Group actions on twin buildings

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Introduction

Though parts of the present paper perfectly look like abstract general nonsense, its origin is a rather concrete question. Given a simply connected almost simple Chevalley group $\mathcal{G}$ and a field k, what is a fundamental domain (in an appropriate sense) for the action of $\mathcal{G}(k[t,t^{-1}])$ on the product $\Delta_+ \times \Delta_-$ of the two associated Bruhat–Tits buildings $\Delta_+ = \Delta(\mathcal{G}(k((t^{-1}))))$ and $\Delta_- = \Delta(\mathcal{G}(k((t))))$? The solution of this problem is of interest if one wants to determine the finiteness properties of the S–arithmetic group $\mathcal{G}(\mathbb{F}_q[t,t^{-1}])$ by similar methods as used in [A1] and [A2]. An answer to the above question is given in Section 3, Proposition 5 below. I first derived this result by applying Theorem 1 of [So] which describes a simplicial fundamental domain (in the strictest sense) for the action of $\mathcal{G}(k[t])$ on Δ_+ together with Lemma 4 of Section 3. Then the proof of Lemma 4 also yielded a preliminary version of Lemma 2 involving both affine buildings Δ_+ and Δ_-. Thus I was led to considering twin buildings and twin BN–pairs which turned out to constitute the most natural framework for the original problem. For example, Soulé's result can easily be deduced and generalized in this context (see Proposition 6 and Remark 8). As I discovered afterwards, this possibility is already indicated in [T2], §15.5 and §15.7.

The action of $\mathcal{G}(k[t,t^{-1}])$ on $\Delta_+ \times \Delta_-$ provides an example for a group acting "strongly transitively" (cf. Definition 3) on a twin building. In Section 2, we shall study arbitrary actions of this type in an abstract framework. In particular, we shall see that they always admit certain easily describable fundamental domains (cf. Proposition 3 and its corollaries). However, these results can be derived under assumptions on the pair (Δ_+, Δ_-) which are weaker than requiring it to be a twin

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building in the sense of [T5]; one of the characteristic features of twin buildings, namely the existence of "many" pairs of opposite chambers, plays no role in the present paper. In order to make plain which axioms we really need, generalizations of twin buildings and twin BN–pairs, called "pre–twin buildings" (the adjective "weak" would be misleading in connection with the notion of "building") and "pre–twin BN-pairs", respectively, are introduced in Sections 1 and 2. Furthermore, every building Δ gives rise to a pre–twin building (by simply "doubling" Δ, cf. Example 1) but not to a twin building, in general. Hence pre–twin buildings seem to be the appropriate framework if one wants to deduce results which can be simultaneously applied to ordinary and to twin buildings (compare the proofs of Proposition 4 and Proposition 6).

At last, I note that the axiomatic approach towards twin buildings used here is that of [AR] which is different from that of [T5]. In the latter paper, a twin building is considered as a pair (Δ_+, Δ_-) of buildings of the same type together with a "codistance" δ^* between the chambers of Δ_+ and Δ_- satisfiying certain conditions. For our purposes, it is more convenient to regard a twin building as a pair Δ_+, Δ_- of buildings together with a set of "twin apartments" and an opposition relation between the chambers of Δ_+ and Δ_-. It is pointed out in [AR] that this approach is equivalent to that of [T5] if the axioms for the twin apartments are chosen appropriately. Again, one axiom (namely (TA4) of [AR]) is not of interest for us in the following, and we are led to the definition of a pre–twin building given in Section 1, eventually.

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1 Pre–twin buildings

In the following, we shall not exploit the full strength of the axioms for twin buildings as they are stated for example in [T5], §2.2. Instead, we introduce the more general concept of "pre–twin buildings" which also admits ordinary buildings as special cases. Before doing this, we have to fix some notations:

Let $M = (m_{ij})_{i,j\in I}$ be a Coxeter matrix over the finite index set I and $W = W(M) = \langle s_i\,;\, i \in I \mid s_i^2 = (s_i s_j)^{m_{ij}} = 1\,;\, i,j \in I, m_{ij} \neq \infty\rangle$ the corresponding Coxeter group. Denote by $\ell : W \longrightarrow \mathbb{N}_0$ the length function with respect to $S = \{s_i \,|\, i \in I\}$ and set $W_J := \langle s_i \,|\, i \in J\rangle$ for every $J \subseteq I$.

