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Coincidences of Simplex Centers and Related Facial Structures

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Abstract. This is an investigation of the geometric properties of simplices in Euclidean d-dimensional space for which analogues of the classical triangle centers coincide. A presentation of related results is given, partially unifying known results for $d = 2$ and $d = 3$.

Keywords: barycentric coordinates, centroid, cevian, circumcenter, Cholesky decomposition, Fermat-Torricelli point, Gram matrix, incenter, Monge point, orthocenter, (regular) simplex, simplex center

0. Introduction

Mainly by methods from linear algebra, we study the analogues of the classical triangle centers (cf. [23]) for general simplices in Euclidean d-dimensional space $\mathbb{E}^d, d \geq 2$. We focus on interpreting the significance of two or more of the classical centers coinciding. Also we give several instructive constructions of examples. The centers under study include the centroid, the circumcenter, the incenter, the orthocenter (or its proper higher dimensional

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generalization, the Monge point), and the Fermat-Torricelli point. We also consider two new centers with clear geometric meanings. In the last section we examine classes of simplices whose cevians through certain centers have equal lengths.

In dimension $d = 2$ it is a standard fact that if two classical centers coincide, then the triangle is equilateral; see, e.g., [22].

In dimension $d = 3$, the parallel conclusion is that if two classical centers coincide, then the tetrahedron is equiareal (i.e., it has faces of equal area implying that these faces are even congruent), but not necessarily equilateral (regular). One noteworthy point is that the orthocenter does not necessarily exist for $d \geq 3$. But when it does, one can often make much stronger conclusions, both for $d = 3$ and in higher dimensions. We hope to return to this point in a subsequent paper, where we plan to make a detailed analysis of orthocentric simplices.

In dimension $d \geq 4$ the situation becomes yet more complicated. When two of the classical centers coincide one can give a meaningful geometric description in terms of the facial structure of the studied simplices. But various examples of various degrees of subtlety show that in general, when two centers coincide, one cannot usually infer much about other centers. After having done some of the work described here we learned of the not-well-known papers of V. Devide (see [10] and [11]) where he also treated some of these problems. We give our own proofs, however. We also resolve questions he posed but did not answer.

Some of the material discussed here exists in various forms in older, scattered literature. We intend to give a unified presentation, collecting a number of related results, with proofs as well as references to the known literature. We also would like to mention the papers [12], [13], [33], [38] and [32], where special types of simplices are investigated, but only regarding their facial structure and not in view of coincidence of certain centers; see also the survey [30], § 9. Furthermore, in [2] some related results are given for simplices in normed linear spaces.

1. Terminology and notation

A *d-simplex* $S = [A_1, \ldots, A_{d+1}]$ in the Euclidean d-dimensional space $\mathbb{E}^d$, $d \geq 2$, with origin $\mathbf{0}$ is defined as the convex hull of $d+1$ affinely independent points (or position vectors) $A_1, \ldots, A_{d+1}$ in $\mathbb{E}^d$. Thus the vectors $A_i - A_j$, $1 \leq i \leq d+1$, $i \neq j$, are linearly independent for every $j \in \{1, \ldots, d+1\}$, and therefore the linear dependence relation

$$c_1 A_1 + \cdots + c_{d+1} A_{d+1} = \mathbf{0}$$

is unique up to multiplication by a scalar. The points $A_1, \ldots, A_{d+1}$ are called the *vertices* of S. A line segment that joins two vertices of S is called an *edge* of S, and a j-simplex whose vertices are any $j+1$ vertices of S is said to be a *j-face* of S. The $(d-1)$-simplex whose vertices are all vertices of S except for A_j, $j \in \{1, \ldots, d+1\}$, is called the jth *facet* of S, or the *facet opposite to* A_j. The d-simplex S is *regular* if all its edges have equal length, it is *equiareal* if all its facets have the same $(d-1)$-volume, and it is called *equifacetal* if all its facets are congruent (i.e., isometric, see [12] for interesting new results on equifacetal simplices). Moreover, we say that S is *equiradial* if all its facets have the same circumradius (see (4) below for a definition of the circumradius).

Every point P in the convex hull of $A_1, \ldots, A_{d+1}$ can be represented in the form

$$P = \frac{v_1 A_1 + \cdots + v_{d+1} A_{d+1}}{v}\,, \tag{1}$$

where v_j is the d-volume of the d-simplex obtained from S by replacing A_j by P. So the d-volume v of S is represented by $v = v_1 + \cdots + v_{d+1}$. Note also that $v_j = \frac{1}{d} a_j h_j$, where a_j is the $(d-1)$-volume of the jth facet of S, and h_j is the altitude from A_j to the jth facet, see, e.g., [3], Theorem 9.12.4.4, page 260. The numbers $\frac{v_j}{v}$ in (1) are called the *barycentric coordinates* of P with respect to $A_1, \ldots, A_{d+1}$.

The *centroid* $\mathcal{G}$ of the d-simplex $S = [A_1, \ldots, A_{d+1}]$ is defined as the average

$$\mathcal{G} = \frac{A_1 + \cdots + A_{d+1}}{d+1} \tag{2}$$

of its vertices.

The *insphere* of S is the sphere that is tangent to all $d+1$ facets of S; its center is the *incenter* $\mathcal{I}$ of S, and its radius is the *inradius* of S. Since $\mathcal{I}$ is equidistant to all the facets of S, it follows that the jth barycentric coordinate of $\mathcal{I}$ is proportional to the $(d-1)$-volume of the jth facet, i.e., the incenter is algebraically defined by

$$\mathcal{I} = \frac{a_1 A_1 + \cdots + a_{d+1} A_{d+1}}{a_1 + \cdots + a_{d+1}}\,, \tag{3}$$

where a_i is the volume of the ith facet of S. The *circumsphere* of S is the sphere passing through all vertices of S, and the center $\mathcal{C}$ of that sphere is called the *circumcenter* of S. Thus $\mathcal{C}$ is defined by the requirement that

$$||\mathcal{C} - A_i|| = ||\mathcal{C} - A_j|| \ \text{ for } \ 1 \le i \le j \le d+1\,, \tag{4}$$

where $||\mathcal{C} - A_i||$ is said to be the *circumradius* of S. Unlike the centroid and the incenter, the circumcenter may lie outside of S.

The *Fermat-Torricelli point* $\mathcal{F}$ of S is defined to be the point whose distances to all the vertices of S have minimal sum. Such a point exists and is unique, and setting

$$f(i) = \sum \left(\frac{A_j - A_i}{||A_j - A_i||} : 1 \le j \le d+1,\, j \ne i \right)$$

it is known (see [26], Theorem 1.1, and [4], Theorem 18.3 and Reformulation 18.4, cf. further also [9]) that if $||f(i)|| > 1$ for all $i \in \{1, \ldots, d+1\}$, then $\mathcal{F}$ is an interior point of S (floating case), and that if $||f(i)|| \le 1$ for some i then this i is unique and $\mathcal{F} = A_i$ (absorbed case). In the floating case, $\mathcal{F}$ is characterized by the property

$$\sum_{i=1}^{d+1} \frac{\mathcal{F} - A_i}{||\mathcal{F} - A_i||} = \mathbf{0}\,. \tag{5}$$

For an interesting application of the Fermat-Torricelli point in classical geometry, namely an extension of Napoleon's theorem to a d-dimensional space, we refer to [31].

The *orthocenter* $\mathcal{O}$ of a d-simplex S is, if it exists, the intersection of the $d+1$ altitudes of S. In stark contrast to the case $d=2$, a d-simplex might not have an orthocenter when $d \geq 3$, see [1] for $d=3$ and [19], [29] as well as [34] for higher dimensions.

The last center we want to define here is discussed only in the next section, i.e., for $d=3$ (but we introduce it for d arbitrary). For each edge $E_{ij}=A_iA_j$ of $S=[A_1,\dots,A_{d+1}]$ there is a unique hyperplane H_{ij} containing the centroid $\mathcal{G}_{ij}$ of the remaining $d-1$ vertices and perpendicular to E_{ij}. These $\binom{d+1}{2}$ hyperplanes have a common point, the *Monge point* $\mathcal{M}$ of S. This point $\mathcal{M}$ is a reflection of $\mathcal{C}$ in $\mathcal{G}$ and coincides, if S is orthocentric, with the orthocenter $\mathcal{O}$, see also [34].

