Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 46 (2005), No. 2, 545-558.

# The Optimal Ball and Horoball Packings of the Coxeter Tilings in the Hyperbolic 3-space

*To the Memory of Professor H. S. M. Coxeter*

**Jenő Szirmai**

*Budapest University of Technology and Economics*
*Institute of Mathematics, Department of Geometry*
*H-1521 Budapest, Hungary*
*e-mail: szirmai@math.bme.hu*

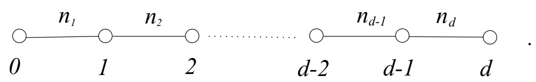
**Abstract.** In this paper I describe a method – based on the projective interpretation of the hyperbolic geometry – that determines the data and the density of the optimal ball and horoball packings of each well-known Coxeter tiling (Coxeter honeycomb) in the hyperbolic space $\mathbb{H}^3$.

## 1. Introduction

The regular Coxeter tilings or regular Coxeter honeycombs $\mathcal{P}$ are partitions of the hyperbolic space $\mathbb{H}^n$ ($n \geqq 2$) into congruent regular polytopes. A honeycomb with cells congruent to a given regular polyhedron $P$ exists if and only if the dihedral angle of $P$ is a submultiple of $2\pi$. All honeycombs for $n = 3$ with bounded cells were first found by Schlegel in 1883, those with unbounded cells by H. S. M. Coxeter in his famous article [5].

Another approach to describing honeycombs involves the analysis of their symmetry groups. If $\mathcal{P}$ is such a honeycomb, then any motion taking one cell into another takes the whole honeycomb into itself. The symmetry group of a honeycomb is denoted by $Sym\mathcal{P}$. Therefore the characteristic simplex $\mathcal{F}$ of any cell $P \in \mathcal{P}$ is a fundamental domain of the group $Sym\mathcal{P}$ generated by reflections in its facets (hyperfaces).

The scheme of a regular polytope $P$ is a weighted graph (characterizing $P \subset \mathbb{H}^n$ up to congruence) in which the nodes, numbered by $0, 1, \dots, d$ correspond to the bounding hyperplanes of $\mathcal{F}$. Two nodes are joined by an edge if the corresponding hyperplanes are not orthogonal. Let the set of weights $(n_1, n_2, n_3, \dots, n_{d-1})$ be the Schläfli symbol of $P$, and $n_d$ the weight describing the dihedral angle of $P$ that equals $\frac{2\pi}{n_d}$, and $\mathcal{F}$ the Coxeter simplex with the scheme




The ordered set $(n_1, n_2, n_3, \dots, n_{d-1}, n_d)$ is said to be the Schläfli symbol of the honeycomb $\mathcal{P}$. To every scheme there is a corresponding symmetric matrix $(a^{ij})$ of size $(d+1) \times (d+1)$ where $a^{ii} = 1$ and, for $i \neq j \in \{0, 1, 2, \dots, d\}$, $a^{ij}$ equals $-\cos \frac{\pi}{n_{ij}}$ with all angles between the facets $i$,$j$ of $\mathcal{F}$; then $n_k =: n_{k-1,k}$, too. Reversing the numbers of the nodes in the scheme of $\mathcal{P}$ (but keeping the weights), leads to the so called dual honeycomb $\mathcal{P}^*$ whose symmetry group coincides with $Sym\mathcal{P}$.

In [3], Böröczky and Florian determined the densest horosphere packing of $\mathbb{H}^3$ without any symmetry assumption. They proved that this provides the general density upper bound for all sphere packings (more precisely ball packings) of $\mathbb{H}^3$, where the density is related to the Dirichlet-Voronoi cell of every ball, as follows:

$$s_0 = (1 + \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} - \ - \ + \ + \cdots)^{-1} \approx 0.85327609.$$

This limit is achieved by the 4 horoballs touching each other in the ideal regular simplex whose honeycomb has the Schläfli symbol $(3, 3, 6)$, the horoball centres are just in the 4 ideal vertices of the simplex. Beyond the universal upper bound there are a few results in this topic ([4], [14], [15], [16]), therefore our method seems to be suited for determining local optimal ball and horoball packings for given hyperbolic tilings.

In this paper we investigate regular Coxeter honeycombs and their optimal ball and horoball packings in the hyperbolic space $\mathbb{H}^3$. By $Sym\mathcal{P}_{pqr}$ we denote the symmetry group of the honeycomb $\mathcal{P}_{pqr}$, $((p, q, r) = (n_1, n_2, n_3))$, thus

$$P_{pqr} = \{ \bigcup_{\gamma \ \in \ Sym\mathcal{P}_{pq}} \gamma(\mathcal{F}_{pqr}) \}.$$

Thus, for the density, we relate each ball or horoball, respectively, to its regular polytope $P_{pqr}$ which contains it, assumed not to be a Dirichlet-Voronoi cell.

These Coxeter-tilings are the following (according to the notation of H. S. M. Coxeter):

$$
\begin{aligned}
(p,\ q,\ r) = (3,5,3),\ (4,3,5),\ (5,3,4),\ (5,3,5), \quad & (1.1)\\
(3,3,6),\ (3,4,4),\ (4,3,6),\ (5,3,6), \quad & (1.2)\\
(3,6,3),\ (4,4,4),\ (6,3,6), \quad & (1.3)\\
(4,4,3),\ (6,3,3),\ (6,3,4),\ (6,3,5). \quad & (1.4)
\end{aligned}
$$

From these, in the first part of this paper, we shall consider every tiling, where a horosphere is inscribed in each regular polyhedron which is infinite centred and has proper or ideal vertices. Thus we obtain of the parameters (1.3–1.4) satisfying the above mentioned properties.