Let Δ_+ and Δ_- be two buildings of type M, i.e. with apartments isomorphic to the Coxeter complex $\Sigma(W, S)$, type: $\Delta_\varepsilon \longrightarrow \{J \,|\, J \subseteq I\}$ the corresponding type functions and $\mathcal{C}_\varepsilon := \mathrm{Ch}(\Delta_\varepsilon)$ their sets of chambers ($\varepsilon \in \{+,-\}$). Morphisms between buildings of type M are always assumed to be type preserving. Throughout this paper the notion of "building" is usually used in the "classical" sense (cf. [T1], ch. 3, or [Br], ch. IV), but the W–distance functions $\delta_\varepsilon : \mathcal{C}_\varepsilon \times \mathcal{C}_\varepsilon \longrightarrow W$ ($\varepsilon \in \{+,-\}$) will also be considered sometimes.

In the following, we assume that we are given a symmetric **opposition relation**

$$\mathrm{op} \subseteq \mathcal{C}_+ \times \mathcal{C}_- \cup \mathcal{C}_- \times \mathcal{C}_+$$

and a subset $\mathcal{A}$ of $\{(\Sigma_+, \Sigma_-) \mid \Sigma_\varepsilon$ is an apartment in Δ_ε for $\varepsilon \in \{+,-\}\}$. Define

$$\mathcal{A}_+ := \{\Sigma_+ \mid \exists\, \Sigma_- : (\Sigma_+, \Sigma_-) \in \mathcal{A}\} \text{ and } \mathcal{A}_- := \{\Sigma_- \mid \exists\, \Sigma_+ : (\Sigma_+, \Sigma_-) \in \mathcal{A}\}.$$

The fact that two chambers $c_+ \in \mathcal{C}_+, c_- \in \mathcal{C}_-$ are opposite will be denoted by $c_+ \operatorname{op} c_-$ or $c_- \operatorname{op} c_+$, respectively. The elements of $\mathcal{A}$ are called **twin apartments**. An isomorphism α between two twin apartments Σ and Σ' consists of two (type preserving) isomorphisms $\alpha_\varepsilon : \Sigma_\varepsilon \longrightarrow \Sigma'_\varepsilon$ of Coxeter complexes such that $c_+ \operatorname{op} c_- \iff \alpha_+(c_+) \operatorname{op} \alpha_-(c_-)$ for all $c_\varepsilon \in \operatorname{Ch}(\Sigma_\varepsilon)$, $\varepsilon \in \{+,-\}$.

It is shown in [AR] that every twin building in the sense of [T5], §2.2, admits a system $(\Delta_+, \Delta_-, \mathcal{A}, \operatorname{op})$ satisfying certain axioms (TA1) – (TA4) and that, conversely, every such system gives rise to a twin building. (TA4) requires that intersections of twin apartments are always "coconvex", i.e. closed under formation of (co-)projections. We can dispense with that condition in the following if we insert (TA0), which follows from (TA1) – (TA4) but not from (TA1) – (TA3):

Definition 1: *A quadruple $\Delta = (\Delta_+, \Delta_-, \mathcal{A}, \operatorname{op})$ with $\Delta_+, \Delta_-, \mathcal{A}, \operatorname{op}$ as above is called a* ***pre–twin building*** *(of type M) if it satisfies the following conditions:*

(TA0) *$\mathcal{A}_\varepsilon$ is a system of apartments for Δ_ε ($\varepsilon \in \{+,-\}$), i.e. for all $c_\varepsilon, d_\varepsilon \in \mathcal{C}_\varepsilon$, there is a $\Sigma_\varepsilon \in \mathcal{A}_\varepsilon$ such that $c_\varepsilon, d_\varepsilon \in \Sigma_\varepsilon$.*

