''\mathbb{R} = (\mathbf{p}_1-\mathbf{p}_4)\mathbb{R} \quad (21)$$

and on the other hand

$$e := \{E_u \mid u \in \mathbb{R} \cup \{\infty\}\} \text{ with}$$

$$E_u := (-\mathbf{p}_3''u + \mathbf{p}_4''))\mathbb{R} = (-\mathbf{p}_0 u + \mathbf{p}_1 + \mathbf{p}_3 u - \mathbf{p}_4)\mathbb{R} \text{ for } u \in \mathbb{R} \text{ and } E_\infty = \mathbf{p}_3''\mathbb{R} = (\mathbf{p}_0 - \mathbf{p}_3)\mathbb{R}. \quad (22)$$

Thus we have

$$T_{05} = \{\beta_u := C_u \vee D_u \vee E_u \mid u \in \mathbb{R} \cup \{\infty\}\}. \quad (23)$$

The first and second row of (12), (13) result from (19) and (21). Note that c and d are skew.

Remark 11. For the planes β_0 and β_∞ holds:

$$\beta_0 \cap H_5 = \{C_0\} \text{ and } \beta_\infty \cap H_5 = \{C_\infty\}, \quad (24)$$

i.e., $\lambda^{-1}(\beta_0)$ and $\lambda^{-1}(\beta_\infty)$ are improper reguli.

Next we replace the line e from (22) with a new linearly parametrized line which we also call e. This new e shall satisfy following four demands:

Demand 1. The lines c, d, and e shall be mutually skew, at least in the general case.
Demand 2. We want that (24) is valid also for the new linearly parametrized line e.
Demand 3. At least in the general case, the new line e shall not be contained in the 3-space $\pi_5(c)$. By the way, $\pi_5(c)$ is described by the equations $p_2'' = p_5'' = 0$.
Demand 4. At least in the general case, e and $\pi_5(c)$ shall span Π_5.

Remark 12. Aim of Demand 1 is that $B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ becomes a 2-regulus. For sake of convenience we pose Demand 2. To justify Demand 3 we consider the involutoric collineation ι_λ of Π_5 which fixes each point of c and each point of the 3-space $\pi_5(c)$. If $e \subset \pi_5(c)$, then each plane of $B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ is invariant under ι_λ which together with $\iota_\lambda(H_5) = H_5$ implies[7] $\iota(\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}) = \mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$. Since we aim at hyperrigid spreads we try to avoid the described situation by Demand 3. Finally, we sharpen Demand 3 by Demand 4.

[7] The collineation ι_λ of PG$(5, \mathbb{R})$ is induced by a transformation of PG$(3, \mathbb{R})$; compare Section 5.3.

For the new line e we make the subsequent setting:

$$e = \{(-\mathbf{f}\,u + \mathbf{g})\mathbb{R} \mid u \in \mathbb{R} \cup \{\infty\}\} \text{ with}$$

$$\mathbf{f} := \mathbf{p}''_3 + \mathbf{p}''_0 r_0 + \mathbf{p}''_1 r_1 + \mathbf{p}''_2 r_2 + \mathbf{p}''_5 r_5 \text{ and } \mathbf{g} := \mathbf{p}''_4 + \mathbf{p}''_0 s_0 + \mathbf{p}''_1 s_1 + \mathbf{p}''_2 s_2 + \mathbf{p}''_5 s_5, \quad r_i, s_i \in \mathbb{R}. \tag{25}$$

We note that e is skew to c in any case. Demand 1 is satisfied, if

$$r_0 s_1 - r_1 s_0 \neq 0. \tag{26}$$

The plane corresponding to the parameter 0 is spanned by the points C_0, D_0, and $\mathbf{g}\mathbb{R}$. Clearly, $(C_0 \vee D_0 \vee \mathbf{g}\mathbb{R}) \cap H_5 = \{C_0\}$ implies $\mathbf{g}\mathbb{R} \in \pi_5(C_0)$, hence $s_2 = s_5$; if $s_2 = s_5$, then $(C_0 \vee D_0 \vee \mathbf{g}\mathbb{R}) \cap H_5 = \{C_0\}$ is equivalent to $|\, s_0^2 + s_1^2| < 1$. From $(C_\infty \vee D_\infty \vee \mathbf{f}\mathbb{R}) \cap H_5 = \{C_\infty\}$ we deduce $r_2 = -r_5$ and $|\, r_0^2 + r_1^2| < 1$. Demand 2 is fulfilled, if

$$r_2 = -r_5, \quad s_2 = s_5, \quad |\, r_0^2 + r_1^2| < 1, \quad \text{and} \quad |\, s_0^2 + s_1^2| < 1. \tag{27}$$

As $e \subset \pi_5(c) \Leftrightarrow (r_2, r_5, s_2, s_5) = (0, 0, 0, 0)$, so Demand 3 is satisfied, if

$$(r_2, r_5, s_2, s_5) \neq (0, 0, 0, 0). \tag{28}$$

Finally, Demand 4 is fulfilled, if

$$r_2 s_5 - r_5 s_2 \neq 0. \tag{29}$$

In order to avoid an overboarding number of parameters we put $r_1 = s_0 = 0$. As new line e we use

$$(e_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4} =)\ e := \{E_u \mid u \in \mathbb{R} \cup \{\infty\}\} \text{ with}$$

$$E_u := \Big(-(\mathbf{p}''_3 + \mathbf{p}''_0 \varepsilon_1 + \mathbf{p}''_2 \frac{\varepsilon_3}{2} - \mathbf{p}''_5 \frac{\varepsilon_3}{2})u + \mathbf{p}''_4 + \mathbf{p}''_1 \varepsilon_2 + \mathbf{p}''_2 \frac{\varepsilon_4}{2} + \mathbf{p}''_5 \frac{\varepsilon_4}{2}\Big)\mathbb{R} =$$

$$\Big(-\mathbf{p}_0 u(1+\varepsilon_1) + \mathbf{p}_1(1+\varepsilon_2) + \mathbf{p}_2 \varepsilon_4 + \mathbf{p}_3 u(1-\varepsilon_1) + \mathbf{p}_4(-1+\varepsilon_2) - \mathbf{p}_5 u \varepsilon_3\Big)\mathbb{R} \text{ and} \tag{30}$$

$$E_\infty := (\mathbf{p}''_3 + \mathbf{p}''_0 \varepsilon_1 + \mathbf{p}''_2 \frac{\varepsilon_3}{2} - \mathbf{p}''_5 \frac{\varepsilon_3}{2})\mathbb{R} = -\mathbf{p}_0(1+\varepsilon_1) + \mathbf{p}_3(1-\varepsilon_1) - \mathbf{p}_5 \varepsilon_3, \quad \varepsilon_j \in \mathbb{R}. \tag{31}$$

The line e with (30) and (31) satisfies Demand 1 for $\varepsilon_1\varepsilon_2 \neq 0$, Demand 2 for $|\varepsilon_1| < 1$ and $|\varepsilon_2| < 1$, and Demand 4 for $\varepsilon_3\varepsilon_4 \neq 0$. The third rows of (12) and (13) result from (30) and (31), respectively.

Remark 13. If $\varepsilon_1\varepsilon_2 \neq 0$, then c, d, e are mutually skew and $B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ is a 2-regulus; this case will be discussed in Section 5 in detail. As we aim also at non-regular symplectic spreads, so we have to guarantee that our setting (12), (13) comprises also the special situation in which $B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ is contained in a 4-space, but not in a 3-space. If $\varepsilon_2 = \varepsilon_4 = 0$ and $\varepsilon_1\varepsilon_3 \neq 0$, then d and e have exactly the point $\mathbf{p}''_4\mathbb{R}$ in common and $\dim(c \vee d \vee e) = 4$; this special case is dealt with shortly in Section 7.

In the following lemma, we are content with appropriate bounds for the four deviations ε_j.

Lemma 4. *If $|\varepsilon_j| < 10^{-4}$ for $j = 1, 2, 3, 4$, then $B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ is a Thas-Walker plane set of Latin type with respect to the Klein quadric* (3) *and $\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ from* (15) *is a spread of* Π.

Proof. By Lemma 3, we have to check the conditions (C2)–(C7) for $B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$. We compute $F_3(u) = 2\,u\,(1+u^2)\,(4\,u^2 - 4\,u^2\varepsilon_1^2 + u\varepsilon_4^2 + 2\,u\varepsilon_4\varepsilon_3 + u\varepsilon_3^2 + 4 - 4\,\varepsilon_2^2)$ for $u \in \mathbb{R}$ and $F_3(\infty) = 0$. An easy estimation of the discriminant of the last factor of $F_3(u)$ shows $\mathbb{U}_e = \{0, \infty\}$, i.e., (C2) holds true.