2. Tetrahedra whose centers coincide

Any 3-simplex (or non-degenerate tetrahedron) has the centers $\mathcal{G},\mathcal{I},\mathcal{C},\mathcal{F}$, and $\mathcal{M}$ (for $\mathcal{F}$ see [25], and for $\mathcal{M}$ see [1], Article 229, pp. 76-77). It follows immediately that the Monge point of the tetrahedron $S=[A,B,C,D]$ coincides with its orthocenter if S is orthocentric, and that this is equivalent to

$$A\cdot B=C\cdot D\,,\ A\cdot C=B\cdot D\,,\ A\cdot D=B\cdot C\,, \tag{6}$$

where "$\cdot$" means the ordinary inner product. Only parts of the following theorem can be found in the basic references [1], [7], [37], and [39], which collect geometric properties of tetrahedra in the Euclidean 3-space.

Theorem 2.1. *For a tetrahedron $T\subset\mathbb{E}^3$ the following conditions are equivalent.*

1) *The tetrahedron T is equifacetal.*
2) *The tetrahedron T is equiareal.*
3) *Every two opposite edges of T are equal.*
4) *The perimeters of the facets of T are equal.*
5) *The circumradii of the facets of T are equal, i.e., S is equiradial.*
6) *The centroid, the incenter, the circumcenter, the Fermat-Torricelli point and the Monge point of T coincide.*
7) *Two of the five centers mentioned above coincide.*

Proof. It is clear that 1) implies all the other statements, because the group of isometries of an equifacetal tetrahedron $[A,B,C,D]$ is transitive, as it contains the Klein 4-group consisting of the permutations $\{(A\,B)\,(C\,D),(A\,C)\,(B\,D),(A\,D)\,(B\,C),e\}$ (see [12], where this is proved even for any dimension). That 2) implies 1) is the well-known theorem usually referred to as Bang's Theorem (cf. [16], [20], pp. 90–97, [5], [1], Article 306, p. 108). The implication 3) $\Rightarrow$ 1) is trivial, 4) implies 3) by solving the corresponding system of linear equations, and 5) implies 1) by [17], Theorem 3. That 6) implies 1) follows from [17], Theorem 5, which states that if any two of $\mathcal{G},\mathcal{I},\mathcal{C}$, and $\mathcal{F}$ coincide, then the tetrahedron T is equifacetal. This also follows from [1], Article 305, page 108, which states that if any two of $\mathcal{G},\mathcal{I},\mathcal{C}$, and $\mathcal{M}$ coincide, then T is equifacetal, cf. also [11]. Thus it remains to show that 7) implies 6), and in view of [17], Theorem 5, and [1], Article 305, page 108, it suffices to show that if $\mathcal{M}$ and

$\mathcal{F}$ coincide, then T is equifacetal. So assume that $\mathcal{M} = \mathcal{F} = \mathbf{0}$, and that $\mathbf{0}$ is in the interior of $T = [A, B, C, D]$. Set

$$||A|| = \frac{1}{a}\,,\ ||B|| = \frac{1}{b}\,,\ ||C|| = \frac{1}{c}\,,\ ||D|| = \frac{1}{d}\,.$$

Then $aA + bB + cC + dD = \mathbf{0}$, yielding by multiplication with aA that if $A \cdot B = A \cdot C = 0$, then $(aA)\cdot(dD) = -1$, and so $\mathbf{0}$ would be on the line AD, contradicting the assumption that $\mathbf{0}$ is an interior point of T. Hence at most one of $A \cdot B, A \cdot C$, and $A \cdot D$ is zero, and we may assume that $A \cdot B \neq 0$ and $A \cdot C \neq 0$. Taking norms of both sides of

$$aA + bB = -cC - dD\,,$$

we obtain $abA \cdot B = cdC \cdot D$. Since $0 \neq A \cdot B = C \cdot D$, it follows that $ab = cd$, and similarly $ac = bd$. Hence $ab = cd = ac = bd$, each being equal to $\sqrt{abcd}$. Therefore $a = b = c = d$, and $\mathbf{0}$ is the centroid $\mathcal{G}$ of T. Thus $\mathcal{G}$ coincides with the Fermat-Torricelli point of T, and by (5) T is equifacetal. □

Remark 1. It should be mentioned that equifacetal tetrahedra can be used to give interesting characterizations of Euclidean motions, see [28]. Also we mention here that equifacetal/equiareal tetrahedra are called *isosceles tetrahedra* by many authors, see, e.g., [1]. We will not follow that usage, since we use the notion of *isosceles simplices* in another sense; see the proofs of Theorems 3.3, 3.4, and in Section 4 below.

Motivated by 5. in Theorem 2.1 one might ask whether equality of the inradii of the facets of a tetrahedron implies equifacetality. Also it should be interesting to check whether all remains valid if more centers are added to the list in 6. of Theorem 2.1. Negative answers to both these questions are supplied in Theorems 2.2 and 2.3 below.

Theorem 2.2. *The inradii of the facets of a non-equifacetal tetrahedron, whose edges have lengths* $1, 1, 1, 1, 1, (3 + \sqrt{33})/6$, *are equal.*

Proof. To justify the existence of a tetrahedron whose edges are as given, and whose edge-lengths are, more generally, even equal to $1, 1, 1, 1, 1, t$ with $t \in (0, \sqrt{3})$, we start with a rhombus $ABCD$ whose sides all have unit length and whose short diagonal is AC. Keeping ABC fixed, we fold $ABCD$ against AC, letting D move towards B. The tetrahedra $T_t = [A, B, C, D]_t$ formed in this way have edges of length $1, 1, 1, 1, 1, t$ with t ranging in $(0, \sqrt{3})$. Now let $r = r(t)$ be the inradius of a triangle whose side-lengths are $1, 1, t$. Then $r = \frac{2a}{p}$, where a is the area and p the perimeter of the triangle. Heron's formula (cf. [8], § 1.5) yields

$$f(t) := 4(r(t))^2 = \frac{16a^2}{p^2} = \frac{t^2(2-t)}{t+2}\,,$$

and solving $f(t) = f(1)$ we find that $t = (3 + \sqrt{33})/6$, as desired. □

It is also interesting to find new natural centers whose coincidence with known ones does not imply that a tetrahedron is equifacetal. For this purpose we set

$$\mathcal{J} = \frac{\alpha A + \beta B + \gamma C + \delta D}{p}$$

for a tetrahedron $T = [A, B, C, D]$, where $\alpha, \beta, \gamma, \delta$ are the perimeters of the facets of T opposite to A, B, C, D, respectively, and $p = \alpha + \beta + \gamma + \delta$. We call $\mathcal{J}$ the *complementary* 1-*centroid* of T.

Theorem 2.3. *The incenter and the complementary* 1*-centroid of a tetrahedron coincide iff the inradii of its facets are equal. Consequently, there exist non-equifacetal tetrahedra T whose incenter and complementary* 1*-centroid coincide.*

Proof. By (3) (replacing $a_1, \dots, a_4$, i.e., the areas of the faces of T, by a, b, c, d) and the definition of the complementary 1-centroid the points $\mathcal{I}$ and $\mathcal{J}$ coincide iff

$$\frac{a}{\alpha} = \frac{b}{\beta} = \frac{c}{\gamma} = \frac{d}{\delta} \,,$$

which is equivalent to the property that the inradii of the facets are equal, since $\frac{2a}{\alpha}$ is the inradius of the facet opposite to A, etc.; the latter statement follows from Theorem 2.2. $\square$

3. Higher-dimensional simplices whose centers coincide

The situation in 3-space does not have exact analogues in higher dimensions. In the following we exhibit various related results for d-simplices if $d \geq 4$ or, in some cases, if $d \geq 3$.