In the second part we consider the Coxeter honeycombs with parameters (1.1). In these cases the cells have proper centres and vertices, too, thus we investigate the ball packings where each ball lies in its regular polyhedron $P_{pqr}$.

In the third section we discuss tilings, where each vertex of the regular polyhedra is at the infinity. These polyhedra with parameters (1.2) will be called total asymptotic. In this part we shall consider two types:

1. The horoball centres lie in the infinite vertices of the cells and each polyhedron of the honeycomb contains only one horoball type.
2. The ball centres lie in the middle of the polyhedra.

With our method, based on the projective interpretation of hyperbolic geometry [11], [13], in each case we have determined the volume of the cells, moreover, we have computed the density of the optimal ball and horoball packings. This method can be generalized to the higher dimensions as well. The computations were carried out by *Maple V Release* 5 up to 30 decimals.

## 2. The optimal horoball packings for honeycombs with parameters (1.3–1.4)

### 2.1. The homogeneous coordinate system

In this section we consider those Coxeter tilings, where the infinite regular polyhedra are circumscribed about a horosphere and the polyhedra have proper or ideal vertices. These honeycombs are given by the parameters $(p,\ q,\ r)$ (Fig. 1) where the faces are regular $p$-gons, $q$ edges of this polyhedron meet in each vertex, and the dihedral angles of two faces are $\frac{2\pi}{r}$. In Fig. 1 we display a part of the infinite regular polyhedron of a Coxeter tiling, where $A_3$ is the centre of a horosphere, the centre of a regular polygon is denoted by $A_2$ ($A_2$ is also the common point of this face and the optimal horosphere), $A_0$ is one of its vertices, and we denote by $A_1$ the footpoint of $A_2$ on an edge of this face. It is sufficient to consider the optimal horoball packing in the orthoscheme $A_0A_1A_2A_3$ because the tiling can be constructed from such orthoschemes as fundamental domain of $Sym\mathcal{P}_{pqr}$.

We consider the real projective 3-space $\mathbb{P}^3(\mathbf{V^4}, V_4^*)$ where the one-, two- and three-dimensional subspaces of the 4-dimensional real vector space $\mathbf{V^4}$ represent the points, lines and planes of $\mathbb{P}^3$, respectively. The point $X(\mathbf{x})$ and the plane $\alpha(a)$ are incident if and only if $\mathbf{x}a = 0$, i.e. the value of the linear form $a$ on the vector $\mathbf{x}$ is equal to zero ($\mathbf{x} \in \mathbf{V^4} \setminus \{0\},\ a \in V_4^* \setminus \{0\}$). The straight lines of $\mathbb{P}^3$ are characterized by 2-subspaces of $\mathbf{V^4}$ or of $V_4^*$, i.e. by 2 points or dually by 2 planes, respectively [11].

We introduce a projective coordinate system, by a vector basis $\mathbf{b}_i$ $(i = 0,1,2,3)$ for $\mathbb{P}^3$, with the following coordinates of the points of the infinite regular polyhedron (see Fig. 1), $A_0(1, x_1, 0, 0),\ A_1(1, t_1, -t_2, 0),\ A_2(1, 0, 0, 0),\ A_3(1, 0, 0, 1)$.

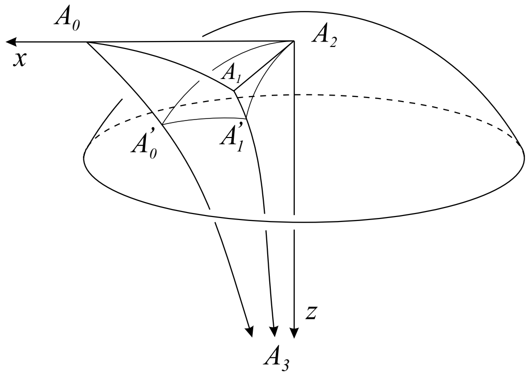



Figure 1.

## 2.2. Description of the horosphere in the hyperbolic space $\mathbb{H}^3$

We shall use the Cayley-Klein ball model of the hyperbolic space $\mathbb{H}^3$ in the Cartesian homogeneous rectangular coordinate system introduced in (2.1) (see Fig. 2). The equation of the horosphere with centre $A_3(1,0,0,1)$ through the point $S(1,0,0,s)$ is obtained [16] by Fig. 2:

$$0 = -2s(x^0)^2 - 2(x^3)^2 + 2(s+1)(x^0x^3) + (s-1)((x^1)^2 + (x^2)^2) \tag{2.1}$$

in the projective coordinates $(x^0, x^1, x^2, x^3)$. In the Cartesian rectangular coordinate system this equation is the following:

$$\frac{2(x^2+y^2)}{1-s} + \frac{4(z-\frac{s+1}{2})^2}{(1-s)^2} = 1, \text{ where } x := \frac{x^1}{x^0}, \ y := \frac{x^2}{x^0}, \ z := \frac{x^3}{x^0}. \tag{2.2}$$

The site of this horosphere in the part of the infinite regular polyhedron is illustrated in Fig. 1.