Let $\mathbf{s}\mathbb{R} = (\sum_{k=0}^{5} \mathbf{p}_k s_k)\mathbb{R}$ be an arbitrary point of H_5, i.e.,

$$s_0 s_3 + s_1 s_4 + s_2 s_5 = 0. \tag{32}$$

Now $\Omega(\mathbf{s}, \mathbf{c}_0) = s_5 = 0$, $\Omega(\mathbf{s}, \mathbf{d}_0) = s_3 - s_0 = 0$, and $\Omega(\mathbf{s}, \mathbf{e}_0) = s_4(1+\varepsilon_2) + s_5\varepsilon_4 + s_1(-1+\varepsilon_2) = 0$ together with (32) imply $s_0^2(1+\varepsilon_2) + s_1^2(1-\varepsilon_2) = 0$ whence $s_0 = s_1 = 0$. Except $\mathbf{p}_2\mathbb{R}$, there is no point $\mathbf{s}\mathbb{R} \in H_5$ with $\Omega(\mathbf{s}, \mathbf{e}_0) = \Omega(\mathbf{s}, \mathbf{d}_0) = \Omega(\mathbf{s}, \mathbf{e}_0) = 0$. From $\Omega(\mathbf{s}, \mathbf{c}_\infty) = s_2 = 0$, $\Omega(\mathbf{s}, \mathbf{d}_\infty) = s_4 - s_1 = 0$, $\Omega(\mathbf{s}, \mathbf{e}_\infty) = s_3(-1-\varepsilon_1) + s_0(1-\varepsilon_1) - s_2\varepsilon_3 = 0$, and (32) we deduce $s_0^2(1-\varepsilon_1) + s_1^2(1+\varepsilon_1) = 0$ and, consequently, $s_0 = s_1 = 0$. Except $\mathbf{p}_5\mathbb{R}$, there is no point $\mathbf{s}\mathbb{R} \in H_5$ with $\Omega(\mathbf{s}, \mathbf{c}_\infty) = \Omega(\mathbf{s}, \mathbf{d}_\infty) = \Omega(\mathbf{s}, \mathbf{e}_\infty) = 0$. Hence (C3) and (24) are valid.

For our setting (12) $C_4(\xi, \eta, \zeta, u) = 0$ becomes the cubic equation

$$Au^3 + Bu^2 + Cu + D = 0 \quad \text{with}$$

$$\begin{aligned}
A &:= -\zeta^2(1+\varepsilon_1) - 1 + \varepsilon_1,\\
B &:= (-\varepsilon_4 - \varepsilon_3)\,\xi\,\zeta + (2\,\varepsilon_1 - 2\,\varepsilon_2)\,\zeta + (\varepsilon_4 + \varepsilon_3)\,\eta + \eta^2\,(1+\varepsilon_1) + (1-\varepsilon_1)\,\xi^2,\\
C &:= -1 + \varepsilon_2 + (\varepsilon_4 + \varepsilon_3)\,\zeta\,\eta + (\varepsilon_4 + \varepsilon_3)\,\xi + (-1-\varepsilon_2)\,\zeta^2 + (2\,\varepsilon_1 + 2\,\varepsilon_2)\,\eta\,\xi,\\
D &:= \xi^2\,(1+\varepsilon_2) - \eta^2\,(-1+\varepsilon_2)
\end{aligned} \tag{33}$$

in the unknown u since $A < 0$ for all $\zeta \in \mathbb{R}$. Using (13) we compute $C_4(\xi, \eta, \zeta, \infty) = A$, hence it suffices to show that (33) has exactly one solution in $\mathbb{R}$. By [10, p. 31], this condition holds, if

$$-18ABCD - B^2C^2 + 27A^2D^2 + 4AC^3 + 4B^3D > 0. \tag{34}$$

We substitute the coefficients of (33) in (34) and get the condition

$$\Gamma(\xi, \eta, \zeta, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) := 4 + \cdots + 4\zeta^8\,(-1-\varepsilon_1)\,(-1-\varepsilon_2)^3 > 0. \tag{35}$$

Thus

$$\Gamma(\xi, \eta, \zeta, 0, 0, 0, 0) = 4\xi^8 + \cdots + 4\eta^8 + \cdots + 4\zeta^8 + \cdots + 4. \tag{36}$$

We compare this with [26, Proof of Lemma 3]: In essential we have the same situation here. By applying the estimation procedure given in [26, Proof of Lemma 3] to $\Gamma(\xi, \eta, \zeta, \varepsilon_1, \ldots, \varepsilon_4)$, we get: $C_4(\xi, \eta, \zeta, u) = 0$ has exactly one solution in $\mathbb{R}$ for all $(\xi, \eta, \zeta) \in \mathbb{R}$. We leave it to the reader to fill in the gaps and to prove the validity of (C5), (C6), and (C7) for the set $B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ of planes. □

For the rest of this paper we assume

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in I^4 \setminus \{(0,0,0,0)\} =: I_\varepsilon \quad \text{with} \quad I := \{x \in \mathbb{R} \mid 10^{-4} > |x|\}. \tag{37}$$

Each spread $\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$, see (15), admits the regulization

$$\Lambda_{\varepsilon_1,\ldots,\varepsilon_4} := \{\lambda^{-1}(\xi) \mid \xi \in (B_{\varepsilon_1,\ldots,\varepsilon_4})'\} \text{ where } (B_{\varepsilon_1,\ldots,\varepsilon_4})' := \{\xi \in B_{\varepsilon_1,\ldots,\varepsilon_4} \mid \xi \cap H_5 \neq \emptyset\}. \tag{38}$$

In the following, the plane of $B_{\varepsilon_1,\dots,\varepsilon_4}$ corresponding to the parameter u is denoted by β_u. The point set

$$\Phi(B_{\varepsilon_1,\dots,\varepsilon_4}) := \bigcup(\beta_u \mid u \in \mathbb{R} \cup \{\infty\}) \tag{39}$$

is a 3-surface in $\mathrm{PG}(5,\mathbb{R})$; we speak of a 2-*ruled surface* with *generating planes* β_u. Using (12) and (13) we get the following parametric representation:

$$\Phi(B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}) = \Big\{ \Big((\mathbf{p}_2 + \mathbf{p}_5 u) + (\mathbf{p}_0 + \mathbf{p}_1 u - \mathbf{p}_3 - \mathbf{p}_4 u)v + (-\mathbf{p}_0(1+\varepsilon_1)u + \mathbf{p}_1(1+\varepsilon_2) + \mathbf{p}_2\varepsilon_4 + \mathbf{p}_3(1-\varepsilon_1)u - \mathbf{p}_4(1-\varepsilon_2) - \mathbf{p}_5\varepsilon_3 u)w \Big)\mathbb{R} \,|\, (u,v,w) \in (\mathbb{R}\cup\{\infty\})^3 \Big\}. \tag{40}$$

The algebraic representation of $\Phi(B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4})$ depends on the mutual situation of the directing lines c, d, e, compare Remark 13, and is given in Section 5 and Section 7. We do not need this ramification for the description of the regulizations $\Lambda_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ in the next Section 4.

4. The regulizations $\Lambda_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$

From the Proof of Lemma 4 we know that $\beta_0 \cap H_5 = \{\mathbf{p}_2\mathbb{R}\}$ and $\beta_\infty \cap H_5 = \{\mathbf{p}_5\mathbb{R}\}$, hence $\lambda^{-1}(\mathbf{p}_2\mathbb{R})$ and $\lambda^{-1}(\mathbf{p}_5\mathbb{R})$ are the improper reguli of $\Lambda_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$.

Proposition 1. *If $u \in \mathbb{R}^{>0}$, then $\beta_u \cap H_5$ is a proper conic. If $u \in \mathbb{R}^{<0}$, then $\beta_u \cap H_5 = \emptyset$. Moreover,*

$$(B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4})' := \Big\{\beta_u \mid u \in \mathbb{R}^{\geq 0} \cup \{\infty\}\Big\} \text{ and } \mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4} = \bigcup\Big(\lambda^{-1}(\beta_u) \mid u \in \mathbb{R}^{\geq 0} \cup \{\infty\}\Big). \tag{41}$$

Proof. We join the point C_u from (19) and an arbitrary point $\mathbf{m}_\mu\mathbb{R} \in D_u \vee E_u$, i.e.,

$$\mathbf{m}_\mu \overset{(21)\wedge(30)}{=} (\mathbf{p}_0 + \mathbf{p}_1 u - \mathbf{p}_3 - \mathbf{p}_4 u) + \Big(-\mathbf{p}_0 u(1+\varepsilon_1) + \mathbf{p}_1(1+\varepsilon_2) + \mathbf{p}_2\varepsilon_4 + \mathbf{p}_3 u(1-\varepsilon_1) + \mathbf{p}_4(-1+\varepsilon_2) - \mathbf{p}_5 u\varepsilon_3\Big)\mu,$$

$\mu \in \mathbb{R}$, and get the line $\ell_\mu = \{\Big((\mathbf{p}_2 + \mathbf{p}_5 u)x + \mathbf{m}_\mu\Big)\mathbb{R} \mid x \in \mathbb{R}\} \cup \{C_u\}$. The determination of $\ell_\mu \cap H_5$ is equivalent to the solution of the quadratic equation

$$G(x) := ux^2 + (\varepsilon_4 - \varepsilon_3)\,\mu\, ux + (1 - \mu\, u\,(1+\varepsilon_1))\,(-1 + \mu\, u\,(1-\varepsilon_1)) + (u + \mu\,(1+\varepsilon_2))\,(-u + \mu\,(-1+\varepsilon_2)) - \varepsilon_3\varepsilon_4\mu^2 u = 0$$

in the unknown x. As discriminant of the above equation we get:

$$D_{G(x)} := uH(u) \quad \text{with} \quad H(u) := 4\Big(\mu^2(1-\varepsilon_1^2) + 1\Big)u^2 + (\varepsilon_3+\varepsilon_4)^2\mu^2 u + 4 + 4\,\mu^2(1-\varepsilon_2^2).$$

For the discriminant $D_{H(u)}$ of the quadratic equation $H(u)$ in the unknown u holds $D_{H(u)} < 0$ as an easy estimation shows. Now $D_{H(u)} < 0$ and $H(0) \geq 4$ imply $H(u) > 0$ for all $u, \mu \in \mathbb{R}$ and all $\varepsilon_1, \dots, \varepsilon_4 \in I$. Hence: $D_{G(x)} > 0 \Leftrightarrow u > 0$. □

If $\varepsilon_1 \neq 0$, then the regulization $\Lambda_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ is composed of the proper reguli