Theorem 3.1. *Let $S = [A_1, \dots, A_{d+1}]$ be a d-simplex. If any two of the centroid, the circumcenter and the Fermat-Torricelli point of S coincide, then all three centers coincide.*

Proof. From the definitions follows that

$$\begin{array}{lcl} \mathbf{0} = \mathcal{G} & \Leftrightarrow & A_1 + \cdots + A_{d+1} = \mathbf{0}\,, \\ \mathbf{0} = \mathcal{F} & \Leftrightarrow & \frac{A_1}{||A_1||} + \cdots + \frac{A_{d+1}}{||A_{d+1}||} = \mathbf{0}\,, \\ \mathbf{0} = \mathcal{C} & \Leftrightarrow & ||A_1|| = \cdots = ||A_{d+1}||\,. \end{array}$$

Since the dependence relation among $A_1, \dots, A_{d+1}$ is unique up to multiplying by a scalar, the proof is complete. $\square$

We remark that the four statements of the following theorem were proven in [10]. However, we give partially new, shorter proofs. Also we need an additional notion. Namely, we say that a d-simplex S has *well-distributed edge-lengths* if *all* its facets have the same sum of squares of all their $\binom{d}{2}$ edge-lengths. (Another proposal would be to call such simplices *equivariant.*)

Theorem 3.2. *For any d-simplex S the following statements hold true.*

(i) *The centroid $\mathcal{G}$ and the circumcenter $\mathcal{C}$ of S coincide iff S has well-distributed edge-lengths.*

(ii) *The circumcenter $\mathcal{C}$ and the incenter $\mathcal{I}$ of S coincide iff $\mathcal{C}$ is interior and S is equiradial.*

(iii) *The incenter $\mathcal{I}$ and the centroid $\mathcal{G}$ of S coincide iff S is equiareal.*

(iv) *The points $\mathcal{G}, \mathcal{C}$, and $\mathcal{I}$ of S coincide iff two of the three conditions {S has well-distributed edge-lengths; S is equiradial; S is equiareal} hold, in each case implying the third.*

Proof. For the proof of statement (iv) we refer to [10], the other three equivalences will now be verified by proofs, which are shorter than those given in [10]. To prove (i), we take $\mathbf{0}$ to be the centroid of S, and we take the scalar product of $A_1 + \cdots + A_{d+1} = \mathbf{0}$ with A_{d+1} to obtain

$$\sum_{1\le i\le d} (A_i \cdot A_{d+1}) = -||A_{d+1}||^2 .$$

Then we have

$$\begin{aligned}
&\sum_{1\le i<j\le d+1} ||A_i - A_j||^2 \\
= &\sum_{1\le i<j\le d} ||A_i - A_j||^2 + \sum_{1\le i\le d} ||A_{d+1} - A_i||^2 \\
= &\sum_{1\le i<j\le d} ||A_i - A_j||^2 + \sum_{1\le i\le d+1} ||A_i||^2 + (d-1)||A_{d+1}||^2 - 2\sum_{1\le i\le d}(A_i \cdot A_{d+1}) \\
= &\sum_{1\le i<j\le d} ||A_i - A_j||^2 + \sum_{1\le i\le d+1} ||A_i||^2 + (d+1)||A_{d+1}||^2 \\
= &\ V_{d+1} + \sum_{1\le i\le d+1} ||A_i||^2 + (d+1)||A_{d+1}||^2 ,
\end{aligned}$$

where V_k denotes the sum of the squares of the edge-lengths of the kth facet. (Note that the last but one line is obtainable by the scalar product considered above.) Thus we have shown that $V_k + (d+1)||A_k||^2$ does not depend on k. Therefore the simplex has well-distributed edge-lengths iff the $||A_k||$'s are equal, i.e., iff $\mathbf{0}$ is the circumcenter.

To see (ii), drop a perpendicular from $\mathcal{C}$ to the facet opposite to A_i, $i \in \{1, \ldots, d+1\}$. The obtained intersection point is the circumcenter $\mathcal{C}_i$ of the ith facet. The distance of $\mathcal{C}$ to any vertex of S is the circumradius R of S, and the distance from $\mathcal{C}_i$ to any vertex of S different from A_i is the circumradius R_i of the ith facet. So the three points $\mathcal{C}, \mathcal{C}_i, A_j$ ($j = 1, \ldots, d+1; i \neq j$) form a right triangle, and $R^2 = {R_i}^2 + |\mathcal{C}\mathcal{C}_i|^2$. But if $\mathcal{I} = \mathcal{C}$, then $|\mathcal{C}\mathcal{C}_i|^2 = r^2$ (the squared inradius of S). Hence ${R_i}^2 = R^2 - r^2$, not depending on the choice of the facet, and S is equiradial. On the other hand, if ${R_i}^2$ does not depend on the choice of the facet, then the formula yields $|\mathcal{C}\mathcal{C}_i|^2 = R^2 - {R_i}^2$, also independent of the choice of the facet. Thus $\mathcal{C}$ has to be the incenter of S, since by assumption $\mathcal{C}$ and $\mathcal{I}$ lie on the interior side of the facet.

Now we show (iii). By (2) and (3) we see that $\mathbf{0}$ is the centroid of S iff $A_1 + \cdots + A_{d+1} = \mathbf{0}$, and that $\mathbf{0}$ is the incenter of S iff $v_1 A_1 + \cdots + v_{d+1} A_{d+1} = \mathbf{0}$, where v_i is the $(d-1)$-volume of the ith facet of S opposite to A_i. Since the dependence relation among $A_1, \ldots, A_{d+1}$ is unique up to multiplying by a scalar, (iii) is obtained.

Regarding (iv) we mention that if two of (i), (ii) and (iii) hold, then either $\mathcal{C} = \mathcal{G}$ or $\mathcal{C} = \mathcal{I}$, in each case yielding $\mathcal{C}$ as interior point. $\square$

M. Hajja and P. Walker [17] have shown that a tetrahedron satisfying $\mathcal{F} = \mathcal{I}$ must be equifacetal, see also Theorem 2.1. It follows that an orthocentric tetrahedron (i.e., a tetrahedron with orthocenter) in which $\mathcal{F} = \mathcal{I}$ must be regular. A similar statement holds in dimension

2. One may conjecture that in any dimension an orthocentric simplex with $\mathcal{F} = \mathcal{I}$ must be regular. This certainly is consistent with dimensional analysis: the space of orthocentric d-simplices is $(d+1)$-dimensional. The equality $\mathcal{F} = \mathcal{I}$ amounts to d equations, one for each coordinate. That leaves one degree of freedom, which can be accounted for by scaling. Since now we will show that a d-simplex, $d \geq 4$, with $\mathcal{F} = \mathcal{I}$ need not be equifacetal, any proof of the conjecture must somehow mix the two hypotheses together more than is done in dimension 3.

Theorem 3.3. *For any $d \geq 4$, a d-simplex whose incenter $\mathcal{I}$ and Fermat-Torricelli point $\mathcal{F}$ coincide need not to be equifacetal.*

Proof. We say that a d-simplex is *isosceles* if it has a vertex P such that all edges emanating from P have the same length. (Note that this definition generalizes the standard notion in dimension 2, but differs from the use of the term for $d = 3$ in some other sources, such as in [1]; see Remark 1 in Section 2 above.) So we denote an isosceles d-simplex S with base T and opposite vertex P by $S = [T, P]$. We will assume that T is equifacetal. It is known that non-regular equifacetal simplices exist in abundance arbitrarily near any regular simplex. As an equifacetal simplex, T has a unique center, which we arrange to lie at the origin $\mathbf{0} \in \mathbb{E}^{d-1}$, where $T \subset \mathbb{E}^{d-1} \subset \mathbb{E}^d$. We also assume that $P = (\mathbf{0}, h)$.

Let R and r denote the circumradius and inradius of T. It is known that $R \geq (n-1)r$, with equality if and only if T is regular. Suppose $T = [A_1, \ldots, A_d]$. Since T is equifacetal, we know that $\sum_{i=1}^d A_i = \mathbf{0}$, since the centroid is $\mathbf{0}$, and also that $|A_1| = \cdots = |A_d|$, since the circumcenter is $\mathbf{0}$. By symmetry all centers of S, such as the incenter and the Fermat-Torricelli point, have the form $(\mathbf{0}, z)$. Let $\mathcal{F} = (\mathbf{0}, f)$ and $\mathcal{I} = (\mathbf{0}, i)$. Now, provided S is not too short, the Fermat-Torricelli point $\mathcal{F}$ of S is characterized by the condition (5) (where we note that $\frac{P-\mathcal{F}}{||P-\mathcal{F}||} = (\mathbf{0}, 1)$).