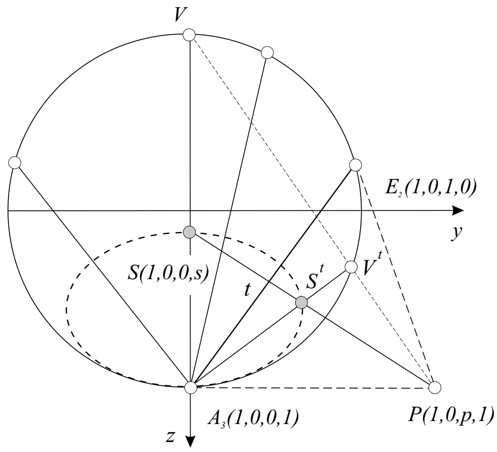



Figure 2.

### 2.3. The data of the cells of the regular honeycombs

By the projective method we can calculate the coordinates which are collected in Table 1.

**Table 1**

| $(p,q,r)$ | $t_1$ | $t_2$ | $x_1$ | $W_{pqr}$ |
|---|---|---|---|---|
| $(3,6,3)$ | $\frac{1}{4}$ | $\frac{\sqrt{3}}{4}$ | $1$ | 0.16915693 |
| $(4,4,3)$ | $\frac{1}{2\sqrt{2}}$ | $\frac{1}{2\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | 0.07633047 |
| $(4,4,4)$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $1$ | 0.22899140 |
| $(6,3,3)$ | $\frac{\sqrt{3}}{4}$ | $\frac{1}{4}$ | $\frac{1}{\sqrt{3}}$ | 0.04228923 |
| $(6,3,4)$ | $\frac{\sqrt{3}}{2\sqrt{2}}$ | $\frac{1}{2\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | 0.10572308 |
| $(6,3,5)$ | $\frac{\sqrt{6}\sqrt{10+\sqrt{2}}}{16}$ | $\frac{\sqrt{2}\sqrt{10+\sqrt{2}}}{16}$ | $\frac{\sqrt{2}\sqrt{7+3\sqrt{5}}}{\sqrt{3}(\sqrt{5}+1)}$ | 0.17150166 |
| $(6,3,6)$ | $\frac{3}{4}$ | $\frac{\sqrt{3}}{4}$ | $1$ | 0.25373540 |

By means of the theorem of N. I. Lobachevsky on the volume of orthoschemes in the hyperbolic 3-space (its application was described in [7] and [14]) we have determined the volume of each orthoscheme $A_0A_1A_2A_3$ for the parameters (1.3–1.4). The volumes $W_{pqr}$ are summarized in Table 1.

### 2.4. On the optimal horoballs

It is clear that the optimal horosphere has to touch the faces of its containing regular polyhedron. Thus the optimal horoball passes through the point $A_2(1,0,0,0)$ and the parameter $s$ in the equation of the optimal horosphere is 0 (see Section 2.2). The orthoscheme $A_0A_1A_2A_3$ and its images under $Sym\mathcal{P}_{pqr}$ divide the optimal horosphere into congruent horospherical triangles (see Fig. 1). The vertices $A'_0, A'_1, A'_2 = A_2(1,0,0,0)$ of such a triangle are in the edges $A_0A_3$, $A_2A_3$, $A_1A_3$, respectively, and on the optimal horosphere. Therefore, their coordinates can be determined in the Cayley-Klein model.

The lengths of the sides of the horospherical triangle (they are horocycles) are determined by the classical formula of J. Bolyai (see Fig. 3.):

$$l(x) = k \sinh \frac{x}{k} \text{ (at present } k = 1). \tag{2.3}$$

The volume of the horoball pieces can be calculated by the formula of J. Bolyai. If the area of the figure $A$ on the horosphere is $\mathcal{A}$, the volume determined by $A$ and the aggregate of axes drawn from $A$ is equal to

$$V = \frac{1}{2}k\mathcal{A} \ \text{ (we assume that } k = 1 \text{ here).} \tag{2.4}$$

It is well known that the intrinsic geometry of the horosphere is Euclidean, therefore, the area $\mathcal{A}_{pqr}$ of the horospherical triangle $A'_0\ A'_1\ A'_2$ is obtained by the formula of Heron.

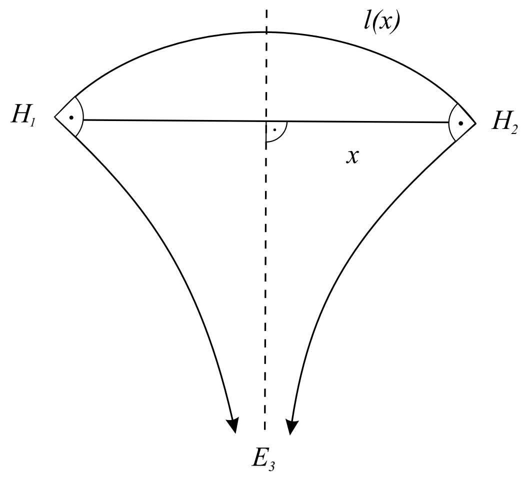



Figure 3.

**Definition 2.1.** *The density of the horoball packing for the regular honeycombs* (1.3− −1.4) *is defined by the following formula:*

$$\delta_{pqr} := \frac{\frac{1}{2}k\mathcal{A}_{pqr}}{W_{pqr}}. \tag{2.5}$$

In Table 2 we have collected the results of the optimal horoball packings for the parameters (1.3–1.4).