$\mathcal{R}_u := \lambda^{-1}(\beta_u) = \lambda^{-1}(\{\mathbf{p}\mathbb{R} \in H_5 \mid \mathbf{p} = \sum_{k=0}^{5} \mathbf{p}_k p_k$ and

$$\Big(u^2(1-\varepsilon_1)+1+\varepsilon_2\Big)p_0 + 2\varepsilon_1 u p_1 + \Big(u^2(1+\varepsilon_1)+1+\varepsilon_2\Big)p_3 = 0$$
$$\wedge \quad \Big(u^2(-1+\varepsilon_1)-1+\varepsilon_2\Big)p_0 + \Big(u^2(-1-\varepsilon_1)-1+\varepsilon_2\Big)p_3 + 2\varepsilon_1 u p_4 = 0$$
$$\wedge \quad -(\varepsilon_3+\varepsilon_4)\,up_0 - 2\varepsilon_1\,u^2 p_2 - (\varepsilon_3+\varepsilon_4)\,up_3 + 2\varepsilon_1 u p_5 = 0\}) \tag{42}$$

with $u > 0$ and the two improper reguli $\{\lambda^{-1}(\mathbf{p}_2\mathbb{R})\}$ and $\{\lambda^{-1}(\mathbf{p}_5\mathbb{R})\}$; in symbols

$$\Lambda_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4} = \{\mathcal{R}_u \mid u \in \mathbb{R}^{>0}\} \cup \{\lambda^{-1}(\mathbf{p}_2\mathbb{R}), \lambda^{-1}(\mathbf{p}_5\mathbb{R})\}. \tag{43}$$

The determination of the equations of β_u for the case $\varepsilon_1 = 0$ is left to the reader.

Proposition 2. *Assume $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) \in I_\varepsilon$, see (37). If $(\varepsilon_1,\varepsilon_2) = (0,0)$ and $\varepsilon_3 = -\varepsilon_4$, then $\Lambda_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ is an elliptic regulization; in all other cases $\Lambda_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ is an asymplecticly complemented regulization.*

Proof. We determine the intersection of the planes β_1, β_2 and β_3 corresponding to $u = 1$, $u = 2$, and $u = 3$. We have to consider various cases.

(a) If $\varepsilon_1\varepsilon_2 \neq 0$, then $\dim(C_1 \vee D_1 \vee E_1 \vee C_2 \vee D_2 \vee E_2) = 5$, i.e., $\beta_1 \cap \beta_2 = \emptyset$. Hence $d' = \dim(\cap(\xi \mid \xi \in (B_{\varepsilon_1,\dots,\varepsilon_4})')) = -1$, $d^p = \dim(\cap(\xi \mid \xi \in (B_{\varepsilon_1,\dots,\varepsilon_4})^p)) = -1$, and the statement follows from [27, (11) and Table 2].

(b) If $\varepsilon_1 \neq 0$ and $\varepsilon_2 = 0$, then we have:

$$\beta_1\cap\beta_2 = \Big\{\Big(\mathbf{p}_0(1+2\,\varepsilon_1)+\mathbf{p}_1(-3)+\mathbf{p}_3(-1+2\,\varepsilon_1)+\mathbf{p}_4\cdot 3+\mathbf{p}_5(2\,\varepsilon_3+2\,\varepsilon_4)\Big)\mathbb{R}\Big\} \not\subset \beta_3 \;\Rightarrow\; d^p = -1.$$

(c) In the case $\varepsilon_1 = 0$ and $\varepsilon_2 \neq 0$ we compute also $\beta_1 \cap \beta_2 \cap \beta_3 = \emptyset$.

(d) If $(\varepsilon_1,\varepsilon_2) = (0,0)$ and $\varepsilon_3 \neq -\varepsilon_4$, then $\beta_1 \cap \beta_2 \cap \beta_3 = \emptyset$.

(e) If $(\varepsilon_1,\varepsilon_2) = (0,0)$ and $\varepsilon_3 = -\varepsilon_4$, then $(-\mathbf{p}_0+\mathbf{p}_3)\mathbb{R} \vee (-\mathbf{p}_1+\mathbf{p}_4)\mathbb{R} \subset \beta_u$ for all $u \in \mathbb{R}$. Thus $d^p = 1$ and the statement follows from [27, (11) and Table 2]. □

Now we are able to define the family $\mathcal{F}$ mentioned in the abstract[8]:

$$\mathcal{F} := \{\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4} \mid (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) \in I_\mathcal{F}\} \text{ with}$$
$$I_\mathcal{F} := I_\varepsilon \setminus \{(x_1,x_2,x_3,x_4) \in I_\varepsilon \mid x_1 = x_2 = 0 \;\wedge\; x_3 = -x_4\}. \tag{44}$$

In the following we investigate only cases where $\Lambda_{\varepsilon_1,\dots,\varepsilon_4}$ is asymplecticly complemented. We thoroughly discuss the general case, i.e., $\varepsilon_1\varepsilon_2 \neq 0$, in Section 5 and throw a short look onto one special case, namely that with $(\varepsilon_2,\varepsilon_4) = (0,0)$ and $\varepsilon_1\varepsilon_3 \neq 0$, in Section 7.

[8] For I_ε see (37).

5. The spreads $\boldsymbol{\mathcal{B}}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ with $\varepsilon_1\varepsilon_2 \neq 0$

5.1. Algebraic representation of $\boldsymbol{\mathcal{B}}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$

By Remark 13, the lines c, d, e are mutually skew, hence

$$v' \text{ resp. } v^p = \dim\Big(\bigvee \xi \mid \xi \in (B_{\varepsilon_1,\ldots,\varepsilon_4})' \text{ resp. } (B_{\varepsilon_1,\ldots,\varepsilon_4})^p\Big) = \dim(c \vee d \vee e) = 5;$$

according to [27, (10) and Table 1], the spreads $\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ are asymplectic. The plane set $B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ is a 2-regulus and the corresponding point set $\Phi(B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4})$, see (39) and (40), is a Segre manifold $S_{2;1}$ of Π_5 whose system Σ_2 of generating planes coincides with $B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4} = \{\beta_u \mid u \in \mathbb{R} \cup \{\infty\}\}$ and whose system Σ_1 of generating lines contains c, d, e, cf. [8, p. 116], [15, p. 190]. In order to change from (40) to the simple description of a Segre manifold as given in [15, p. 192, (25.36)], we use the basis $\{\mathbf{a}_{00}, \mathbf{a}_{01}, \mathbf{a}_{02}, \mathbf{a}_{10}, \mathbf{a}_{11}, \mathbf{a}_{12}\}$ with

$$\begin{aligned} &\mathbf{a}_{00} = \mathbf{p}_2, \quad \mathbf{a}_{01} = \mathbf{p}_0 - \mathbf{p}_3, \quad \mathbf{a}_{02} = \mathbf{p}_1(1+\varepsilon_2) + \mathbf{p}_2\varepsilon_4 + \mathbf{p}_4(-1+\varepsilon_2), \\ &\mathbf{a}_{10} = \mathbf{p}_5, \quad \mathbf{a}_{11} = \mathbf{p}_1 - \mathbf{p}_4, \quad \mathbf{a}_{12} = -\mathbf{p}_0(1+\varepsilon_1) + \mathbf{p}_3(1-\varepsilon_1) - \mathbf{p}_5\varepsilon_3 \end{aligned} \tag{45}$$

such that (19), (21), (30), and (45) imply:

$$C_w = (\mathbf{a}_{00}+\mathbf{a}_{10}w)\mathbb{R},\ D_w = (\mathbf{a}_{01}+\mathbf{a}_{11}w)\mathbb{R},\ E_w = (\mathbf{a}_{02}+\mathbf{a}_{12}w)\mathbb{R} \text{ for all } w \in \mathbb{R}\cup\{\infty\} \text{ and} \tag{46}$$

$$\Phi(B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}) = \{(\mathbf{a}_{00}+\mathbf{a}_{01}u+\mathbf{a}_{02}v+\mathbf{a}_{10}w+\mathbf{a}_{11}uw+\mathbf{a}_{12}vw)\mathbb{R} \mid (u,v,w) \in (\mathbb{R}\cup\{\infty\})^3\}. \tag{47}$$

According to [15, p. 189, Theorem 25.5.1] holds:

$$\Phi(B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}) = Q_1 \cap Q_2 \cap Q_3 \text{ with} \tag{48}$$

$$Q_1 := \{\mathbf{x}\mathbb{R} \in \mathcal{P}_5 \mid \mathbf{x} = \sum_{j=0}^{1}\sum_{k=0}^{2} \mathbf{a}_{jk}x_{jk} \text{ and } x_{00}x_{11} - x_{01}x_{10} = 0\}, \tag{49}$$

$$Q_2 := \{\mathbf{x}\mathbb{R} \in \mathcal{P}_5 \mid x_{01}x_{12} - x_{02}x_{11} = 0\}, \text{ and } Q_3 := \{\mathbf{x}\mathbb{R} \in \mathcal{P}_5 \mid x_{02}x_{10} - x_{00}x_{12} = 0\}. \tag{50}$$

Proposition 3. *For the Klein image of the spread $\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ holds:*

$$\lambda(\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}) = H_5 \cap (Q_1 \cap Q_2 \cap Q_3). \tag{51}$$

Proof. (a) For $X \in \lambda(\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4})$ there exists $u_X \in \mathbb{R} \cup \{\infty\}$ with $X \in \beta_{u_X}$ because of (41). The plane β_{u_X} is a generating plane of the Segre manifold $\Phi(B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}) \overset{(48)}{=} Q_1 \cap Q_2 \cap Q_3$. Hence $X \in Q_1 \cap Q_2 \cap Q_3$.