Now $||A_i - \mathcal{F}|| = \sqrt{R^2 + f^2}$, so this sum yields

$$d(\mathbf{0}, -f)/\sqrt{R^2+f^2} = (\mathbf{0}, -1) \ \text{ or } \ df = \sqrt{R^2+f^2}\,.$$

Hence

$$(d^2-1)f^2 = R^2 \ \text{ or } \ f = \frac{R}{\sqrt{d^2-1}}\,.$$

In particular f and hence $\mathcal{F}$ do not depend on h, provided h is big enough, at least. (Otherwise $\mathcal{F} = P$.) The condition we need is just that

$$h > \frac{R}{\sqrt{d^2-1}}\,.$$

Now as h increases from 0 toward ∞, i increases from 0 toward r. So we can choose h so that $i = \frac{R}{\sqrt{d^2-1}}$ provided that $\frac{R}{\sqrt{d^2-1}} < r$. But we know that $R \geq (d-1)r$, with equality iff T is regular. So the possible region of success is

$$(d-1)r < R < \left(\sqrt{d^2-1}\right) r\,.$$

When $h = \frac{R}{\sqrt{d^2-1}}$, then i is certainly much less than $\frac{R}{\sqrt{d^2-1}}$, since the incenter lies in the interior of S. As h increases, i also increases toward r. If we choose T to be equifacetal but not regular, very near the regular $(d-1)$-simplex (as in [12], using $d \geq 4$), then R is close to, but larger than $(d-1)r$, hence less than $(\sqrt{d^2-1})r$. For this purpose, it is convenient to rescale each simplex under consideration so that $r = 1$ for all $(d-1)$-simplices considered. $\square$

Theorem 3.4. *For any $d \geq 4$, there are equiradial d-simplices which are not equiareal.*

Proof. We first show that if $S = [T, P]$ is an isosceles d-simplex with vertex P, base T (i.e., facet opposite to P), and edge-length h at P, then the circumradius R_S of S is given by

$$R_S = \frac{h^2}{2\sqrt{h^2 - {R_T}^2}}\,.$$

To see this, let $\mathcal{C}_S$ and $\mathcal{C}_T$ denote the circumcenters of S and T, respectively, and note that $\mathcal{C}_S$ lies on the line $P\mathcal{C}_T$, which is perpendicular to T. Let k denote the distance between $\mathcal{C}_S$ and $\mathcal{C}_T$. Let V be a vertex of T. Now, applying the Pythagorean theorem to the triangles $P\mathcal{C}_T V$ and $\mathcal{C}_S\mathcal{C}_T V$, with right angles at $\mathcal{C}_T$, we have $(R_S \pm k)^2 + {R_T}^2 = h^2$ and ${R_S}^2 - {R_T}^2 = k^2$. (Use a plus sign if $\mathcal{C}_S$ lies between P and $\mathcal{C}_T$, and a minus sign if $\mathcal{C}_T$ lies between P and $\mathcal{C}_S$.) One may solve the second equation for k and substitute the result in the first equation, getting

$$\left(R_S \pm \sqrt{{R_S}^2 - {R_T}^2}\right)^2 + {R_T}^2 = h^2\,.$$

Expanding and collecting terms yields $\pm 2R_S\sqrt{{R_S}^2 - {R_T}^2} = h^2 - 2{R_S}^2$. Squaring both sides, we have

$$4{R_S}^4({R_S}^2 - {R_T}^2) = h^4 - 4h^2{R_S}^2 + 4{R_S}^4\,,$$

hence $4(h^2 - {R_T}^2){R_S}^2 = h^4$, which yields the desired formula for R_S. Now we show that for any $d \geq 4$ there is an isosceles d-simplex with an equilateral (or a regular) base that is equiradial but not equiareal. Let T be a regular $(d-1)$-simplex, normalized for convenience to have edge-length 1, say. For isosceles d-simplices of the form $[T, P]$, where P has distance h to each of the vertices of T, we need to calculate what values of h yield equiradial d-simplices. Certainly, $h = 1$ works in any dimension, producing the regular d-simplex. But when $d \geq 4$, there is a second value of h yielding the desired examples. Note that all facets F of T have the same circumradius. We seek an edge-length h such that $R_{[F,P]} = R_T$ or

$$\frac{h^2}{2\sqrt{h^2 - {R_F}^2}} = \frac{1^2}{2\sqrt{1^2 - {R_F}^2}}\,,$$

which has two solutions. The first is $h^2 = 1$, yielding (as mentioned already) the regular d-simplex. The second is given by

$$h = \frac{R_F}{\sqrt{1 - {R_F}^2}}\,.$$

In order that this value of h gives rise to an honest isosceles simplex, it is necessary and sufficient that $h > R_T$, that is

$$\frac{R_F}{\sqrt{1 - {R_F}^2}} > \frac{1}{2\sqrt{1 - {R_F}^2}}$$

or

$$R_F > \frac{1}{2}\,.$$

Since R_F, being the circumradius of the regular $(d-2)$-simplex of edge-length 1, increases with d, and since $R_F = \frac{1}{2}$ for $d-2=1$, it follows that $R_F > \frac{1}{2}$ iff $d-2>1$, i.e., iff $d \geq 4$. □

Remark 2. The simplex constructed in Theorem 3.4 has an exterior circumcenter, which is equivalent to saying that $h^2 < 2{R_T}^2$. To see this, let R_d denote the circumradius of the regular d-simplex having unit edge-length. Then it is easy to see that

$${R_d}^2 = \frac{d}{2(d+1)}\,.$$

Since

$$h^2 = \frac{R_{d-2}^2}{1-{R_{d-2}}^2}\,,\quad {R_T}^2 = {R_{d-1}}^2\,,$$

it follows that

$$h^2 = \frac{d-2}{d}\,,\quad {R_T}^2 = \frac{d-1}{2d}\,,$$

and therefore $h^2 < 2{R_T}^2$, as desired.

Remark 3. One can have an alternative and intuitive view of the latter construction of equiradial d-simplices that are not equiareal for $d \geq 4$. Namely, consider the regular $(d-1)$-simplex T of edge-length 1 inscribed in its circumsphere of radius R_T. Now each facet F of T is also a facet of a second isosceles $(d-1)$-simplex inscribed in the same sphere, hence having the same circumradius as T. These "ears" can be folded up to form the desired d-simplex provided the length of the external edges is large enough or, equivalently, that the height of the "ears" is large enough. One can easily check that $d \geq 4$ suffices to complete the construction. □

We continue with a characterization of regular simplices by two coinciding centers, one of which still has to be defined. Namely, the 1-*center* of $S = [A_1, \ldots, A_{d+1}]$ is the center of the $(d-1)$-sphere which is tangent to all edges A_iA_j of S if it exists (in general it does not exist).