(TA1) *For every $\Sigma = (\Sigma_+, \Sigma_-) \in \mathcal{A}$, the restriction of the opposition relation induces a bijection between $\operatorname{Ch}(\Sigma_+)$ and $\operatorname{Ch}(\Sigma_-)$ and the latter a type preserving isomorphism $\operatorname{op}_\Sigma : \Sigma_+ \longrightarrow \Sigma_-$.*

(TA2) *For all $c_+ \in \mathcal{C}_+$, $d_- \in \mathcal{C}_-$, there exists a $\Sigma \in \mathcal{A}$ with $(c_+, d_-) \in \Sigma$ (which means $c_+ \in \Sigma_+$ and $d_- \in \Sigma_-$).*

(TA3) *For all $\Sigma, \Sigma' \in \mathcal{A}$ and all $a = (a_+, a_-) \in \Sigma \cap \Sigma'$, there exists an isomorphism $\alpha : \Sigma \longrightarrow \Sigma'$ of twin apartments satisfying $\alpha(a) = a$.*

Here are the main examples for pre–twin buildings:

Example 1: Let Δ' be a building of type M and $\mathcal{A}'$ a system of apartments of Δ'. Define $\Delta_+ := \Delta' =: \Delta_-$, $\mathcal{A} := \{(\Sigma, \Sigma) \mid \Sigma \in \mathcal{A}'\}$ and $c \operatorname{op} d :\iff c = d$ for $c, d \in \operatorname{Ch}(\Delta')$. Then $(\Delta_+, \Delta_-, \mathcal{A}, \operatorname{op})$ is a pre–twin building.

Example 2: (cf. [T4], Section 3, [T5], §2.3, and [AR]) Let $(\Delta_+, \Delta_-, \delta^*)$ be a twin building. In particular, we have $c_+ \operatorname{op} c_- \iff \delta^*(c_+, c_-) = 1$ for any $c_+ \in \mathcal{C}_+$, $c_- \in \mathcal{C}_-$. For every pair of opposite chambers $c_\varepsilon \operatorname{op} c_{-\varepsilon}$, we define $\Sigma(c_\varepsilon, c_{-\varepsilon})$ to be the subcomplex of Δ_ε generated by $\{d_\varepsilon \in \mathcal{C}_\varepsilon \mid \delta_\varepsilon(c_\varepsilon, d_\varepsilon) = \delta^*(c_{-\varepsilon}, d_\varepsilon)\}$.
Set $\mathcal{A} := \{(\Sigma(c_+, c_-), \Sigma(c_-, c_+)) \mid c_+ \operatorname{op} c_-\}$. Then $(\Delta_+, \Delta_-, \mathcal{A}, \operatorname{op})$ is a pre–twin building.

Remark 1 (cf. [AR], Section 2): If $\Delta = (\Delta_+, \Delta_-, \mathcal{A}, \mathrm{op})$ is a pre–twin building, a "W–codistance" $\delta^* : \mathcal{C}_+ \times \mathcal{C}_- \cup \mathcal{C}_- \times \mathcal{C}_+ \longrightarrow W$ can be well–defined by setting $\delta^*(c_+, d_-) := \delta_-(\mathrm{op}_\Sigma(c_+), d_-) =: \delta^*(d_-, c_+)^{-1}$ for any $\Sigma \in \mathcal{A}$ such that $(c_+, d_-) \in \Sigma$. Then $\delta^*(c_+, d_-) = 1 \iff c_+ \text{ op } d_-$ and for all $c_\varepsilon \in \mathcal{C}_\varepsilon;\ d_{-\varepsilon}, e_{-\varepsilon} \in \mathcal{C}_{-\varepsilon}$ it holds:

(Tw1) $\delta^*(d_{-\varepsilon}, c_\varepsilon) = \delta^*(c_\varepsilon, d_{-\varepsilon})^{-1}$