**Table 2**

| $(p,q,r)$ | $\mathcal{A}_{pqr}$ | $\delta^{pqr}$ |
|---|---|---|
| $(3,6,3)$ | 0.21650635 | 0.63995706 |
| $(4,4,3)$ | 0.06250000 | 0.81880805 |
| $(4,4,4)$ | 0.25000000 | 0.54587203 |
| $(6,3,3)$ | 0.03608439 | 0.85327609 |
| $(6,3,4)$ | 0.07216878 | 0.68262087 |
| $(6,3,5)$ | 0.09447006 | 0.55084110 |
| $(6,3,6)$ | 0.21650635 | 0.42663804 |

**Remark 2.2.** In the case $(6,3,3)$ we have obtained the arrangement of the densest horosphere packing [3].

## 3. The optimal ball packings to the regular honeycombs with parameters (1.1)

In Fig. 4 we have illustrated a part of the regular polyhedron of a Coxeter tiling, where $A_3$ is the centre of a cell, the centre of a regular polygon is denoted by $A_2$, $A_0$ is one of its vertices and we denote by $A_1$ the midpoint of an edge of this face. The regular polyhedra can be constructed with such orthoschemes. The cells for these parameters have proper vertices and centres. The volume of every regular polyhedron $\mathcal{P}_{pqr}$ is denoted by $V(\mathcal{P}_{pqr})$. In this section we are interested in ball packings, where the congruent balls with radius $R_{pqr}$ lie in cells of the above mentioned tilings.

**Definition 3.1.** *The density of the ball packing to any Coxeter honeycomb* (1.1) *can be defined by the following formula:*

$$\delta_{pqr} := \frac{2\pi\{\sinh(R_{pqr})\cosh(R_{pqr}) - R_{pqr}\}}{V(\mathcal{P}_{pqr})}. \tag{3.1}$$

It is clear that the optimal ball with centre $A_3$ has to touch the faces of its regular polyhedron (see Fig. 4.).

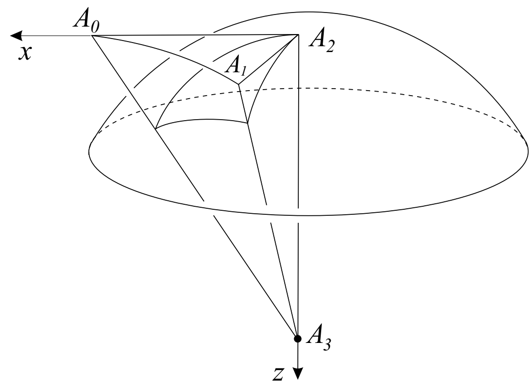



Figure 4.

Thus the optimal ball passes through the point $A_2$, and the optimal radius $A_2A_3$ of these tilings can be calculated by hyperbolic trigonometry. The following equation is obtained from the right-angled triangle $A_0A_2A_3$:

$$R_{pqr}^{opt} := A_2A_3 = \operatorname{arcosh}\frac{\cos\alpha}{\sin\beta} = \operatorname{arcosh}\frac{-a_{23}}{\sqrt{a_{22}a_{33}}}, \tag{3.2}$$

where the angles $\alpha = A_2A_0A_3\angle$ and $\beta = A_0A_3A_2\angle$ can be determined from the regular polytopes. On the other hand $R_{pqr}^{opt}$ can be computed also with our projective method [9], [13], where $(a_{ij}) = (a^{ij})^{-1}$ and $a^{ij} = -\cos\frac{\pi}{n_{ij}}$ (see Section 1).

Again, we have calculated the volume $W_{pqr}$ of the orthoschemes $A_0A_1A_2A_3$ for the parameters (1.1).

The volumes $W_{pqr}$ and the volumes $V(\mathcal{P}_{pqr})$ of the regular polyhedra $\mathcal{P}_{pqr}$ are summarized in Table 3.

**Table 3**

| $(p,q,r)$ | $W_{pqr}$ | $V(\mathcal{P}_{pqr})$ |
|---|---|---|
| $(3,5,3)$ | 0.03905029 | $120 \cdot W_{353} \approx 4.68603427$ |
| $(4,3,5)$ | 0.03588506 | $48 \cdot W_{435} \approx 1.72248304$ |
| $(5,3,4)$ | 0.03588506 | $120 \cdot W_{534} \approx 4.30620760$ |
| $(5,3,5)$ | 0.09332554 | $120 \cdot W_{535} \approx 11.19906474$ |

The optimal radius and optimal density are summarized by the formulas (3.1), (3.2) in the following table:

**Table 4**

| $(p,q,r)$ | $R^{opt}_{pqr}$ | $\delta^{pqr}_{opt}$ |
|---|---|---|
| $(3,5,3)$ | 0.86829804 | 0.68002717 |
| $(4,3,5)$ | 0.53063753 | 0.38437165 |
| $(5,3,4)$ | 0.80846083 | 0.58553917 |
| $(5,3,5)$ | 0.99638450 | 0.45079491 |

## 4. The optimal ball and horoball packings of the honeycombs with parameters (1.2)

In these cases under consideration the cells of the regular tilings have ideal vertices and proper centers. Fig. 5 shows a part of a total asymptotic regular polyhedron of a Coxeter tiling, where $A_3$ is the centre of a cell, the centre of an asymptotic regular polygon is denoted by $A_2$, $A_0$ is one of its ideal vertices and we denote with $A_1$ the "midpoint" (i.e. the footpoint of $A_2$) of an edge of this face.