(b) Assumed $Y \in H_5 \cap (Q_1 \cap Q_2 \cap Q_3)$. By [8, p. 116], the point Y of the Segre manifold $Q_1 \cap Q_2 \cap Q_3$ is on exactly one generating plane, say β_{u_Y}. From $Y \in \beta_{u_Y} \cap H_5 \neq \emptyset$ we deduce via Prop. 1 that $u_Y \in \mathbb{R}^{\geq 0} \cup \{\infty\}$. This and (41) imply: $\lambda^{-1}(Y) \in \mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$. □

Now (51) and (49), (50) show that $\mathcal{B}_{\varepsilon_1,\ldots,\varepsilon_4}$ is an algebraic spread. By Lemma 8 follows that $\mathcal{B}_{\varepsilon_1,\ldots,\varepsilon_4}$ is topological and also a dual spread. Because of $\varepsilon_1\varepsilon_2 \neq 0$ and Proposition 2 the regulization $\Lambda_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ is asymplecticly complemented. We sum up in

Theorem 1. *Put* $\mathcal{Q}_k := \lambda^{-1}(Q_k)$, $k = 1, 2, 3$, *for the three quadratic line complexes which are described in Plücker coordinates by the equations*

$$\begin{aligned}&\varepsilon_2^2\varepsilon_3\,(-1+\varepsilon_1)\,p_0^2 - 2\,\varepsilon_2^2\varepsilon_3 p_0p_3 + 2\,\varepsilon_1\varepsilon_2^2\,(1-\varepsilon_1)\,p_0p_5 + \varepsilon_1^2\varepsilon_4\,(1-\varepsilon_2)\,p_1^2 +\\ &2\,\varepsilon_1^2\varepsilon_2\,(-1+\varepsilon_2)\,p_1p_2 + 2\,\varepsilon_1^2\varepsilon_4 p_1p_4 + 2\,\varepsilon_1^2\varepsilon_2\,(-1-\varepsilon_2)\,p_2p_4 + \varepsilon_2^2\varepsilon_3\,(-1-\varepsilon_1)\,p_3^2 +\\ &2\,\varepsilon_1\varepsilon_2^2\,(1+\varepsilon_1)\,p_3p_5 + \varepsilon_1^2\varepsilon_4\,(1+\varepsilon_2)\,p_4^2 = 0,\end{aligned} \tag{52}$$

$$\varepsilon_2^2\,(1-\varepsilon_1)\,p_0^2 + 2\,\varepsilon_2^2 p_0p_3 + \varepsilon_1^2\,(1-\varepsilon_2)\,p_1^2 + 2\,\varepsilon_1^2 p_1p_4 + \varepsilon_2^2\,(1+\varepsilon_1)\,p_3^2 + \varepsilon_1^2\,(1+\varepsilon_2)\,p_4^2 = 0, \tag{53}$$

and

$$\begin{aligned}&(\varepsilon_3+\varepsilon_4)\,p_0p_1 - 2\,\varepsilon_2 p_0p_2 + (\varepsilon_3+\varepsilon_4)\,p_0p_4 + (\varepsilon_3+\varepsilon_4)\,p_1p_3 - 2\,\varepsilon_1 p_1p_5 - 2\,\varepsilon_2 p_2p_3 +\\ &(\varepsilon_3+\varepsilon_4)\,p_3p_4 - 2\,\varepsilon_1 p_4p_5 = 0,\end{aligned} \tag{54}$$

respectively. If $|\varepsilon_j| < 10^{-4}$ $(j = 1, 2, 3, 4)$ *and* $\varepsilon_1\varepsilon_2 \neq 0$, *then* $\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4} = \mathcal{Q}_1 \cap \mathcal{Q}_2 \cap \mathcal{Q}_3$ *is an asymplectic algebraic spread which admits the asymplecticly complemented regulization* $\Lambda_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ *described in Section* 4. *The spread* $\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ *is topological and a dual spread.*

5.2. Proper reguli contained in $\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$

Lemma 5. *Let* k *be a proper conic contained in the Segre manifold* $S_{2;1}$ *such that* k *does not belong to a generating plane of* $S_{2;1}$. *Then* k *and an arbitrary generating plane* ξ *of* $S_{2;1}$ *have exactly one common point.*

Proof. The underlying space is a real projective 5-space. Let $P \in k$ be an arbitrary point. By [15, p. 190, Theorem 25.5.3], P is on exactly one generating plane α_P of $S_{2;1}$. In case of $\alpha_P = \xi$ there is nothing to do, hence we assume $\alpha_P \neq \xi$. Because of $k \not\subset \alpha_P$, the subspace $A := \alpha_P \vee \operatorname{span} k$ is either of dimension 3 or 4. If $\dim A = 3$, then $A \cap S_{2;1}$ consists of α_p and a line $S_{0;1}$ as follows from [8, p. 172, Hilfssatz[9] über lineare Schnitte der $S_{s-1;1}$]; this yields the absurdity that the conic k is contained in the line $S_{0;1}$. Consequently, $\dim A = 4$. Now Burau's Hilfssatz shows that $A \cap S_{2;1}$ consists of α_p and an $S_{1;1}$ which by [8, p. 133] is a hyperbolic quadric of a 3-space. Obviously, $k \subset S_{1;1}$. According to [15, p. 190, Theorem 25.5.3], ξ and α_P are skew which implies $\xi \not\subset A$. Hence $\xi \cap A$ is a line on $S_{1;1}$, in other words, a generatrix of the hyperbolic quadric $S_{1;1}$. This generatrix has exactly one common point with $k(\subset S_{1;1})$. Each common point of ξ and $k(\subset A)$ must belong to $\xi \cap A$. □

Theorem 2. *If the assumptions of Theorem* 1 *are valid, then the spread* $\mathcal{B}_{\varepsilon_1,\ldots,\varepsilon_4}$ *contains no proper regulus off the asymplecticly complemented regulization* $\Lambda_{\varepsilon_1,\ldots,\varepsilon_4}$ *described in Section* 4. *The spread* $\mathcal{B}_{\varepsilon_1,\ldots,\varepsilon_4}$ *admits exactly one regulization, namely* $\Lambda_{\varepsilon_1,\ldots,\varepsilon_4}$.

Proof. Let $\mathcal{R} \subset \mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ be a proper regulus. From (41) follows

$$\lambda(\mathcal{R}) \subset \bigcup\Big(\beta_u \mid u \in \mathbb{R}^{\geq 0} \cup \{\infty\}\Big) \overset{(39)}{\subset} \Phi(B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}).$$

[9] We add that this Hilfssatz is valid for real projective spaces, too.

This implies that for $u_{\_} \in \mathbb{R}^{<0}$ the generating plane $\beta_{u_{\_}}$ of the Segre manifold $\Phi(B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4})$ and the proper conic $\lambda(\mathcal{R})$ have no common point. Consequently, $\lambda(\mathcal{R})$ is contained in a generating plane of $\Phi(B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4})$ by Lemma 5. □

5.3. Collineations and dualities which leave $\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ invariant

By $\mathrm{PGL}_e(4,\mathbb{R})$ we denote the extended collineation group of $\Pi = \mathrm{PG}(3,\mathbb{R})$ which consists of all collineations and all dualities of Π. Each $\tau \in \mathrm{PGL}_e(4,\mathbb{R})$ induces a collineation $\tau_\lambda \in \mathrm{PGL}(6,\mathbb{R})$ of $\Pi_5 = \mathrm{PG}(5,\mathbb{R})$ with $\tau_\lambda(H_5) = H_5$, i.e., $\lambda \circ \tau = \tau_\lambda \circ \lambda$. For a spread $\mathcal{S}$ of Π we put $\mathrm{Aut}\,\mathcal{S} := \{\kappa \in \mathrm{PGL}(4,\mathbb{R}) \mid \kappa(\mathcal{S}) = \mathcal{S}\}$ for the group of all automorphic collineations of $\mathcal{S}$ and $\mathrm{Aut}_e\,\mathcal{S} := \{\tau \in \mathrm{PGL}_e(4,\mathbb{R}) \mid \tau(\mathcal{S}) = \mathcal{S}\}$ for the group of all automorphic collineations and dualities of $\mathcal{S}$.