Theorem 3.5. *If the* 1*-center of a* d*-simplex* S *exists and coincides with the circumcenter of* S*, then* S *is regular.*

Proof. The circumcenter of S can be viewed as the intersection of the hyperplanes perpendicular to the edges at their midpoints. On the other hand, the 1-center is the intersection of hyperplanes perpendicular to the edges A_iA_j at points dividing any such edge into lengths $\overline{a}_i$ and $\overline{a}_j$ such that $|A_iA_j| = \overline{a}_i + \overline{a}_j$, depending only on its endpoints. (Note that all tangential segments from an exterior point of a $(d-1)$-sphere to the respective touching points have equal lengths.) The hyperplanes defining the 1-center are obviously parallel to the hyperplanes defining the circumcenter. Thus, if the 1-center exists and coincides with the circumcenter, the two families of hyperplanes must coincide, and it follows that $\overline{a}_i = \overline{a}_j = \frac{1}{2}|A_iA_j|$ for all different $i, j \in \{1, \ldots, d+1\}$. Hence all edge-lengths of S are equal to $2\overline{a}_i$, i.e., S is regular. □

Finally we mention three characterizations of regular simplices within the restricted family of orthocentric simplices (since these statements are related to our considerations). The first one was proved in [35]: *An orthocentric d-simplex is regular iff its orthocenter and its Fermat-Torricelli point coincide.* In [14] it was shown that *an orthocentric d-simplex is regular iff its centroid and its orthocenter coincide*, and [15] contains the observation that *an equiareal orthocentric d-simplex is regular.*

4. Constructions in dimension 4

In view of further results restricted to dimension 4, we continue with the representation of a tool that relates the geometry of a simplex S to the algebraic properties of a certain matrix associated to S (see [21] and [27]). Namely, for a d-simplex $S = [A_1, \ldots, A_{d+1}]$ in $\mathbb{E}^d$ one defines the Gram matrix $G(S)$ to be the symmetric, positive semidefinite $(d+1) \times (d+1)$ matrix of rank d whose (i, j)th entry is the inner product $A_i \cdot A_j$ (we mean the ordinary inner product), cf. [21], p. 407. Given $G(S)$, one can calculate the distances $d(A_i, A_j)$ for every i, j using the formula

$$(d(A_i, A_j))^2 = (A_i - A_j) \cdot (A_i - A_j) \ .$$

According to the last part of Proposition 9.7.1 in [3], $G(S)$ determines S up to an isometry of $\mathbb{E}^d$. Also one recovers S from $G(S)$ via the Cholesky factorization $G(S) = HH^t$, where the rows of H are the vectors A_i coordinatized with respect to some orthonormal basis of $\mathbb{E}^d$. In fact, if G is a symmetric, semidefinite, real matrix of rank r, say, then there exists a unique symmetric, positive semidefinite, real matrix of rank r with $H^2 = G$, cf. [21], Theorem 7.2.6, p. 405, and the symmetry of H implies $G = HH^t$.

A 4-simplex $S = [A, B, C, D, E]$ whose circumcenter and centroid coincide (with circumradius 1 and the origin as circumcenter, say) is thus determined by unit vectors A, B, C, D, E satisfying $A + B + C + D + E = \mathbf{0}$. The Gram matrix $G(S)$ is then a symmetric, positive semidefinite matrix of rank 4 having the form

$$G(S) = \begin{pmatrix} 1 & x & y & z & \bullet \\ \bullet & 1 & Z & Y & \bullet \\ \bullet & \bullet & 1 & X & \bullet \\ \bullet & \bullet & \bullet & 1 & \bullet \\ \bullet & \bullet & \bullet & \bullet & 1 \end{pmatrix} , \tag{7}$$

where $x = A \cdot B$, $y = A \cdot C$, $z = A \cdot D$, $X = C \cdot D$, $Y = B \cdot D$, $Z = B \cdot C$, and the $\bullet$'s are defined by the symmetry of $G(S)$ and the fact that the entries of every row add up to zero. Thus, to construct a simplex $S = [A, B, C, D, E]$ whose circumcenter and centroid coincide (at $\mathbf{0}$, say), we need to construct a matrix $G(S)$ of the form described in (7) and satisfying the conditions formulated after (7). We then define A, B, C, D, E to be the rows of the matrix H that satisfies $HH^t = G$. To the assumption that the resulting simplex S have $\mathbf{0}$ as its incenter we add the extra requirement that all facets of S have the same 3-volume. Denoting the 3-volume of the facet $[A, B, C, D]$ by V_E, we have $4V_E = \det M_E M_E^t$, where M_E is the matrix whose rows are the vectors $B - A$, $C - A$, $D - A$. In terms of x, y, z, X, Y, Z this is written as

$$\begin{aligned} 4{V_E}^2 &= \det \begin{pmatrix} (B-A)\cdot(B-A) & (B-A)\cdot(C-A) & (B-A)\cdot(D-A) \\ \bullet & (C-A)\cdot(C-A) & (C-A)\cdot(D-A) \\ \bullet & \bullet & (D-A)\cdot(D-A) \end{pmatrix} \\ &= \det \begin{pmatrix} 2-2x & 1-x-y+Z & 1-x-z+Y \\ \bullet & 2-2y & 1-y-z+X \\ \bullet & \bullet & 2-2z \end{pmatrix}, \end{aligned}$$

where the $\bullet$'s are to be filled in by symmetry. Defining V_A, V_B, V_C, V_D analogously and recalling that $E = -A-B-C-D$, we get

$$M_D = \begin{pmatrix} B-A \\ C-A \\ E-A \end{pmatrix} = \begin{pmatrix} B-A \\ C-A \\ -2A-B-C-D \end{pmatrix}$$

$$4{V_D}^2 = \det \begin{pmatrix} 2-2x & 1-x-y+Z & 1-x+y+z-Z-Y \\ \bullet & 2-2y & 1-y+x+z-X-Z \\ \bullet & \bullet & 4+2x+2y+2z \end{pmatrix}. \tag{8}$$

The matrices M_C, M_B, M_A (and $4{V_C}^2, 4{V_B}^2, 4{V_A}^2$) are obtained from M_D (and $4{V_D}^2$) by applying the permutations $(z\,y)(Z\,Y)(z\,x)(Z\,X)(y\,X)(Y\,x)$, respectively. Thus a simplex $S = [A, B, C, D, E]$ corresponds to a symmetric, positive semidefinite matrix $G(S)$ as in (7) each of whose rows adds up to zero with

$$4{V_E}^2 = 4{V_D}^2 = 4{V_C}^2 = 4{V_B}^2 = 4{V_A}^2\,. \tag{9}$$

Theorem 4.1. *Let G be a symmetric matrix of the form* (7) *each row of which adds up to zero, and let G_0 be obtained from G by taking*

$$y = Y = x\,,\ \ z = Z = X = -\frac{1}{2} - x\,,$$

where x is such that

$$\frac{-\sqrt{5}-1}{4} < x < \frac{\sqrt{5}-1}{4} \quad (\textit{or, equivalently, } 72^\circ < \cos^{-1} x < 144^\circ)\,. \tag{10}$$

Let A, B, C, D, E be the row vectors of the matrix H defined by $HH^t = G_0$, and let $S = [A, B, C, D, E]$. Then the centroid, the circumcenter, the incenter and the Fermat-Torricelli point of S coincide.

Proof. The characteristic polynomial of G_0 is $T(T-r_1)^2(T-r_2)^2$ with

$$r_1, r_2 = \frac{5 \pm \sqrt{5}(4x+1)}{4}\,,$$

as one can immediately check. By (10), r_1 and r_2 are positive, and therefore G_0 is positive semidefinite and of rank 4. Thus there exists a real matrix H of rank 4 with $HH^t = G_0$.

The row vectors A, B, C, D, E of H are unit vectors since the diagonal entries of G_0 are 1's. From the discussion above we still need only to check that $V_A = V_B = V_C = V_D = V_E$, which is immediate, with $V_A = \frac{5}{2}\sqrt{1-2x-4x^2}$. This completes the proof. □

Remark 4. According to a Maple search, the simplices constructed in Theorem 4.1 are the *only* 4-simplices whose incenter, circumcenter, and centroid coincide. One can easily check that these simplices are equifacetal for all x in the specified interval, thus proving that the three traditional centers of a 4-simplex S coincide if and only if S is equifacetal. One wonders whether such a statement is valid in higher dimensions. □

Theorem 4.2. *There exists an equiareal 4-simplex whose centroid, circumcenter and Fermat-Torricelli point are pairwise distinct.*

Proof. Let

$$G = \begin{pmatrix} 1 & x & x & -1-2x & x \\ x & 1 & 5x & x & x \\ x & 5x & 1 & x & x \\ -1-2x & x & x & 1 & x \\ x & x & x & x & 1 \end{pmatrix}.$$

It is routine to check that the characteristic polynomial of G is $g(T) = (T-(2x+2))(T-(1-5x))f(T)$, where

$$f(T) = T^3 - (2+3x)T^2 + (1+x-18x^2)T - 2x(x^2-8x-1),$$

and that exactly one of the zeros of $g(T)$ represents 0 while the others are non-negative iff $x = 4-\sqrt{17}$. Let $S = [A, B, C, D, E]$ be the 4-simplex that corresponds to G for this value of x. Thus, again A, B, C, D, E are the rows of the matrix H with $G = HH^t$. Since the diagonal of G consists of 1's, the circumcenter of S is $\mathbf{0}$. And since the rows of G do not add to zero, the centroid of S is not $\mathbf{0}$. From this and Theorem 4.1 it follows that the centroid, the circumcenter and the Fermat-Torricelli point are pairwise distinct. It remains to check equiareality. Again it is routine to show that the volumes of all facets of S are equal to $4-20x-4x^2+20x^3$. □

On the other hand one might ask how the properties (i)–(iv) in Theorem 3.2 are connected with each other. The following statements refer to this in 4-space.