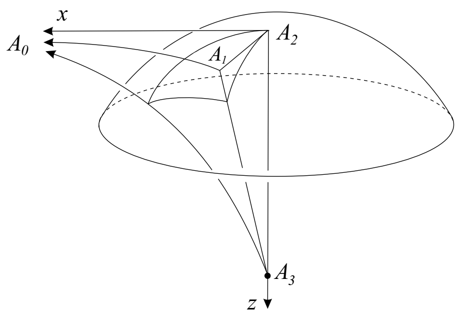



Figure 5.

### 4.1. The optimal ball packings

In this subsection we consider the ball packings where the congruent balls with radius $R_{pqr}$ lie in cells of the above mentioned Coxeter honeycombs. The volume of each regular polyhedron is denoted by $V(\mathcal{P}_{pqr})$. As in Section 3, the density can be defined by the formula (3.1). It is clear that the optimal ball passes through the point $A_2$, and the optimal radius $A_2A_3$ of these tilings can be calculated by hyperbolic trigonometry. The optimal radius $R^{opt}_{pqr} = A_2A_3$ is the *distance of parallelism* of the angle $A_0A_3A_2\angle$, thus the equation (4.1) follows from the formula of J. Bolyai (see (3.2)).

$$\tanh R_{pqr} = \cos\beta_i \ (i = 1,\ 2,\ 3,\ 4) \ \Leftrightarrow$$
$$\Leftrightarrow R^{opt}_{pqr} = A_2A_3 = \operatorname{arcosh}\frac{1}{\sin\beta_i} = \operatorname{arcosh}\frac{-a^i_{23}}{\sqrt{a^i_{22}a^i_{33}}}. \tag{4.1}$$

We obtain the values $\beta_i$ from the metric data of the regular polytopes:

1. Tetrahedron $\{3,\ 3\}$: $\beta_1 = \arccos\frac{1}{3}$,
2. Cube $\{4,\ 3\}$: $\beta_2 = \arccos\frac{1}{\sqrt{3}}$,
3. Octahedron $\{3,\ 4\}$: $\beta_3 = \arccos\frac{1}{\sqrt{3}}$,
4. Dodecahedron $\{5,\ 3\}$: $\beta_4 = \arccos\sqrt{\frac{5+2\sqrt{5}}{15}}$.

The volumes $W_{pqr}$ of the orthoschemes $A_0A_1A_2A_3$ can be calculated for the parameters (1.2), similarly to Sections 2 and 3. The regular, total asymptotic polyhedra of $\mathcal{P}_{pqr}$ can be constructed from these orthoschemes, thus the volume $V(\mathcal{P}_{pqr})$ can be determined. The optimal radius and the optimal density, respectively, is obtained by formulas (4.1) and (3.1). The results are collected in Table 5.

**Table 5**

| $(p,q,r)$ | $R^{opt}_{pqr} = \text{artanh}\beta_i$ | $V(\mathcal{P}_{pqr})$ | $\delta^{pqr}_{opt}$ |
|---|---|---|---|
| $(3,3,6)$ | 0.34657359 | 1.01494161 | 0.17597899 |
| $(4,3,6)$ | 0.65847895 | 5.07470803 | 0.25697101 |
| $(3,4,4)$ | 0.65847895 | 3.66386238 | 0.35592299 |
| $(5,3,6)$ | 1.08393686 | 20.58019935 | 0.32739972 |

### 4.2. The optimal horoball packings

In our cases (1.2) the vertices of a regular cell $E_i$, $i = 0,1,2,3,4\dots$, (Fig. 6) lie on the absolute of $\mathbb{H}^3$, therefore these vertices can be centres of some horoballs.

If the symmetry group $Sym\mathcal{P}_{pqr}$ of these tilings coincides with the symmetry group of the horospheres, then the optimal horoball packing corresponds to the optimal horoball packing of the dual Coxeter tilings $\mathcal{P}^*_{pqr}$. Thus we have not obtained any new optimal horosphere packings. Therefore, we investigate the horoball packings with one horoball type in each polyhedron of $\mathcal{P}_{pqr}$. We shall use the Cayley-Klein ball model of the hyperbolic space $\mathbb{H}^3$ in the Cartesian homogeneous rectangular coordinate system. We introduce for each Coxeter tiling a projective coordinate system, by vector bases $\mathbf{b}_i$ $(i = 0,1,2,3)$ for $\mathbb{P}^3$.

#### 4.2.1. The tetrahedron (3,3,6)

It is clear that in this case the optimal horoball packing corresponds to the optimal horoball packing of the Coxeter honeycomb with parameter $(3,6,3)$, as we have illustrated with horoball centre $E_3$ in the Fig. 6.