Consider the improper reguli $\{\lambda^{-1}(\mathbf{p}_2\mathbb{R})\}$ and $\{\lambda^{-1}(\mathbf{p}_5\mathbb{R})\}$ of the regulization $\Lambda_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ satisfying (RZ1) by definition, cf. [23, p. 140]. Because of (RZ1) and Theorem 2 the lines $\lambda^{-1}(\mathbf{p}_2\mathbb{R})$ and $\lambda^{-1}(\mathbf{p}_5\mathbb{R})$ are the only lines of the spread $\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ that do not belong to a proper regulus of $\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$. Thus we have

Corollary 1. *Let $\tau \in \mathrm{Aut}_e\,\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$. Then τ either fixes or interchanges the lines*

$$\lambda^{-1}(\mathbf{p}_2\mathbb{R}) = \lambda^{-1}(\mathbf{a}_{00}\mathbb{R}) \quad \text{and} \quad \lambda^{-1}(\mathbf{p}_5\mathbb{R}) = \lambda^{-1}(\mathbf{a}_{10}\mathbb{R}).$$

Lemma 6. *Let $\tau \in \mathrm{Aut}_e\,\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$. Then the Segre manifold $\Phi(B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4})$ is invariant under the induced collineation τ_λ.*

Proof. By Theorem 2 the collineation τ_λ permutes the proper conics $H_5 \cap \beta_{u_p}$, i.e., $u_p \in \mathbb{R}^{>0}$ by Prop. 1. Hence τ_λ permutes the generating planes β_{u_p} of $\Phi(B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4})$. By [8, p. 135, Satz], a Segre manifold $S_{2;1}$ is uniquely determined by three generating planes $\gamma_1, \gamma_2, \gamma_3$ with

$$5 = \dim(\gamma_1 \vee \gamma_2 \vee \gamma_3) = \dim(\gamma_i \vee \gamma_k) \quad \text{for all} \quad (i,k) \in \{1,2,3\}^2 \quad \text{and} \quad i \neq k. \tag{55}$$

For the planes $\beta_1, \beta_2, \beta_3$ used in the proof of Prop. 2 the conditions (55) and $1, 2, 3 \in \mathbb{R}^{>0}$ are valid. Consequently, $\beta_1, \beta_2, \beta_3$ as well as $\tau_\lambda(\beta_1), \tau_\lambda(\beta_2), \tau_\lambda(\beta_3)$ uniquely determine the Segre manifold $\Phi(B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4})$ and thus $\tau_\lambda(\Phi(B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4})) = \Phi(B_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4})$. □

From (48)–(50), [15, p. 192, (25.36)], Lemma 6, and [15, p. 193, (25.37)] follows that the collineation τ_λ is described by[10]

$$y_{jk}\rho = \sum_{r=0}^{1}\sum_{s=0}^{2} b_{jr}c_{ks}x_{rs}, \qquad \rho \in \mathbb{R}\setminus\{0\}, \quad b_{jr}, c_{ks} \in \mathbb{R} \quad j = 0,1, \quad k = 0,1,2,$$

[10]Note that in [15] a left vector space is used and here a right one.

and the corresponding (6×6)-matrix

$$T := \begin{pmatrix} b_{00}c_{00} & b_{00}c_{01} & b_{00}c_{02} & b_{01}c_{00} & b_{01}c_{01} & b_{01}c_{02} \\ b_{00}c_{10} & b_{00}c_{11} & b_{00}c_{12} & b_{01}c_{10} & b_{01}c_{11} & b_{01}c_{12} \\ b_{00}c_{20} & b_{00}c_{21} & b_{00}c_{22} & b_{01}c_{20} & b_{01}c_{21} & b_{01}c_{22} \\ b_{10}c_{00} & b_{10}c_{01} & b_{10}c_{02} & b_{11}c_{00} & b_{11}c_{01} & b_{11}c_{02} \\ b_{10}c_{10} & b_{10}c_{11} & b_{10}c_{12} & b_{11}c_{10} & b_{11}c_{11} & b_{11}c_{12} \\ b_{10}c_{20} & b_{10}c_{21} & b_{10}c_{22} & b_{11}c_{20} & b_{11}c_{21} & b_{11}c_{22} \end{pmatrix} \tag{56}$$

is the Kronecker product of the (2×2)-matrix (b_{jr}) and the (3×3)-matrix (c_{ks}) therefore holds $|T| = |(b_{jr})|^3 |(c_{ks})|^2 \neq 0$ and consequently

$$|(b_{jr})| \neq 0 \quad \text{and} \quad |(c_{ks})| \neq 0. \tag{57}$$

According to Corollary 1 we have the alternatives

$$\textit{Case}\ \mathrm{A} : \tau_\lambda(\mathbf{a}_{00}\mathbb{R}) = \mathbf{a}_{00}\mathbb{R} \quad \text{and} \quad \tau_\lambda(\mathbf{a}_{10}\mathbb{R}) = \mathbf{a}_{10}\mathbb{R}$$

$$\textit{Case}\ \mathrm{B} : \tau_\lambda(\mathbf{a}_{00}\mathbb{R}) = \mathbf{a}_{10}\mathbb{R} \quad \text{and} \quad \tau_\lambda(\mathbf{a}_{10}\mathbb{R}) = \mathbf{a}_{00}\mathbb{R}.$$

Case A. From (56) we read off the following ten conditions:

$$k_1 := b_{00}c_{10} = 0,\ k_2 := b_{00}c_{20} = 0,\ k_3 := b_{10}c_{00} = 0,\ k_4 := b_{10}c_{10} = 0,\ k_5 := b_{10}c_{20} = 0,$$
$$k_6 := b_{01}c_{00} = 0,\ k_7 := b_{01}c_{10} = 0,\ k_8 := b_{01}c_{20} = 0,\ k_9 := b_{11}c_{10} = 0,\ k_{10} := b_{11}c_{20} = 0. \tag{58}$$

Because of k_1 we get the ramification

Subcase A.A: $b_{00} = 0$ *Subcase* A.B: $b_{00} \neq 0$ and $c_{10} = 0$.

Subcase A.A: By (57) we have $b_{01}b_{10} \neq 0$, consequently, $b_{01} \neq 0$ and $b_{10} \neq 0$, hence k_3, k_4, k_5 imply $c_{00} = c_{10} = c_{20} = 0$. This contradicts $|(c_{ks})| \neq 0$ from (57).

Subcase A.B: From k_2 we deduce $c_{20} = 0$. Now (57) yields $c_{00}(c_{11}c_{22} - c_{12}c_{21}) \neq 0$, i.e., $c_{00} \neq 0$. Hence k_3 and k_6 imply $b_{10} = b_{01} = 0$. Thus we have $k_1 = \cdots = k_{10} = 0$.
From (58) and (57) follows necessarily:

$$b_{01} = b_{10} = c_{10} = c_{20} = 0 \quad \text{and} \tag{59}$$

$$b_{00} \neq 0, \quad b_{11} \neq 0, \quad c_{00} \neq 0, \quad c_{11}c_{22} - c_{12}c_{21} \neq 0. \tag{60}$$

Conversely, we verify easily that (59), (60) is also sufficient for $\tau_\lambda(\mathbf{a}_{j0}\mathbb{R}) = \mathbf{a}_{j0}\mathbb{R}$, $j = 0, 1$.

Case B. Analogously to *Case* A we get the subsequent necessary and sufficient conditions:

$$b_{00} = b_{11} = c_{10} = c_{20} = 0 \quad \text{and} \tag{61}$$

$$b_{01} \neq 0, \quad b_{10} \neq 0, \quad c_{00} \neq 0, \quad c_{11}c_{22} - c_{12}c_{21} \neq 0. \tag{62}$$

Further conditions for the matrix T, cf. (56), we deduce from the fact $\tau_\lambda(H_5) = H_5$. With (3) and (45) we get:

$$H_5 = \Big\{\mathbf{x}\mathbb{R} \in \mathcal{P}_5 \mid \mathbf{x} = \sum_{j=0}^{1}\sum_{k=0}^{2} \mathbf{a}_{jk}x_{jk} \text{ and } \Big(x_{01} + (-1-\varepsilon_1)\,x_{12}\Big)\Big((1-\varepsilon_1)\,x_{12} - x_{01}\Big) + \Big((1+\varepsilon_2)\,x_{02} + x_{11}\Big)\Big((-1+\varepsilon_2)\,x_{02} - x_{11}\Big) + \Big(x_{00} + \varepsilon_4 x_{02}\Big)\Big(x_{10} - \varepsilon_3 x_{12}\Big) = 0\Big\}. \quad (63)$$

The point $K(t,u,v,w) :=$

$$\Big(\mathbf{a}_{00}\varepsilon_1(-\varepsilon_4 u - \varepsilon_4 w + 2\varepsilon_2) + \mathbf{a}_{01}\varepsilon_2(-t + \varepsilon_1 t - v - \varepsilon_1 v) + \mathbf{a}_{02}\varepsilon_1(u+w) + \mathbf{a}_{10}\varepsilon_2(-\varepsilon_3 t - 2\,\varepsilon_1 tv - 2\,\varepsilon_1 uw - \varepsilon_3 v) + \mathbf{a}_{11}\varepsilon_1(-u + \varepsilon_2 u - w - \varepsilon_2 w) + \mathbf{a}_{12}\varepsilon_2(-t-v)\Big)\mathbb{R} \quad (64)$$

belongs to H_5 for all $(t,u,v,w) \in \mathbb{R}^4$, roughly spoken, (64) is a parametric representation of H_5. Hence $\tau_\lambda(H_5) = H_5$ implies $\tau_\lambda\Big(K(t,u,v,w)\Big) \in H_5$ for all $(t,u,v,w) \in \mathbb{R}^4$. With the help of a computer and via (56), (64) we calculate the doubly indexed coordinates of $\tau_\lambda\Big(K(t,u,v,w)\Big)$ and these coordinates have to satisfy the equation from (63). Thus we get a polynomial[11]

$$p(t,u,v,w) := t^2v^2\Big(4\,\varepsilon_1^2\varepsilon_2^2(-b_{01}^2c_{10}^2 - b_{11}^2c_{20}^2 + b_{11}^2c_{20}^2\varepsilon_1^2 + \cdots)\Big) + \cdots \quad (65)$$

which has to vanish for all $(t,u,v,w) \in \mathbb{R}^4$. Consequently, we have to compare coefficients. The coefficient of $p(t,u,v,w)$ at $t^iu^jv^kw^\ell$ will be denoted by $C_p(t^i,u^j,v^k,w^\ell)$.

In the subsequent we write down only those coefficients of $p(t,u,v,w)$ that are essential for the progress of the determination of $\mathrm{Aut}_e(\mathcal{B}_{\varepsilon_1,\dots,\varepsilon_4})$. Nevertheless, we roughly sketch the strategy how to find these essential stations. It is useful to make a routine which yields the non-vanishing coefficients decomposed into factors. In order to maintain control it is advisable to collect at the one hand the vanishing b_{ik}'s and c_{ik}'s in a list and on the other hand the non-vanishing b_{ik}'s and c_{ik}'s in another list. Note that vanishing and non-vanishing b_{ik}'s and c_{ik}'s are of the same significance for the conclusions.