Theorem 4.3. *For the centroid $\mathcal{G}$, the circumcenter $\mathcal{C}$, and the incenter $\mathcal{I}$ of a 4-simplex, the property $\{\mathcal{G} = \mathcal{C}\}$ does not imply any of $\{\mathcal{C} = \mathcal{I}, \mathcal{I} = \mathcal{G}\}$, and the property $\{\mathcal{I} = \mathcal{G}\}$ does not imply any of $\{\mathcal{G} = \mathcal{C}, \mathcal{C} = \mathcal{I}\}$.*

Proof. The existence of a non-equiareal 4-simplex with coinciding centroid and circumcenter was verified in [18]. Thus $\mathcal{G} = \mathcal{C} \nRightarrow \mathcal{C} = \mathcal{I}$ and $\mathcal{G} = \mathcal{C} \nRightarrow \mathcal{I} = \mathcal{G}$. Theorem 4.2 above shows that $\mathcal{I} = \mathcal{G} \nRightarrow \mathcal{G} = \mathcal{C}$ and $\mathcal{I} = \mathcal{G} \nRightarrow \mathcal{C} = \mathcal{I}$. □

V. Devidé [11] asked *whether there are 4-simplices which are both equiradial and equiareal, but not equifacetal.* In the following we will answer that question in the affirmative, by constructing a 1-parameter family of non-equifacetal 4-simplices that are both equiradial and equiareal. Namely, we will consider isosceles 4-simplices (for this notion see the proof of Theorem 3.3) whose bases are equifacetal 3-simplices $T = [A_1, A_2, A_3, A_4]$ with three distinct edge-lengths a, b, c, say. We denote such a tetrahedron by $T = (a, b, c, a, b, c)$, where the edges are written in the order $a_{12}, a_{23}, a_{13}, a_{34}, a_{14}, a_{24}$, with $a_{ij} = |A_iA_j|$. It is well-known that opposite edges of T are congruent, and that equifacetal tetrahedra exist iff a, b, c are the edge-lengths of an acute triangle, see, e.g., [1]. This is characterized by the conditions $a^2 < b^2+c^2$, $b^2 < a^2+c^2$, and $c^2 < a^2+b^2$. Using the notation from the proof of Theorem 3.3, T is the base of certain isosceles 4-simplices $S = [T, P]$, where P lies on a line perpendicular to the affine hull of T at T's circumcenter. Thus P has to be equidistant to the four vertices of T, and so $S = [T, P]$ can be described, in terms of edge-lengths, by $S = (a, b, c, a, b, c, h, h, h, h)$ for an appropriate h. We will show how to choose a, b, c, h for getting 4-simplices that are equiradial and equiareal, but not equifacetal. We also mention that these simplices cannot be obtained as perturbations of regular simplices.

Lemma 4.4. *Let D be an acute (or right) triangle with side lengths a, b, c and with circumradius R. Then*

$$\frac{a^2+b^2+c^2}{8} \geq R^2 \geq \frac{a^2+b^2+c^2}{9},$$

with the extreme values attained when D is right-angled and when D is equilateral.

Proof. Let $\mathbf{0}$ be the circumcenter of D. Applying the Law of Cosines to the triangles BOC, COA, and AOB, and using the facts that $\angle BOC = 2A$, etc., we obtain

$$a^2+b^2+c^2 = 6R^2 - 2R^2\sigma,$$

where $\sigma = \cos 2A + \cos 2B + \cos 2C$. We have $\sigma = -4\cos A \cos B \cos C - 1$ by [6], formula 682, page 166, and therefore

$$a^2+b^2+c^2 = 8R^2(1+\cos A \cos B \cos C). \tag{11}$$

Since D is acute, it follows that the minimum of $\cos A \cos B \cos C$ is 0, and is attained when D is right-angled. Also, the maximum of $\cos A \cos B \cos C$ is $1/8$, and is attained when $A = B = C$. This follows from the fact that if $x \neq y$, then

$$2\cos x \cos y = \cos(x-y) + \cos(x+y) < 1 + \cos(x+y) = 2\cos^2 \frac{x+y}{2}.$$

Thus $0 \leq \cos A \cos B \cos C \leq 1/8$, with the extreme values attained at right-angled and equilateral triangles. The rest follows from (11). □

Theorem 4.5. *Let $D = (a, b, c)$ be an acute triangle with side lengths a, b, c and with circumradius R. Let $T = (a, b, c, a, b, c)$ be the equifacetal tetrahedron having D as a facet, and let $S = (a, b, c, a, b, c, h, h, h)$ be the isosceles 4-simplex obtained by adjoining to T a vertex at feasible distance h from each vertex of T. Then S is equiareal and equiradial iff S is regular or*

$$R^2 = \frac{3(a^2+b^2+c^2)}{25} \text{ and } h^2 = \frac{a^2+b^2+c^2}{5}. \tag{12}$$

Consequently, there exist equiareal, equiradial 4*-simplices that are not equifacetal.*

Proof. Let K be the area of the triangle D and let $Q = 16K^2$. It is well-known that

$$\begin{aligned} Q &= 16K^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4) \\ R^2 &= \frac{a^2b^2c^2}{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}\,; \end{aligned}$$

see [22], page 69. Letting

$$u = a^2 + b^2 + c^2\,,\ v = a^2b^2 + b^2c^2 + c^2a^2\,,\ w = a^2b^2c^2\,,$$

we see that

$$Q = 4v - u^2\,,\ \ w = R^2Q\,. \tag{13}$$

We find it more convenient to work with the parameters u, R, and Q instead of a, b, and c, and we freely use the relations in (13). Recalling that T is the equifacetal tetrahedron (a, b, c, a, b, c), we let T' be the isosceles tetrahedron (a, b, c, h, h, h) and note that each facet of S other than T is congruent to T'.

By the well-known volume formula for d-simplices in terms of their edge-lengths (see, e.g., [36], Problem 1.18, page 29), the volume V of the tetrahedron with edge-lengths x, y, z, X, Y, Z is given by

$$288V^2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & x^2 & y^2 & Z^2 \\ 1 & x^2 & 0 & z^2 & Y^2 \\ 1 & y^2 & z^2 & 0 & X^2 \\ 1 & Z^2 & y^2 & X^2 & 0 \end{bmatrix}.$$

In particular, with $x = X = a, y = Y = b, z = Z = c$ the volume V_T of T is determined by

$$\begin{aligned} 288{V_T}^2 &= 4(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)(a^2 + b^2 - c^2) \\ &= 4(u - 2a^2)(u - 2b^2)(u - 2c^2) = 4(-u^3 + 4uv + 8w) \\ &= 4(u(4v - u^2) - 8w) = 4(uQ - 8QR^2) = 4Q(u - 8R^2)\,, \end{aligned}$$

see [1], page 102. Similarly, with $x = a, y = b, z = c, X = Y = Z = H$ the volume $V_{T'}$ of T' is determined by

$$\begin{aligned} 288{V_{T'}}^2 &= 4a^2b^2h^2 + c^2b^2h^2 - 2h^2b^4 + 4c^2a^2h^2 - 2c^4h^2 - 2a^4h^2 - 2a^2b^2c^2 \\ &= 2h^2Q - 2w = 2h^2Q - 2QR^2 = 2Q(h^2 - R^2)\,. \end{aligned}$$