By the notation of Section 2 and by Definition 2.1 (see Fig. 1, Fig. 6)

$$W_{363} = W^1_{336} \approx 0.16915693, \quad \mathcal{A}_{363} = \mathcal{A}^1_{336} \approx 0.21650635,$$

$$\delta_{363} = \delta^1_{336} \approx 0.63995706.$$

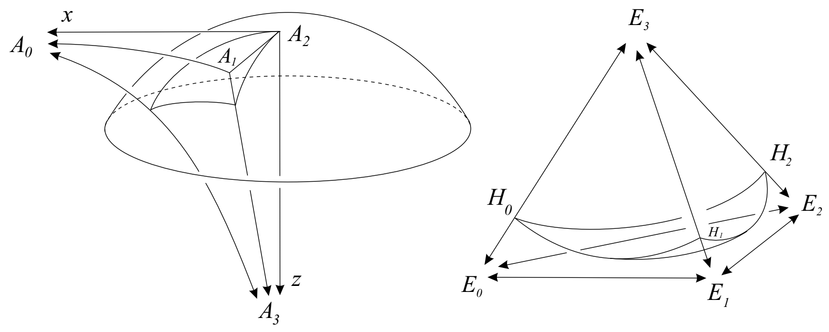



Figure 6.

### 4.2.2. The octahedron (3,4,4)

Fig. 7.a shows a projective coordinate system introduced by a Cartesian rectangular coordinate system with the homogeneous coordinates $E_0(1,0,0,0)$, $E_1(1,1,0,0)$, $E_2(1,0,1,0)$, $E_3(1,0,0,1)$. We consider the horoball packings with one horoball type whose center is $E_3(1,0,0,1)$. The equation of such horospheres were determined in the Subsection 2.2. It is clear that the optimal horosphere has to touch those faces of the octahedron that do not include the vertex $E_3(1,0,0,1)$ (Fig. 7.a). By the projective method (see [11], [14], [15], [16]) wee can calculate the coordinates of a footpoint $Y(\mathbf{y})$, the intersection of the perpendicular from the point $E_3(\mathbf{e}_3)$ on the plane $(u)$ where the plane $(u)$ is a side plane of the octahedron. The coordinates of this footpoint are $Y(\mathbf{y}) = (1, \frac{1}{2}, \frac{1}{2}, 0)$. This point is the “midpoint” of the edge $E_1E_2$. In order to find the equation of the optimal horosphere with centre $E_3(1,0,0,1)$ we have substituted the coordinates of the footpoints $Y(\mathbf{y})$ into the equation of the horosphere, and we have obtained the value of the parameter $s$ and so the equation of the optimal horosphere (see Fig. 7.a):

$$s = -\frac{1}{3}; \quad \frac{3}{2}x^2 + \frac{3}{2}y^2 + \frac{9}{4}(z - \frac{1}{3})^2 - 1 = 0. \tag{4.2}$$

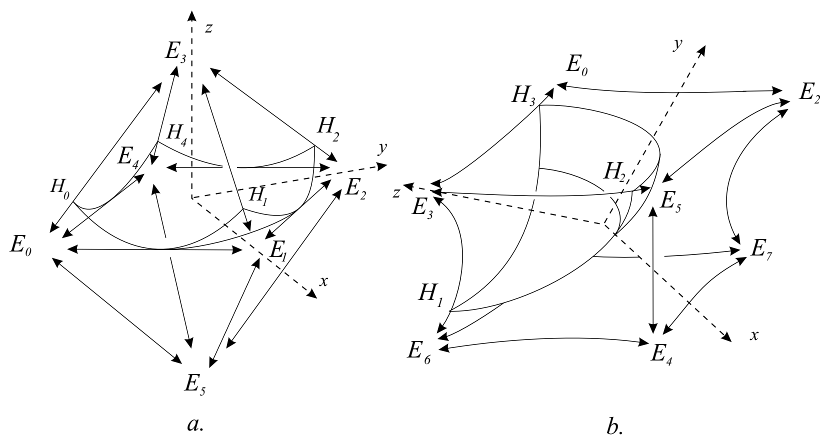



Figure 7.

The octahedra with common vertex $E_3$ divide the optimal horosphere into congruent horospherical quadrangles. The vertices $H_0, H_1, H_2, H_4$ of such a quadrangle are in the edges $E_3E_0$, $E_3E_1$, $E_3E_2$, $E_3E_4$, respectively, and on the optimal horosphere. Therefore, their coordinates can be determined in the Cayley-Klein model. They are summarized in Table 6. The area of the horospherical quadrilateral $H_0H_1H_2H_4$ is denoted by $\mathcal{A}_{344}^{opt}$ (see Fig. 7.a).

**Table 6**

| $H_i(\mathbf{h}_i)/$ | Octahedron |
|---|---|
| $H_0(\mathbf{h}_0)$ | $(1, 0, -\frac{4}{5}, \frac{1}{5})$ |
| $H_1(\mathbf{h}_1)$ | $(1, \frac{4}{5}, 0, \frac{1}{5})$ |
| $H_2(\mathbf{h}_2)$ | $(1, 0, \frac{4}{5}, \frac{1}{5})$ |
| $H_4(\mathbf{h}_4)$ | $(1, -\frac{4}{5}, 0, \frac{1}{5})$ |

Similar to the above sections we have calculated the volume $V(\mathcal{P}_{344})$ of the regular octahedron $P_{344}$ and we have determined the density of the optimal horoball packing by formulas (2.2), (2.3), (2.4), and according to Definition 2.1

$$\delta_{344}^{opt} = \frac{\frac{1}{2}\mathcal{A}_{344}^{opt}}{V(\mathcal{P}_{344})} \approx \frac{2.00000000}{3.66386238} \approx 0.54587203. \tag{4.3}$$

**Remark 4.1.** The optimal density of the horoball packing of the Coxeter honeycomb $(3, 4, 4)$ corresponds to the optimal density of $(4, 4, 4)$ (see 4.3 and Table 2.).