Continuation of Case A. Now

$$C_p(t^0,u^1,v^0,w^0) = 2\,\varepsilon_1^2\varepsilon_2\,(1-\varepsilon_2)\,b_{00}b_{11}\,c_{00}\,(\varepsilon_3 c_{21} - c_{01}) = 0 \text{ and}$$
$$C_p(t^0,u^1,v^1,w^1) = 2\,\varepsilon_1\varepsilon_2^2\,(1+\varepsilon_1)\,b_{00}b_{11}\,c_{00}\,(\varepsilon_4 c_{21} + c_{01}) = 0.$$

Because of $\varepsilon_1\varepsilon_2 \neq 0$ according to the title of this section, $(-1+\varepsilon_2) \neq 0$ and $(1+\varepsilon_1) \neq 0$ by (37), and $b_{00}b_{11}\,c_{00} \neq 0$ by (60) we have $(\varepsilon_3 c_{21} - c_{01}) = 0$ and $(\varepsilon_4 c_{21} + c_{01}) = 0$. For $\varepsilon_4 \neq -\varepsilon_3$ these two equations yield

$$c_{01} = c_{21} = 0. \quad (66)$$

For the rest of the discussion of *Case* A and *Case* B

we exclude the case with $\varepsilon_4 = -\varepsilon_3$. (67)

$$|\,c_{ik}| \overset{(59),(66)}{=} c_{00}c_{11}c_{22} \neq 0 \quad\Rightarrow\quad c_{11} \neq 0 \text{ and } c_{22} \neq 0. \quad (68)$$

[11]Since this polynomial is rather voluminous, it is commendable to refrain from displaying it completely on the screen. It suffices to display certain coefficients.

Now $C_p(t^0,u^2,v^0,w^1) = 2\varepsilon_1^2\varepsilon_2 b_{00}b_{11}c_{00}(\varepsilon_4 c_{00} - c_{02} - \varepsilon_4 c_{22}) = 0$,
and $C_p(t^1,u^0,v^0,w^0) = -2\varepsilon_1\varepsilon_2^2 b_{00}b_{11}c_{00}(\varepsilon_3 c_{00} + c_{02} - \varepsilon_3 c_{22}) = 0$, and (67) yield

$$c_{22} = c_{00} \quad \text{and} \quad c_{02} = 0. \tag{69}$$

Thus we have
$C_p(t^1,u^1,v^0,w^0) = 2\varepsilon_1\varepsilon_2 c_{11}c_{12}\Big((1-\varepsilon_1)b_{00}^2 + (-1+\varepsilon_2)b_{11}^2\Big) = 0$ and
$C_p(t^1,u^0,v^0,w^1) = -2\varepsilon_1\varepsilon_2 c_{11}c_{12}\Big((-1+\varepsilon_1)b_{00}^2 + (1+\varepsilon_2)b_{11}^2\Big) = 0$ and, consequently, the alternatives

Subcase A.1: $\Big((1-\varepsilon_1)b_{00}^2 + (-1+\varepsilon_2)b_{11}^2\Big) = 0$ and $\Big((-1+\varepsilon_1)b_{00}^2 + (1+\varepsilon_2)b_{11}^2\Big) = 0$
Subcase A.2: $c_{12} = 0$.

Subcase A.1: By adding the two conditions we get $2\varepsilon_2 b_{11}^2 = 0$, a contradiction to $\varepsilon_2 \neq 0$ and (60).

Subcase A.2:
$$C_p(t^0,u^0,v^0,w^2) = \varepsilon_1^2\,(1+\varepsilon_2)\,(b_{00}\,c_{00} - b_{11}\,c_{11})\,\Big(b_{00}\,c_{00}(-1+\varepsilon_2) + b_{11}\,c_{11}(1+\varepsilon_2)\Big) = 0$$

Subsubcase A.2.1: $c_{11} = c_{00}b_{00}b_{11}^{-1} \;\Rightarrow\; C_p(t^0,u^1,v^0,w^1) = 4\varepsilon_1^2\varepsilon_2^2 b_{00}c_{00}^2(b_{00} - b_{11}) = 0 \Rightarrow\; b_{11} = b_{00} \wedge c_{11} = c_{00}$. Now $T = \mathrm{diag}(b_{00}c_{00}, b_{00}c_{00}, b_{00}c_{00}, b_{00}c_{00}, b_{00}c_{00}, b_{00}c_{00})$, i.e., τ_λ is the identity.

Subsubcase A.2.2: $c_{11} = b_{00}c_{00}(1-\varepsilon_2)\Big(b_{11}(1+\varepsilon_2)\Big)^{-1}$. Now $C_p(t^0,u^2,v^0,w^0) = 8b_{00}^2c_{00}^2\varepsilon_1^2\varepsilon_2^2(-1+\varepsilon_2)(1+\varepsilon_2)^{-2}$ never vanishes.

With the exception of the extra case $\varepsilon_4 = -\varepsilon_3$ the discussion of *Case* A is completed now; as only result we get the identity.

Continuation of Case B. From $C_p(t^0,u^1,v^0,w^0) = C_p(t^0,u^1,v^1,w^1) = 0$ and $\varepsilon_4 \neq -\varepsilon_3$ by (67) we deduce $c_{01} = c_{21} = 0$. Thus $|\,c_{ik}| \overset{(61)}{=} c_{00}c_{11}c_{22} \neq 0 \;\Rightarrow\; c_{11} \neq 0$ and $c_{22} \neq 0$. Now: $C_p(t^0,u^2,v^0,w^1) = C_p(t^1,u^0,v^0,w^0) = 0 \;\Rightarrow\; \varepsilon_4 c_{00} + \varepsilon_3 c_{22} - c_{02} = \varepsilon_3 c_{00} + \varepsilon_4 c_{22} + c_{02} = 0 \;\Rightarrow\; c_{22} = -c_{00}$ and $c_{02} = (\varepsilon_4 - \varepsilon_3)c_{00}$. Hence we get:
$C_p(t^0,u^0,v^1,w^1) = -2\varepsilon_1\varepsilon_2 c_{11}c_{12}\Big((1+\varepsilon_2)b_{01}^2 + (-1-\varepsilon_1)b_{10}^2\Big) = 0$ and $C_p(t^0,u^1,v^1,w^0) = 2\varepsilon_1\varepsilon_2 c_{11}c_{12}\Big((-1+\varepsilon_2)b_{01}^2 + (1+\varepsilon_1)b_{10}^2\Big) = 0$. As the alternative $\Big((1+\varepsilon_2)b_{01}^2 + (-1-\varepsilon_1)b_{10}^2\Big) = \Big((-1+\varepsilon_2)b_{01}^2 + (1+\varepsilon_1)b_{10}^2\Big) = 0$ yields the contradiction $2\varepsilon_2 b_{01}^2 = 0$, compare (62), so $c_{12} = 0$ must hold. Now $C_p(t^0,u^0,v^0,w^2) = -\varepsilon_1^2\Big((1+\varepsilon_2)\,b_{01}\,c_{11} + (-1+\varepsilon_1)\,b_{10}\,c_{00}\Big)\Big((1+\varepsilon_2)\,b_{01}\,c_{11} + (-1-\varepsilon_1)\,b_{10}\,c_{00}\Big) = 0$ leads to the ramification:

Subcase B.1: $c_{11} = (1-\varepsilon_1)H$ with $H := b_{10}c_{00}\Big((1+\varepsilon_2)b_{01}\Big)^{-1}$. (70)
Subcase B.2: $c_{11} = (1+\varepsilon_1)H$.

Subcase B.1: Now $C_p(t^0,u^2,v^0,w^0) = \underbrace{4b_{10}^2c_{00}^2\varepsilon_1^2\varepsilon_2(-1+\varepsilon_1)(1+\varepsilon_2)^{-2}}_{\neq 0}(\varepsilon_1 + \varepsilon_2) = 0$ implies $\varepsilon_2 = -\varepsilon_1$. Thus $C_p(t^0,u^1,v^0,w^1) = -4\varepsilon_1^4 b_{10}c_{00}^2(b_{01} - b_{10}) = 0 \Rightarrow b_{10} = b_{01} \overset{(70)}{\Rightarrow} c_{11} = c_{00}$, whereby $p(t,u,v,w)$ becomes the zero polynomial. We substitute all found conditions in (56) and see that we may assume $b_{01}c_{00}=1$. Thus τ_λ is described by:

$$\begin{aligned} y_{00}\rho &= x_{10}+(\varepsilon_4-\varepsilon_3)x_{12}, \quad y_{01}\rho = x_{11}, \quad y_{02}\rho = -x_{12}, \\ y_{10}\rho &= x_{00}+(\varepsilon_4-\varepsilon_3)x_{02}, \quad y_{11}\rho = x_{01}, \quad y_{12}\rho = -x_{02}. \end{aligned} \tag{71}$$

Using (45) with $\varepsilon_2=-\varepsilon_1$ we return to the basis $\{\mathbf{p}_0,\dots,\mathbf{p}_5\}$ and get:

$$(\mathbf{p}_0p_0+\cdots+\mathbf{p}_5p_5)\mathbb{R} \overset{\tau_\lambda}{\longmapsto} \Big(\mathbf{p}_0p_4+\mathbf{p}_1p_3+\mathbf{p}_2(-p_5)+\mathbf{p}_3p_1+\mathbf{p}_4p_0+\mathbf{p}_5(-p_2)\Big)\mathbb{R},$$

wherefrom we read off that is τ_λ involutoric. The collineation τ_λ of $\mathrm{PG}(5,\mathbb{R})$ is induced by the polarity τ of $\mathrm{PG}(3,\mathbb{R})$ with

$$(\mathbf{b}_0a_0+\cdots+\mathbf{b}_3a_3)\mathbb{R} \overset{\tau}{\longmapsto} \{\mathbf{x}\mathbb{R}\in\mathrm{PG}(3,\mathbb{R}) \mid \mathbf{x}=\sum_{k=0}^{3}\mathbf{b}_kx_k \text{ and } a_0x_0+a_2x_1+a_1x_2-a_3x_3=0\}. \tag{72}$$

Subcase B.2: $C_p(t^0,u^2,v^0,w^0) = \underbrace{4b_{10}^2c_{00}^2\varepsilon_1^2\varepsilon_2(1+\varepsilon_1)(1+\varepsilon_2)^{-2}}_{\neq 0}(\varepsilon_1-\varepsilon_2) = 0 \Rightarrow \varepsilon_2=\varepsilon_1$.