Thus equiareality of S, given by $V_T = V_{T'}$, is equivalent to the condition

$$h^2 = 2u - 15R^2\,. \tag{14}$$

Equiradiality of S is equivalent to the condition $R_T = R_{T'}$, where $R_T, R_{T'}$ are the circumradii of T and T', respectively. By [1], p. 102, R_T is given by

$${R_T}^2 = \frac{a^2 + b^2 + c^2}{8} = \frac{u}{8}\,. \tag{15}$$

For $R_{T'}$, we use the formula obtained in the proof of Theorem 3.3, yielding

$$R_{T'}{}^2 = \frac{h^4}{4(h^2 - R^2)}\,.$$

From this and (15) it follows that equiradiality is equivalent to the condition

$$2h^4 = u(h^2 - R^2)\,. \tag{16}$$

It is easy to verify that (12) satisfies both (14) and (16). Conversely, if (14) and (16) hold then, eliminating h, we obtain

$$2(2u - 15R^2)^2 = u(2u - 15R^2 - R^2)\,,$$

which simplifies into

$$0 = 3u^2 - 52uR^2 + 225R^4 = (u - 9R^2)(3u - 25R^2)\,.$$

By Lemma 4.4, the solution $u = 9R^2$ corresponds to the equilateral triangle $a = b = c$. In view of (14) this corresponds to the case when $h = a$, i.e., to the case when S is the regular 4-simplex. The solution $3u = 25R^2$ corresponds, again in view of (14), to the case $h^2 = u/5$.

To prove the last statement, note that if $D = (a, b, c)$ runs over all acute triangles inscribed in a circle of radius R, then, by Lemma 4.4 and continuity arguments, $a^2 + b^2 + c^2$ will take all values between $8R^2$ and $9R^2$. Thus, given any $R > 0$, there exists an acute triangle whose side lengths a, b, c satisfy $25R^2 = 3(a^2 + b^2 + c^2)$. Finally, to guarantee the existence of the isosceles 4-simplex $(a, b, c, a, b, c, h, h, h, h)$, h can take any value that is greater than the circumradius R_T of T. Thus, in view of (15), the only restriction on h is given by

$$h^2 > \frac{a^2 + b^2 + c^2}{8}\,,$$

and the choice $h^2 = (a^2 + b^2 + c^2)/5$ falls within this restriction.

Remark 5. The simplex $S = [T, h]$ constructed in Theorem 4.5 has an exterior circumcenter. In fact, it follows from

$$h^2 = \frac{a^2 + b^2 + c^2}{5}\,, \quad R_T{}^2 = \frac{a^2 + b^2 + c^2}{8}$$

that

$$h^2 < 2R_T{}^2 < R_T{}^2 + R_S{}^2\,,$$

implying that the circumcenter of S cannot be interior.

Remark 6. Within the family of isosceles 4-simplices with an equifacetal base, the degree of freedom in constructing an equiareal, equiradial, but non-equifacetal simplex is embodied in our freedom in choosing an acute triangle whose side lengths a, b, c and circumradius R satisfy the relation

$$R^2 = \frac{3(a^2 + b^2 + c^2)}{25}\,.$$

It would be interesting to investigate whether this freedom can be exploited in constructing 4-simplices that have, beside equiareality, equiradiality and non-equifacetality, additional significant properties.

5. Center coincidence and cevians

This section will refer to the relationship between coinciding centers on the one hand, and the lengths of cevians associated with these centers on the other hand.

As already mentioned, the affine independence of the vertex set of $S = [A_1, \ldots, A_{d+1}]$ allows a unique (up to a constant) linear combination of the origin $\mathbf{0}$, i.e.,

$$\mathbf{0} = a_1 A_1 + \cdots + a_{d+1} A_{d+1} \tag{17}$$

with

$$s = a_1 + \cdots + a_{d+1} \neq 0\,. \tag{18}$$

Namely, otherwise we would have

$$\mathbf{0} = a_1 (A_1 - A_{d+1}) + \cdots + a_d (A_d - A_{d+1})$$

with $a_1 = \cdots = a_d = 0$ and hence $a_{d+1} = 0$, contradicting the non-triviality of (17). We shall also assume that none of the vertices of S is $\mathbf{0}$, and that the lines through the vertices and $\mathbf{0}$ intersect the opposite facets. To say that the line joining A_{d+1} and $\mathbf{0}$ intersects the opposite facet is equivalent to the existence of numbers $c_1, \ldots, c_{d+1}$ such that

$$c_1 A_1 + \cdots + c_d A_d = c_{d+1} A_{d+1} \ \text{ and } \ c_1 + \cdots + c_d = 1\,.$$

From the uniqueness of (17) it follows that $a_1 + \cdots + a_d \neq 0$. Therefore we may also assume that

$$\text{no } d \text{ of the numbers } a_1, \ldots, a_{d+1} \text{ add up to } 0\,. \tag{19}$$

Under the assumptions (17), (18), and (19) we let A_j^* be the point where the line through A_j and $\mathbf{0}$ intersects the jth facet of S. The line segment $A_j A_j^*$ is usually called the *cevian* through A_j relative to $\mathbf{0}$. Since

$$\frac{-a_{d+1} A_{d+1}}{a_1 + \cdots + a_d} = \frac{a_1 A_1 + \cdots + a_d A_d}{a_1 + \cdots + a_d}\,,$$

and since the left hand side lies on the cevian and the right hand side lies on the facet, it follows that

$$A_{d+1}^* = -\frac{-a_{d+1} A_{d+1}}{a_1 + \cdots + a_d}$$

and

$$\|A_{d+1} - A_{d+1}^*\| = \frac{|a_1 + \cdots + a_{d+1}|}{|a_1 + \cdots + a_d|} \|A_{d+1}\|\,.$$

Thus we have that

$$\text{the } d+1 \text{ cevians through } \mathbf{0} \text{ are equal } \Leftrightarrow \frac{|s|}{|s - a_j|} \|A_j\| \text{ is independent of } j\,, \tag{20}$$

where $s = a_1 + \cdots + a_{d+1}$.

Theorem 5.1. *Let $S = [A_1, \ldots, A_{d+1}]$ be a d-simplex. Then the following properties of S are equivalent.*

1. *The centroid $\mathcal{G}$ and the circumcenter $\mathcal{C}$ of S coincide.*
2. *The cevians through the centroid $\mathcal{G}$ have equal lengths.*
3. *The cevians through the Fermat-Torricelli point $\mathcal{F}$ have equal lengths.*
4. *The circumcenter $\mathcal{C}$ lies in S and the cevians through $\mathcal{C}$ have equal lengths.*

Proof. Assume that the centroid is $\mathbf{0}$. Then $a_j = \frac{1}{d+1}$ for all j in (17). Using (20), we get the following equivalences.

The cevians through the centroid of S are of equal lengths

$\Leftrightarrow \frac{d+1}{d}||A_j||$ is independent of j

$\Leftrightarrow ||A_1|| = \cdots = ||A_{d+1}||$

$\Leftrightarrow \mathbf{0}$ is the circumcenter

$\Leftrightarrow$ the circumenter coincides with the centroid.

Assume that the Fermat-Torricelli point is $\mathbf{0}$. Then we may suppose that $a_j = \frac{1}{||A_j||}$ for all j in (17). Again using (20), we see the following equivalences.

The cevians through the Fermat-Torricelli point of S have equal lengths

$\Leftrightarrow a_i(s - a_i) = a_j(s - a_j) \Leftrightarrow (a_i - a_j)(s - a_i - a_j) = 0 \Leftrightarrow a_i = a_j$ for all i, j

$\Leftrightarrow a_1 = \cdots = a_{d+1}$

$\Leftrightarrow \mathbf{0}$ is the centroid of S

$\Leftrightarrow$ the Fermat-Torricelli point of S coincides with the centroid of S

$\Leftrightarrow$ the centroid of S coincides with the circumcenter of S (by Theorem 4.1).

Finally, assume that the circumcenter is $\mathbf{0}$ and that it lies in S. Then in (17) $||A_1|| = \cdots = ||A_{d+1}||$, and $a_j \geq 0$ for all j. Using (20), we get the following equivalences.