### 4.2.3. The cube (4,3,6)

Analogous to 4.2.2 we introduce a projective coordinate system, by an orthogonal vector basis $\mathbf{b}_i$ $(i = 0, 1, 2, 3)$ with signature $(-1, 1, 1, 1)$ for $\mathbb{P}^3$, with the following coordinates of the vertices of the infinite regular cube (see Fig. 7.b), in the Cayley-Klein ball model:

$$E_0(1, -\frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}),\ E_1(1, -\frac{\sqrt{2}}{\sqrt{3}}, -\frac{\sqrt{2}}{3}, -\frac{1}{3}),\ E_2(1, 0, 2\frac{\sqrt{2}}{3}, -\frac{1}{3}),$$

$$E_3(1, 0, 0, 1),\ E_4(1, \frac{\sqrt{2}}{\sqrt{3}}, -\frac{\sqrt{2}}{3}, -\frac{1}{3}).$$

Similar to 4.2.2 we have obtained the following results:

1. The equation of the optimal horosphere with centre $E_3$ corresponds to the formula (4.2). The site of this horosphere in the part of the infinite regular polyhedron is illustrated in Fig. 7.b.
2. The cubes with common vertex $E_3$ divide the optimal horosphere into congruent horospherical triangles. The coordinates of the vertices $H_1$, $H_2$, $H_3$ of such a triangle are collected in the following table:

| **Table 7** | |
|---|---|
| $H_i(\mathbf{h}_i)$ | Cube |
| $H_1(\mathbf{h}_1)$ | $(1, 0, -\frac{4\sqrt{2}}{7}, \frac{3}{7})$ |
| $H_2(\mathbf{h}_2)$ | $(1, \frac{2\sqrt{6}}{7}, \frac{2\sqrt{2}}{7}, \frac{3}{7})$ |
| $H_3(\mathbf{h}_3)$ | $(1, -\frac{2\sqrt{6}}{7}, \frac{2\sqrt{2}}{7}, \frac{3}{7})$ |

3. We have calculated the volume $V(\mathcal{P}_{436})$ of the regular cube $P_{436}$ and the area of the horospherical triangle $H_1$ $H_2$ $H_3$ which is denoted by $\mathcal{A}_{436}^{opt}$. Thus the density of the optimal horoball packing for cube (4,3,6) with one horoball type is

$$\delta_{436}^{opt} = \frac{\frac{1}{2}\mathcal{A}_{436}^{opt}}{V(\mathcal{P}_{436})} \approx \frac{2.59807621}{5.07470803} \approx 0.51196565. \tag{4.4}$$

### 4.2.4. The dodecahedron (5,3,6)

Similar to 4.2.2 we introduce a projective coordinate system for $\mathbb{P}^3$, with the following coordinates of the vertices of the infinite regular dodecahedron (see Fig. 8), in the Cayley-Klein ball model:

$$E_0(1, -\frac{\sqrt{5}-1}{2\sqrt{6}}, \frac{\sqrt{5}+3}{2\sqrt{6}}, \frac{\sqrt{5}}{3}),\ E_1(1, 0, \frac{2\sqrt{2}}{3}, -\frac{1}{3}),$$

$$E_2(1, \frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}), E_3(1, 0, 0, 1).$$

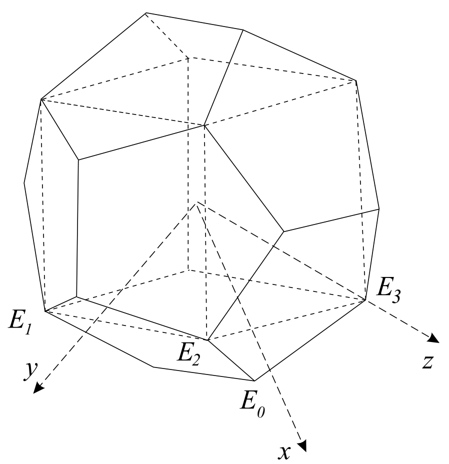



Figure 8.

Analogous to 4.2.2 and 4.2.3 we have obtained the following results:

1. The optimal horosphere has to touch some faces of the dodecahedron which do not include the vertex $E_3(1, 0, 0, 1)$ (Fig. 8), thus, in order to find the equation of the optimal horosphere, we have to calculate the coordinates of the footpoint $Y(\mathbf{y})$ of the

perpendicular from the point $E_3(\mathbf{e}_3)$ on the side plane $E_0E_1E_2$ (see Fig. 8) of the dodecahedron:

$$Y(\mathbf{y}) = (1, -\frac{(-3+\sqrt{5})^2\sqrt{6}}{4\,(-17+7\sqrt{5})}, \frac{(-3+\sqrt{5})\sqrt{2}(1+\sqrt{5})}{4\,(-17+7\sqrt{5})}, \frac{-3+\sqrt{5}}{(-17+7\sqrt{5})}).$$

2. The equation of the optimal horosphere with centre $E_3$ is

$$s = 0; \quad 2x^2 + 2y^2 + 4(z - \frac{1}{2})^2 - 1 = 0. \tag{4.5}$$

This horosphere touches, for example, the face $E_0E_1E_2$ of the regular dodecahedron and passes through the centre of the Cayley-Klein model.