(Tw2)′ $\delta^*(c_\varepsilon, d_{-\varepsilon}) = w \in W, \delta_{-\varepsilon}(d_{-\varepsilon}, e_{-\varepsilon}) = s \in S \Rightarrow \delta^*(c_\varepsilon, e_{-\varepsilon}) \in \{w, ws\}$

(Tw3) $\delta^*(c_\varepsilon, d_{-\varepsilon}) = w \in W, s \in S$

$$\Rightarrow \exists\, x_{-\varepsilon} \in \mathcal{C}_{-\varepsilon} : \delta_{-\varepsilon}(d_{-\varepsilon}, x_{-\varepsilon}) = s \text{ and } \delta^*(c_\varepsilon, x_{-\varepsilon}) = ws$$

Furthermore, the stronger axiom

(Tw2) $\delta^*(c_\varepsilon, d_{-\varepsilon}) = w \in W,\ \delta_{-\varepsilon}(d_{-\varepsilon}, e_{-\varepsilon}) = s \in S$ and $\ell(ws) < \ell(w)$

$$\Rightarrow \delta^*(c_\varepsilon, e_{-\varepsilon}) = ws$$

ist satisfied if and only if $(\Delta_+, \Delta_-, \delta^*)$ is a twin building from which Δ arises as in Example 2.

2 Pre–twin BN–pairs and strongly transitive actions

Let G be a group together with subgroups B_+, B_-, N such that

i) $B_+ \cap N = B_- \cap N =: H$ is normal in N.

ii) $N/H = W = \langle S \rangle$ is the Coxeter group introduced in Section 1.

iii) (G, B_ε, N, S) is a Tits system for $\varepsilon \in \{+, -\}$.

Definition 2: *A system (G, B_+, B_-, N, S) as above is called a* ***pre–twin BN–pair*** *(with Weyl group W) if it satisfies*

(TBN1)′ $B_\varepsilon w B_{-\varepsilon} s B_{-\varepsilon} \subseteq B_\varepsilon \{w, ws\} B_{-\varepsilon} \quad \forall w \in W,\ s \in S,\ \varepsilon \in \{+, -\}$

(TBN2)′ $B_+(W \setminus \{1\}) \cap B_- = \emptyset$

Corresponding to Examples 1 and 2 we obtain:

Example 3: If (G, B, N, S) is a Tits system then (G, B, B, N, S) is a pre–twin BN–pair.

Example 4: Let (G, B_+, B_-, N, S) be a twin BN–pair in the sense of [T5], §3.2, i.e. a system as above satisfying

(TBN1) $B_\varepsilon w B_{-\varepsilon} s B_{-\varepsilon} = B_\varepsilon ws B_{-\varepsilon}$ for $\varepsilon \in \{+, -\}$ and all $w \in W, s \in S$ such that $\ell(ws) < \ell(w)$

(TBN2) $B_+ s \cap B_- = \emptyset$ for all $s \in S$

Then (TBN1)$'$ follows from (TBN1) together with $sB_{-\varepsilon}s \subseteq B_{-\varepsilon} \cup B_{-\varepsilon}sB_{-\varepsilon}$ and $B_+w \cap B_- = \emptyset \quad \forall w \in W \setminus \{1\}$ from (TBN2) by applying (TBN1) as in [T5].

Remark 2: If (G, B_+, B_-, N, S) is a pre–twin BN–pair, then one can deduce as in [T5], §3.2, a "Birkhoff decomposition" for G. This means, more precisely, that the map $W \longrightarrow B_+ \setminus G/B_-,\ w \longmapsto B_+wB_-$, is bijective.

We are now going to associate a pre–twin building to every pre–twin BN–pair. Recall (cf. [T1], Theorem 3.2.6, or [Br], Section V.3) that there is a thick building corresponding to (G, B_ε, N, S) ($\varepsilon \in \{+,-\}$), namely $\Delta_\varepsilon := \Delta(G, B_\varepsilon) = \{g\,P_J^\varepsilon \mid g \in G,\ J \subseteq I\}$ ($P_J^\varepsilon := B_\varepsilon W_J B_\varepsilon$). Denote by $\Sigma_\varepsilon^0 := \{n\,P_J^\varepsilon \mid n \in N,\ J \subseteq I\}$ the standard apartment of Δ_ε and set $\mathcal{A} := \{(g\Sigma_+^0,\ g\Sigma_-^0) \mid g \in G\}$.