Finally, $C_p(t^0,u^1,v^0,w^1)=0 \Rightarrow b_{10}=b_{01} \Rightarrow c_{11}=c_{00}$, which makes $p(t,u,v,w)$ to the zero polynomial. Also in this case τ_λ is described by (71). Using (45) with $\varepsilon_2=\varepsilon_1$ we return to the basis $\{\mathbf{p}_0,\dots,\mathbf{p}_5\}$ and get:

$$(\mathbf{p}_0p_0+\cdots+\mathbf{p}_5p_5)\mathbb{R} \overset{\tau_\lambda}{\longmapsto} \Big(\mathbf{p}_0p_1+\mathbf{p}_1p_0+\mathbf{p}_2p_5+\mathbf{p}_3p_4+\mathbf{p}_4p_3+\mathbf{p}_5p_2\Big)\mathbb{R},$$

i.e., τ_λ is involutoric, too, and τ_λ is induced by the polarity τ with

$$(\mathbf{b}_0a_0+\cdots+\mathbf{b}_3a_3)\mathbb{R} \overset{\tau}{\longmapsto} \{\mathbf{x}\mathbb{R}\in\mathrm{PG}(3,\mathbb{R}) \mid \mathbf{x}=\sum_{k=0}^{3}\mathbf{b}_kx_k \text{ and } a_3x_0-a_1x_1+a_2x_2+a_0x_3=0\}. \tag{73}$$

With the exception of the extra case $\varepsilon_4=-\varepsilon_3$ the discussion of *Case* B is completed now; only in two special cases we get a non-trivial automorphism: for $\varepsilon_2=-\varepsilon_1$ the polarity (72) and for $\varepsilon_2=\varepsilon_1$ the polarity (73).

We sum up in

Theorem 3. *Assume* $|\varepsilon_j|<10^{-4}$ $(j=1,2,3,4)$, $\varepsilon_1\varepsilon_2\neq 0$, *and* $\varepsilon_4\neq-\varepsilon_3$. *If* $\varepsilon_2\neq\pm\varepsilon_1$, *then the spread* $\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ *from Theorem* 1 *is hyperrigid. If* $\varepsilon_2=-\varepsilon_1$ *or* $\varepsilon_2=\varepsilon_1$, *then* $\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ *is rigid, but not hyperrigid, and* $\mathrm{Aut}_e\,\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ *consists of two elements, namely the identity and polarity from* (72) *or* (73), *respectively.*

6. The translation planes represented by the spreads $\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ with $\varepsilon_1\varepsilon_2\neq 0$

Theorem 4. *Assume* $|\varepsilon_j|<10^{-4}$ $(j=1,2,3,4)$, $\varepsilon_1\varepsilon_2\neq 0$, *and* $\varepsilon_4\neq-\varepsilon_3$. *Let* $\mathbf{P}(\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4})$ *be the (projective) translation plane represented by the spread* $\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ *of Theorem* 1. *Then:*

A) $\mathbf{P}(\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4})$ *is a rigid* 4*-dimensional translation plane.*

B) *The full collineation group of* $\mathbf{P}(\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4})$ *is* 5*-dimensional.*

C) $\mathbf{P}(\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4})$ *is not Bol.*

Proof. A) By Theorem 1, $\mathcal{B}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ is a topological spread and, by definition, a topological spread represents a 4-dimensional translation plane. For the rigidity compare Theorem 3.

B) Use [2, Satz 2].

C) See [26, Section 9, pp. 336–337]. □

7. The spreads $\mathcal{B}_{\varepsilon_1,0,\varepsilon_3,0}$ with $\varepsilon_1\varepsilon_3 \neq 0$

By Remark 13, $d \cap e = \{\mathbf{p}_4''\mathbb{R}\}$ and $\dim(c \vee d \vee e) = 4$; moreover,

$$v' \text{ resp. } v^p = \dim\Big(\bigvee \xi \mid \xi \in (B_{\varepsilon_1,\ldots,\varepsilon_4})' \text{ resp. } (B_{\varepsilon_1,\ldots,\varepsilon_4})^p\Big) = \dim(c \vee d \vee e) = 4;$$

according to [27, (10) and Table 1], the spreads $\mathcal{B}_{\varepsilon_1,0,\varepsilon_3,0}$ are symplectic.

Lemma 7. *A symplectic spread $\mathcal{S}$ of* $\mathrm{PG}(3,\mathbb{R})$ *is not hyperrigid.*

Proof. Let $\mathcal{G}$ be a linear complex of lines with $\mathcal{S} \subset \mathcal{G}$. By [23, p. 151, Rem. 4.1.3], $\mathcal{G}$ must be general. For the null polarity γ associated with $\mathcal{G}$ holds: $\gamma(x) = x$ for all $x \in \mathcal{G}$. Consequently, $\gamma(\mathcal{S}) = \mathcal{S}$. □

In order to get a simple description of the point set $\Phi(\mathcal{B}_{\varepsilon_1,0,\varepsilon_3,0})$ we use the basis $\{\mathbf{c}_{00}, \mathbf{c}_{01}, \mathbf{c}_{02}, \mathbf{c}_{10}, \mathbf{c}_{11}, \mathbf{c}_{12}\}$ with

$$\begin{aligned} \mathbf{c}_{00} &= \mathbf{p}_2, \quad \mathbf{c}_{01} = \mathbf{p}_0 - \mathbf{p}_3, \quad \mathbf{c}_{02} = \mathbf{p}_1, \\ \mathbf{c}_{10} &= \mathbf{p}_5, \quad \mathbf{c}_{11} = \mathbf{p}_1 - \mathbf{p}_4, \quad \mathbf{c}_{12} = -\mathbf{p}_0(1+\varepsilon_1) + \mathbf{p}_3(1-\varepsilon_1) - \mathbf{p}_5\varepsilon_3 \end{aligned} \tag{74}$$

such that (19), (21), (30), and (74) imply:

$$C_w = (\mathbf{c}_{00} + \mathbf{c}_{10}w)\mathbb{R},\ D_w = (\mathbf{c}_{01} + \mathbf{c}_{11}w)\mathbb{R},\ E_w = (\mathbf{c}_{11} + \mathbf{c}_{12}w)\mathbb{R} \text{ for all } w \in \mathbb{R} \cup \{\infty\} \tag{75}$$

and

$$\Phi(B_{\varepsilon_1,0,\varepsilon_3,0}) = \{(\mathbf{c}_{00} + \mathbf{c}_{01}u + \mathbf{c}_{10}w + \mathbf{c}_{11}(uw+v) + \mathbf{c}_{12}vw)\mathbb{R} \mid (u,v,w) \in (\mathbb{R}\cup\{\infty\})^3\}. \tag{76}$$

Immediately we see:

$$\Phi(B_{\varepsilon_1,0,\varepsilon_3,0}) \subseteq C_1 \cap C_2 \text{ where} \tag{77}$$

$$C_1 := \{\mathbf{z}\mathbb{R} \in \mathcal{P}_5 \mid \mathbf{z} = \sum_{j=0}^{1}\sum_{k=0}^{2} \mathbf{c}_{jk}z_{jk} \text{ and } z_{02} = 0\} \text{ and} \tag{78}$$

$$C_2 := \{\mathbf{z}\mathbb{R} \in \mathcal{P}_5 \mid z_{00}^2 z_{12} - z_{00}z_{10}z_{11} + z_{01}z_{10}^2 = 0\}. \tag{79}$$

Next we sharpen (77).