The cevians through the circumcenter of S are of equal lengths

$\Leftrightarrow |s - a_j|$ is independent of j

$\Leftrightarrow s - a_j$ is independent of j (because $s - a_j \geq 0$)

$\Leftrightarrow a_1 = \cdots = a_{d+1}$

$\Leftrightarrow \mathbf{0}$ is the centroid of S

$\Leftrightarrow$ the centroid of S and the circumcenter of S coincide. □

The last statement in Theorem 5.1 triggers the question whether there exists a d-simplex S whose circumcenter $\mathcal{C}$ is outside of S and whose cevians through $\mathcal{C}$ have equal lengths. Theorem 5.5 will show that this can happen if and only if $d \geq 4$. As preparation for that theorem, we need three lemmas.

Lemma 5.2. *Let $S = [A_1, \ldots, A_{d+1}]$ be a d-simplex whose circumcenter is $\mathbf{0}$ and whose circumradius is 1. Then the cevians through the circumcenter are equal if and only if there exists a suitable r with $0 \leq r < \frac{d+1}{2}$ such that, after some rearrangement,*

$$(2d - 2r + 1)(A_1 + \cdots + A_r) - (2r - 1)(A_{r+1} + \cdots + A_{d+1}) = \mathbf{0}\,. \tag{21}$$

Proof. Let the dependence relation among the A_i's be given as in (17), and let $s = a_1 + \cdots + a_{d+1}$. By rearranging the A_j's and by multiplying (17) with -1, if necessary, we may assume that $s - a_j \geq 0$ for $1 \leq j \leq r$ and $s - a_j < 0$ for $j > r$, where $0 \leq r \leq \frac{d+1}{2}$. By (20) we obtain the following equivalences.

The cevians through the circumcenter of S are of equal length
$\Leftrightarrow |s - a_j|$ is independent of j
$\Leftrightarrow s - a_1 = \cdots = s - a_r = -(s - a_{r+1}) = \cdots = -(s - a_{d+1})$
$\Leftrightarrow a_1 = \cdots = a_r,\ a_{r+1} = \cdots = a_{d+1} = 2s - a_1$
$\Leftrightarrow sa_1 = \cdots = sa_r = s\frac{2d-2r+1}{d+1-2r},\ sa_{r+1} = \cdots = sa_{d+1} = s\frac{1-2r}{d+1-2r}$
$\Leftrightarrow (2d - 2r + 1)(A_1 + \cdots + A_r) - (2r - 1)(A_{r+1} + \cdots + A_{d+1}) = \mathbf{0}$, as desired.

Note that the possibility $d+1-2r = 0$ is excluded since it leads to the contradiction $s = 0$.□

Lemma 5.3. *Let V be a unit vector in $\mathbb{E}^d$, and let $t \neq 0$ be in the open interval $(-d, d)$. If $d \geq 2$, then there exists a basis $B_1, \ldots, B_d$ of $\mathbb{E}^d$ consisting of unit vectors with $B_1 + \cdots + B_d = tV$. In fact, one can choose $B_1, \ldots, B_d$ to be equally inclined in the sense that $B_i \cdot B_j = B_k \cdot B_l$ whenever $i \neq j$ and $k \neq l$.*

Proof. Let U be the orthogonal complement of V, and let $[E_1, \ldots, E_d]$ be a regular $(d-1)$-simplex in U centred at $\mathbf{0}$ and having circumradius 1. Then $E_1, \ldots, E_d$ are equally inclined, affinely independent unit vectors with $E_1 + \cdots + E_d = \mathbf{0}$. As x takes all non-zero values, $\frac{dx}{\sqrt{1+x3^2}}$ takes all non-zero values in the open interval $(-d, d)$. It follows that there exists an x such that $\frac{dx}{\sqrt{1+x^2}} = t$. Let

$$B_j = \frac{E_j + xV}{\sqrt{1+x^2}}\,.$$

Then the B_j's are equally inclined unit vectors with $B_1 + \cdots + B_d = tV$. It remains to show that they are linearly independent. We use the fact that if a linear combination of equally inclined unit vectors vanishes, then all the coefficients are equal, see also [24]. Thus

$$\begin{aligned} c_1B_1 + \cdots + c_dB_d = \mathbf{0} \quad &\Rightarrow \quad (c_1E_1 + \cdots + c_dE_d) + x\,(c_1 + \cdots + c_d)\,V = \mathbf{0} \\ &\Rightarrow \quad c_1E_1 + \cdots + c_dE_d = (c_1 + \cdots + c_d)\,V = \mathbf{0} \\ & \quad \text{(since } V \perp U) \\ &\Rightarrow \quad c_1 = \cdots = c_d \text{ (by [24]) and } c_1 + \cdots + c_d = 0 \\ &\Rightarrow \quad c_1 = \cdots = c_d = 0, \text{ as desired.} \end{aligned}$$

□

Lemma 5.4. *Suppose that $2 \leq r \leq d-1$ and that b, c are non-zero real numbers such that $br + c(d - r + 1) \neq 0$. Then there exist affinely independent unit vectors $A_1, \ldots, A_{d+1}$ in $\mathbb{E}^d$ such that*

$$b\,(A_1 + \cdots + A_r) + c\,(A_{r+1} + \cdots + A_{d+1}) = \mathbf{0}\,. \tag{22}$$

Proof. Dividing (22) by an appropriate number, one may assume that b, c are non-zero small numbers, say in $(-1, 1)$. We decompose $\mathbb{E}^d$ into the direct sum of three mutually orthogonal subspaces U_{r-1}, U_1, U_{d-r} of dimensions $r-1, 1, d-r$, respectively, and we let V be a unit vector in U_1. By the previous lemma, the direct sum $U_1 \oplus U_{r-1}$ has a basis consisting of unit vectors $A_1, \ldots, A_r$ with $A_1 + \cdots + A_r = -cV$. Similarly, $U_1 \oplus U_{d-r}$ has a basis consisting of unit vectors $A_{r+1}, \dots, A_{d+1}$ with $A_{r+1} + \dots + A_{d+1} = bV$. Then $A_1, \dots, A_{d+1}$ satisfy (22). Also, since the sum of the coefficients in (22) is not zero, being nothing but $br + c(d - r + 1)$, it follows that $\mathbf{0}$ is in the affine hull of $A_1, \dots, A_{d+1}$. Hence the affine hull of $A_1, \dots, A_{d+1}$ is their linear span and thus has dimension $r + (d - r + 1) - 1 = d$. Therefore $A_1, \dots, A_{d+1}$ are affinely independent, as desired. □

Theorem 5.5. *There exists a d-simplex whose circumcenter is exterior and whose cevians through the circumcenter are of equal lengths if and only if* $d \geq 4$.

Proof. We use Lemma 5.2. Taking $d = 2$ and $r = 0$ in (21), we obtain $A_1 + A_2 + A_3 = \mathbf{0}$, and hence $\mathbf{0}$ is the centroid and cannot be exterior. Taking $d = 2$ and $r = 1$, we obtain $3A_1 - (A_2 + A_3) = \mathbf{0}$, which is impossible since $||3A_1|| = 3$. Similarly for the cases $d = 3, r = 0$ and $d = 3, r = 1$. For $d \geq 4$, let r be any number such that $2 \leq r < \frac{d+1}{2}$, and let $b = 2d - 2r + 1$, $c = -(2r - 1)$. Then $r \leq d - 1$ and $br + c(d - r + 1) \neq 0$. Hence, by Lemma 5.4 there exist affinely independent unit vectors $A_1, \dots, A_{d+1}$ such that $b(A_1 + \dots + A_r) + c(A_{r+1} + \dots + A_{d+1}) = \mathbf{0}$. Now the simplex $S = [A_1, \dots, A_{d+1}]$ has the desired properties. □

Unfortunately, we were not able to prove a statement in the spirit of Theorem 5.1 that refers to cevians of equal lengths going through the incenter of S. Such a result would be the natural generalization of the well-known Steiner-Lehmus theorem; see [8], pages 9 and 420.

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