Finally, we define the opposition relation by

$$g\,B_+ \ \text{op}\ h\,B_- :\Longleftrightarrow\ g\,B_+ \cap h\,B_- \neq \emptyset \quad (g, h \in G)\ .$$

Proposition 1: *The system $\Delta = (\Delta_+, \Delta_-, \mathcal{A}, \mathrm{op})$ introduced above is a pre–twin building. It is a twin building (in the sense of Example 2 and Remark 1) if and only if (G, B_+, B_-, N, S) is a twin BN–pair.*

Proof: First we verify axioms (TA0) – (TA3) for Δ:

(TA0) It is well known that $\mathcal{A}_\varepsilon = \{g\Sigma_\varepsilon^0 \mid g \in G\}$ is a system of apartments for Δ_ε.

(TA1) It suffices to consider $\Sigma^0 := (\Sigma_+^0, \Sigma_-^0)$. Using (TBN2)$'$, we obtain for all

$$\begin{aligned} n_1, n_2 \in N: \qquad n_1B_+ \ \text{op}\ n_2B_- \quad &\Longleftrightarrow\ n_1^{-1}n_2 \in N \cap B_+B_- = H \\ &\Longleftrightarrow\ n_1^{-1}n_2 \in N \cap B_- \\ &\Longleftrightarrow\ n_2B_- = n_1B_- \end{aligned}$$

Therefore, op induces the bijection

$$\mathrm{Ch}(\Sigma_+^0) \longrightarrow \mathrm{Ch}(\Sigma_-^0),\ nB_+ \longmapsto nB_- \quad (n \in N)$$

and hence the isomorphism

$$\Sigma_+^0 \longrightarrow \Sigma_-^0,\ nP_J^+ \longmapsto nP_J^- \quad (n \in N,\ J \subseteq I)\ .$$

(TA2) This is an immediate consequence of the decomposition $G = B_+NB_-$: Let $gB_+ \in \mathcal{C}_+$ and $hB_- \in \mathcal{C}_-$ be given. Write $g^{-1}h = b_+nb_-$ for some $b_+ \in B_+$, $n \in N,\ b_- \in B_-$. Then $(gB_+, hB_-) \in (gb_+\Sigma_+^0, gb_+\Sigma_-^0)$.

(TA3) In view of (TA2), it suffices to prove (TA3) under the additional assumption that $a_+ \in \mathcal{C}_+$ or $a_- \in \mathcal{C}_-$. So let two twin apartments Σ, Σ' and a pair $a = (a_+, a_-) \in \Sigma \cap \Sigma'$ be given such that (without loss of generality) $a_+ \in \mathcal{C}_+$. We may assume $\Sigma = \Sigma^0$ and $a_+ = B_+$. Choose a $g \in G$ such that $\Sigma' = g\Sigma$. Because $a_+ = B_+ \in g\Sigma_+^0 = \Sigma_+'$ implies $g \in B_+N$, we can even achieve $g \in B_+$ here.

Now there are $w_1, w_2 \in W$ and a $J \subseteq I$ such that $a_- \in \Sigma_-^0 \cap \Sigma'_-$ is of the form $a_- = w_1 P_J^- = g w_2 P_J^-$. In view of $g \in B_+$ and of (TBN1)$'$, this implies $B_+ w_1 W_J B_- = B_+ w_2 W_J B_-$. Using the Birkhoff decomposition (cf. Remark 2), we obtain $w_1 W_J = w_2 W_J$ and hence $w_1 P_J^- = w_2 P_J^-$. Therefore, $g \in \mathrm{Stab}_G(a_-)$, and our desired isomorphism $\alpha : \Sigma \longrightarrow \Sigma'$ is given by multiplication with g.