Proposition 4. *In the plane $d \vee e = \{\mathbf{z}\mathbb{R} \in \mathcal{P}_5 \mid z_{02} = z_{00} = z_{10} = 0\} \subset C_2$ the line set $\{D_w \vee E_w \mid w \in \mathbb{R}\cup\{\infty\}\}$ envelops the conic*

$$\begin{aligned} k := \{\mathbf{z}\mathbb{R} \in \mathcal{P}_5 \mid z_{02} = z_{00} = z_{10} = z_{11}^2 - 4z_{01}z_{12} = 0\} = \\ \{(\mathbf{c}_{01} + \mathbf{c}_{11}\cdot 2t + \mathbf{c}_{12}\cdot t^2)\mathbb{R} \mid t \in \mathbb{R}\cup\{\infty\}\}. \end{aligned} \tag{80}$$

Those points of $d \vee e$ which are interior points of k form the set

$$\mathrm{Int}(k) := \{\mathbf{z}\mathbb{R} \in \mathcal{P}_5 \mid z_{02} = z_{00} = z_{10} \text{ and } z_{11}^2 - 4z_{01}z_{12} < 0\} \tag{81}$$

and following equality is valid:

$$(C_1 \cap C_2) \setminus \mathrm{Int}(k) = \Phi(B_{\varepsilon_1,0,\varepsilon_3,0}). \tag{82}$$

Proof. The point $Y = (y_{00}, \ldots, y_{12})\mathbb{R} \in C_1 \cap C_2$ belongs to the plane $\beta_w = C_w \vee D_w \vee E_w$ if, and only if, the (4×5)-matrix

$$K := \begin{pmatrix} 1 & 0 & w & 0 & 0 \\ 0 & 1 & 0 & w & 0 \\ 0 & 0 & 0 & 1 & w \\ y_{00} & y_{01} & y_{10} & y_{11} & y_{12} \end{pmatrix}$$

is of rank 3, i.e., iff the determinants of all (4×4)-submatrices of K vanish:

$$G_1(w) := -w\left(y_{01}\, w^2 - y_{11} w + y_{12}\right) = 0, \quad G_2(w) := -w^2\left(y_{00}\, w - y_{10}\right) = 0,$$

$$G_3(w) := y_{01}\, w^2 - y_{11} w + y_{12} = 0, \quad G_4(w) := w\left(y_{00}\, w - y_{10}\right) = 0, \quad G_5(w) := y_{00}\, w - y_{10} = 0.$$

We consider $G_5(w) = 0$.

If $y_{00} \neq 0$, then we get: $w = y_{10}y_{00}^{-1}$. Now $G_j(y_{10}y_{00}^{-1}) = 0$ for $j = 2, 4, 5$ and

$$G_1(y_{10}y_{00}^{-1}) = G_3(y_{10}y_{00}^{-1}) = 0 \quad \Leftrightarrow \quad y_{00}^2 y_{12} - y_{00}y_{10}y_{11} + y_{01}y_{10}^2 = 0 \quad \overset{(79)}{\Leftrightarrow} \quad Y \in C_2.$$

Hence there exists $w \in \mathbb{R}$, namely $w = y_{10}y_{00}^{-1}$, such that $Y \in \beta_w$, i.e., $Y \in \Phi(B_{\varepsilon_1,0,\varepsilon_3,0})$.

If $y_{00} = 0$, then $Y \in C_2 \overset{(79)}{\Rightarrow} y_{01}y_{10}^2 = 0$.

Case $y_{01} = 0$: Now $Y = (0, 0, 0, y_{10}, y_{11}, y_{12})\mathbb{R}$ belongs to the plane $\mathbf{c}_{10}\mathbb{R} \vee \mathbf{c}_{11}\mathbb{R} \vee \mathbf{c}_{12}\mathbb{R} \overset{(75)}{=} C_\infty \vee D_\infty \vee E_\infty \overset{(9)}{=} \beta_\infty$, i.e., $Y \in \Phi(B_{\varepsilon_1,0,\varepsilon_3,0})$.

Case $y_{10} = 0$: Now $Y = (0, y_{01}, 0, 0, y_{11}, y_{12})\mathbb{R}$ belongs to the plane $\mathbf{c}_{01}\mathbb{R} \vee \mathbf{c}_{11}\mathbb{R} \vee \mathbf{c}_{12}\mathbb{R} \overset{(75)}{=} d \vee e$. We consider $G_3(w)$ and $G_1(w)$. There exists a $w \in \mathbb{R}$ such that $Y \in \beta_w$ if, and only if, the discriminant of $G_3(w)$ is not negative, in symbols: $y_{11}^2 - 4y_{01}y_{12} \geq 0$. □

Thus we have: $\lambda(\mathcal{B}_{\varepsilon_1,0,\varepsilon_3,0}) \overset{(15)}{=} \bigcup \left(\xi \cap H_5 \mid \xi \in B_{\varepsilon_1,0,\varepsilon_3,0}\right) =$

$$\Phi(B_{\varepsilon_1,0,\varepsilon_3,0}) \cap H_5 \overset{(82)}{=} (C_1 \cap C_2 \cap H_5) \setminus (\mathrm{Int}(k) \cap H_5).$$

Hence we compute $\mathrm{Int}(k) \cap H_5$. With

$$H_5 = \{\mathbf{z}\mathbb{R} \in \mathcal{P}_5 \mid z_{00}\, z_{10} - z_{00}\, z_{12}\, \varepsilon_3 - z_{01}^2 + 2\, z_{01}\, z_{12} - z_{02} z_{11} - z_{11}^2 + z_{12}^2(-1 + \varepsilon_1^2) = 0\} \tag{83}$$

and (80) we see that the determination of $H_5 \cap k$ is equivalent to the solution of the equation

$$f(t) := \underbrace{\left(-1 + \varepsilon_1^2\right)}_{<0} t^4 - 2\, t^2 - 1 = 0$$

in the unknown t. As $f(t) < -1$ for all $t \in \mathbb{R}$ and $\mathbf{c}_{12}\mathbb{R} \notin H_5$ $(t = \infty)$, so $H_5 \cap k = \emptyset$. This and $(\mathbf{c}_{01} + \mathbf{c}_{11} \cdot \varepsilon_1 + \mathbf{c}_{12})\mathbb{R} \in \mathrm{Int}(k) \cap H_5$ imply

$$\mathrm{Int}(k) \cap H_5 = (d \vee e) \cap H_5 = \{\mathbf{z}\mathbb{R} \in \mathcal{P}_5 \mid z_{02} = z_{00} = z_{10} = -z_{01}^2 + 2z_{01}z_{12} - z_{11}^2 + z_{12}^2(-1 + \varepsilon_1^2) = 0\}. \tag{84}$$

We sum up in

Theorem 5. *Assume $|\varepsilon_j| < 10^{-4}$, $j = 1, 3$, and $\varepsilon_1\varepsilon_3 \neq 0$. Consider the general linear complex*

$$\mathcal{C}_1 := \lambda^{-1}(\{\mathbf{p}\mathbb{R} \in \mathcal{P}_5 \mid p_1 + p_4 = 0\}) \tag{85}$$

of lines and the cubic complex

$$\mathcal{C}_2 := \lambda^{-1}\Big(\{\mathbf{p}\mathbb{R} \in \mathcal{P}_5 \mid -4\varepsilon_1^2 p_2^2(p_0 + p_3) + 4\varepsilon_1^2(-\varepsilon_3 p_0 - \varepsilon_3 p_3 + 2\varepsilon_1 p_5)p_2 p_4 + \Big(p_0(-1 + \varepsilon_1) + p_3(-1 - \varepsilon_1)\Big)(-\varepsilon_3 p_0 - \varepsilon_3 p_3 + 2\varepsilon_1 p_5)^2 = 0\}\Big) \tag{86}$$

of lines. The algebraic line congruence $\mathcal{C}_1 \cap \mathcal{C}_2$ contains the proper regulus

$$\mathcal{R}_{d \vee e} := \lambda^{-1}(\{\mathbf{p}\mathbb{R} \in \mathcal{P}_5 \mid p_1 + p_4 = p_2 = \varepsilon_3 p_0 + \varepsilon_3 p_3 - 2\varepsilon_1 p_5 = 0\}). \tag{87}$$

The line set $(\mathcal{C}_1 \cap \mathcal{C}_2) \setminus \mathcal{R}_{d \vee e}$ coincides with the symplectic spread $\mathcal{B}_{\varepsilon_1,0,\varepsilon_3,0}$ which[12] *admits the asymplecticly complemented regulization $\Lambda_{\varepsilon_1,0,\varepsilon_3,0}$ described in Section* 4. *The spread $\mathcal{B}_{\varepsilon_1,0,\varepsilon_3,0}$ is not hyperrigid.*

8. Algebraic spreads of PG(3, ℝ) are topological

Lemma 8. *Let $\mathcal{S}$ be an algebric spread of* $\mathrm{PG}(3,\mathbb{R})$. *Then $\mathcal{S}$ is topological, i.e., $\mathcal{S}$ represents a topological translation plane, and $\mathcal{S}$ is also a dual spread.*

Proof. The algebraic spread $\mathcal{S}$ is described by a finite number of algebraic forms $f_k : \Pi_5 \to \mathbb{R}$, $k = 1, \dots, N$, in the Plücker coordinates $p_0, \dots, p_5$; recall that Π_5 is the projective space on $\mathbb{R}^4 \wedge \mathbb{R}^4$. The forms f_k and the quadratic form $h_5 : \Pi_5 \to \mathbb{R}$ are continuous mappings from the compact space Π_5, cf. [30, 64.3, p. 351], into $\mathbb{R}$. Hence $\lambda(\mathcal{S})$ is the intersection of the zero-sets $f_1^{-1}(0), \dots, f_N^{-1}(0)$ and $h_5^{-1}(0)$. By [11, p. 327], each of these zero-sets is closed and, consequently, $\lambda(\mathcal{S})$ and H_5 are closed. According to [11, Theorem 1.4(3), p. 224], $\lambda(\mathcal{S})$ and H_5 are compact subspaces of the compact space Π_5. As the Klein mapping $\lambda : \mathcal{L} = \mathcal{G}_{3,1} \to H_5$ is a homeomorphism, cf. [29, Theorem 2.2.(d), p. 19], so $\mathcal{S}$ is a compact subset of the compact set $\mathcal{L} = \mathcal{G}_{3,1}$, cf. [30, 64.3, p. 351]. With [17, Prop. 1.26, p. 22] follows that $\mathcal{S}$ represents a topological translation plane. By [7], each topological spread of $\mathrm{PG}(3,\mathbb{R})$ is a dual spread. □

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[12] Note that we did not answer the question whether $\mathcal{B}_{\varepsilon_1,0,\varepsilon_3,0}$ is algebraic or not.

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