3. The dodecahedra with common vertex $E_3$ divide the optimal horosphere into congruent horospherical triangles. The coordinates of the vertices $H_1$, $H_2$, $H_3$ of such a triangle are collected in the following table:

**Table 8**

| $H_i(\mathbf{h}_i)$ | Dodecahedron |
|---|---|
| $H_1(\mathbf{h}_1)$ | $(1, \frac{\sqrt{6}(2\sqrt{5}-1)}{38}, \frac{\sqrt{2}(3\sqrt{5}+8)}{38}, \frac{3\sqrt{5}+8}{19})$ |
| $H_2(\mathbf{h}_2)$ | $(1, -\frac{\sqrt{6}(5\sqrt{5}+7)}{76}, \frac{\sqrt{2}(3\sqrt{5}-11)}{76}, \frac{3\sqrt{5}+8}{19})$ |
| $H_3(\mathbf{h}_3)$ | $(1, \frac{\sqrt{6}(\sqrt{5}+9)}{76}, -\frac{\sqrt{2}(9\sqrt{5}+5)}{76}, \frac{3\sqrt{5}+8}{19})$ |

4. We have determined the volume $V(\mathcal{P}_{436})$ of the regular dodecahedron of $\mathcal{P}_{536}$ and the area of the horospherical triangle $H_1\ H_2\ H_3$ which is denoted by $\mathcal{A}_{536}^{opt}$. Thus the density of the optimal horoball packing for honeycomb (5,3,6) with one horoball type is

$$\delta_{536}^{opt} = \frac{\frac{1}{2}\mathcal{A}_{536}^{opt}}{V(\mathcal{P}_{536})} \approx \frac{8.90373963}{20.58019935} \approx 0.43263622. \tag{4.6}$$

The way of putting any analog questions for determining the optimal ball and horoball packings of tilings in hyperbolic $n$-space ($n > 2$) seems to be interesting and timely. Our projective method is suited to study and to solve these problems. We shall consider the optimal horoball packings for the higher dimensional Coxeter honeycombs in a forthcoming paper.

**Acknowledgement.** I thank Prof. Emil Molnár for helpful comments to this paper.

## References

[1] Böhm, J.; Hertel, E.: *Polyedergeometrie in n-dimensionalen Räumen konstanter Krümmung*. Birkhäuser, Basel 1981. Zbl 466.52001

[2] Böröczky, K.: *Packing of spheres in spaces of constant curvature*. Acta Math. Acad. Sci. Hung. **32** (1978), 243–261. Zbl 0422.52011

[3] Böröczky, K.; Florian, A.: *Über die dichteste Kugelpackung im hyperbolischen Raum*. Acta Math. Acad. Sci. Hung. **15** (1964), 237–245. Zbl 0125.39803

[4] Brezovich, L.; Molnár, E.: *Extremal ball packings with given symmetry groups generated by reflections in the hyperbolic 3-space* (in Hungarian). Mat. Lapok **34**(1–3) (1987), 61–91. Zbl 0762.52008

[5] Coxeter, H. S. M.: *Regular honeycombs in hyperbolic space*. Proc. internat. Congr. Math., Amsterdam 1954, **III**, 155–169. Zbl 0073.36603

[6] Dress, A. W. M.; Huson, D. H.; Molnár,E.: *The classification of face-transitive periodic three-dimensional tilings*. Acta Crystallographica **A.49** (1993), 806–819.

[7] Kellerhals, R.: *The Dilogarithm and Volumes of Hyperbolic Polytopes*. AMS Mathematical Surveys and Monographs **37** (1991), 301–336.

[8] Kellerhals, R.: *Ball packings in spaces of constant curvature and the simplicial density function*. J. Reine Angew. Math. **494** (1998), 189–203. Zbl 0884.52017

[9] Molnár, E.: *Projective metrics and hyperbolic volume*. Ann. Univ. Sci. Budap., Sect. Math. **32** (1989), 127–157. Zbl 0722.51016

[10] Molnár, E.: *Klassifikation der hyperbolischen Dodekaederpflasterungen von flächentransitiven Bewegungsgruppen*. Math. Pannonica **4**(1) (1993), 113–136. Zbl 0784.52022

[11] Molnár, E.: *The projective interpretation of the eight 3-dimensional homogeneous geometries*. Beitr. Algebra Geom. **38**(2) (1997), 261–288. Zbl 0889.51021

[12] Szirmai, J.: *Typen von flächentransitiven Würfelpflasterungen*. Ann. Univ. Sci. Budap. **37** (1994), 171–184. Zbl 0832.52006

[13] Szirmai, J.: *Über eine unendliche Serie der Polyederpflasterungen von flächentransitiven Bewegungsgruppen*. Acta Math. Hung. **73**(3) (1996), 247–261. Zbl 0933.52021

[14] Szirmai, J.: *Flächentransitive Lambert-Würfel-Typen und ihre optimale Kugelpackungen*. Acta Math. Hungarica **100** (2003), 101–116.

[15] Szirmai, J.: *Determining the optimal horoball packings to some famous tilings in the hyperbolic 3-space*. Stud. Univ. Zilina, Math. Ser. **16** (2003), 89–98. Zbl pre02114109

[16] Szirmai, J.: *Horoball packings for the Lambert-cube tilings in the hyperbolic 3-space*. Beitr. Algebra Geom. **46**(1) (2004), 43–60. Zbl pre02153146

Received November 10, 2004