Since Δ is a pre–twin building, we may consider the function δ^* introduced in Remark 1. It is clear from the definitions that the action of G preserves δ^*. Given $g, h \in G$, we choose a decomposition

$$g^{-1}h = b_\varepsilon n b_{-\varepsilon} \in B_\varepsilon w B_{-\varepsilon} \ (w = nH \in N/H = W,\ \varepsilon \in \{+,-\})$$

and obtain

$$\begin{aligned} \delta^*(gB_\varepsilon, hB_{-\varepsilon}) &= \delta^*(B_\varepsilon, g^{-1}hB_{-\varepsilon}) = \delta^*(B_\varepsilon, b_\varepsilon n B_{-\varepsilon}) = \delta^*(B_\varepsilon, nB_{-\varepsilon}) \\ &= \delta_{-\varepsilon}(B_{-\varepsilon}, nB_{-\varepsilon}) = w \end{aligned}$$

This shows

$$\delta^*(gB_\varepsilon, hB_{-\varepsilon}) = w \iff g^{-1}h \in B_\varepsilon w B_{-\varepsilon} \ .$$

Now suppose we are given chambers $gB_\varepsilon \in \mathcal{C}_\varepsilon$; $hB_{-\varepsilon}$, $kB_{-\varepsilon} \in \mathcal{C}_{-\varepsilon}$ such that $\delta^*(gB_\varepsilon, hB_{-\varepsilon}) = w \in W$, $\delta_{-\varepsilon}(hB_{-\varepsilon}, kB_{-\varepsilon}) = s \in S$ and $\ell(ws) < \ell(w)$. The first two equations can be translated into $g^{-1}h \in B_\varepsilon w\, B_{-\varepsilon}$ and $h^{-1}k \in B_{-\varepsilon} s\, B_{-\varepsilon}$. Furthermore, $\delta^*(gB_\varepsilon, kB_{-\varepsilon}) = ws$ if and only if $g^{-1}k = (g^{-1}h)(h^{-1}k) \in B_\varepsilon ws\, B_{-\varepsilon}$. This proves: If (G, B_+, B_-, N, S) satisfies (TBN1), then δ^* satisfies (Tw2). Because $g^{-1}h \in B_\varepsilon w\, B_{-\varepsilon}$ and $h^{-1}k \in B_{-\varepsilon} s\, B_{-\varepsilon}$ may be chosen arbitrarily, the converse is also true. ∎

In the rest of this section, we assume that we are given a pre–twin building $\Delta = (\Delta_+, \Delta_-, \mathcal{A}, \mathrm{op})$ of type M and a group G acting on Δ. We say that G acts on Δ if the following holds:

i) G acts (type preservingly) on Δ_+ and Δ_-.

ii) $g\Sigma := (g\Sigma_+, g\Sigma_-) \in \mathcal{A}$ for any $\Sigma = (\Sigma_+, \Sigma_-) \in \mathcal{A}$ and $g \in G$.

iii) $gc_+ \ \mathrm{op}\ gc_- \iff c_+ \ \mathrm{op}\ c_-$ for any $c_+ \in \mathcal{C}_+$, $c_- \in \mathcal{C}_-$ and $g \in G$.

In particular, G preserves the function δ^* introduced in Remark 1.

We wish to prove a sort of converse of Proposition 1. As in ordinary building theory (cf. [T1], §3, [Br], ch.V, or [R], ch.5), this leads to certain requirements which the action of G on Δ should satisfy.

Definition 3: *G acts **strongly transitively** on Δ, if G acts transitively on $\mathcal{A}$ and $\mathrm{Stab}_G(\Sigma)$ acts transitively on $\mathrm{Ch}(\Sigma_+)$ (and hence on $\mathrm{Ch}(\Sigma_-)$ as well) for every $\Sigma = (\Sigma_+, \Sigma_-) \in \mathcal{A}$.*

Remark 3: If G and Δ are as described in Proposition 1, G acts strongly transitively on Δ.

Lemma 1: *If G acts strongly transitively on Δ then it acts transitively on $\mathcal{C}_w := \{(c_+, d_-) \mid c_+ \in \mathcal{C}_+,\ d_- \in \mathcal{C}_-,\ \delta^*(c_+, d_-) = w\}$ for every $w \in W$.*

Proof: Assume $(c_+, d_-), (c'_+, d'_-) \in \mathcal{C}_w$. Choose $\Sigma, \Sigma' \in \mathcal{A}$ with $(c_+, d_-) \in \Sigma$ and $(c'_+, d'_-) \in \Sigma'$. Since G acts strongly transitively on Δ, there exists a $g \in G$ such that $g\Sigma = \Sigma'$ and $gc_+ = c'_+$. But then $gd_- = d'_-$ holds automatically:

Set $c_- := \mathrm{op}_\Sigma(c_+)$ and $c'_- := \mathrm{op}_{\Sigma'}(c'_+)$. Then $gc_+ = c'_+$ and $g\Sigma = \Sigma'$ imply $gc_- = c'_-$. It follows

$$\begin{aligned} \delta_-(c'_-, gd_-) &= \delta_-(gc_-, gd_-) = \delta_-(c_-, d_-) = \delta^*(c_+, d_-) = w \\ &= \delta^*(c'_+, d'_-) = \delta_-(c'_-, d'_-) \end{aligned}$$

Since $c'_-, gd_-,\ d'_-$ are all contained in the apartment Σ'_- of Δ_-, $\delta_-(c'_-, gd_-) = \delta_-(c'_-, d'_-)$ implies $gd_- = d'_-$. ■

Remark 4: It follows that G acts strongly transitively on Δ if and only if G acts transitively on $\mathcal{C}_1 = \{(c_+, c_-) \mid c_+ \,\mathrm{op}\, c_-\}$ and $\mathrm{Stab}_G(c_+) \cap \mathrm{Stab}_G(c_-)$ acts transitively on $\mathcal{A}(c_+, c_-) := \{\Sigma \in \mathcal{A} \mid (c_+, c_-) \in \Sigma\}$ for every $(c_+, c_-) \in \mathcal{C}_1$. In particular, if Δ is a twin building and hence $\#\,\mathcal{A}(c_+, c_-) = 1$ for all $(c_+, c_-) \in \mathcal{C}_1$ (cf. [T4], Proposition 3(i), or [AR], Lemma 4), the notion "strongly transitive" is equivalent to "transitive on $\mathcal{C}_1$". This is exactly what Tits requires in [T5], §3.2. Consequently, that paragraph already contains the following proposition in the case of twin buildings.

We assume that G acts strongly transitively on Δ in the following. Choose a twin apartment $\Sigma \in \mathcal{A}$ and chambers $c_+ \in \Sigma_+,\ c_- \in \Sigma_-$ satisfying $c_+ \,\mathrm{op}\, c_-$ and set

$$\begin{aligned} N &:= \mathrm{Stab}_G(\Sigma)\ (= \mathrm{Stab}_G(\Sigma_+) \cap \mathrm{Stab}_G(\Sigma_-)) \\ B_\varepsilon &:= \mathrm{Stab}_G(c_\varepsilon)\ \ \varepsilon \in \{+,-\}\,. \end{aligned}$$

Proposition 2: *Assume that Δ_+ and Δ_- are thick buildings. Then (G, B_+, B_-, N, S) is a pre–twin BN–pair. The pre–twin building $\widetilde{\Delta} = \Delta(G, B_+, B_-, N)$ associated to it as described in Proposition 1 is isomorphic to Δ. In particular, (G, B_+, B_-, N, S) is a twin BN–pair if and only if Δ is a twin building.*

Proof: We successively check the conditions defining a pre–twin BN–pair:

i) $B_+ \cap N = B_- \cap N$ follows from the definitions and (TA1).

ii) The homomorphism $\nu : N \longrightarrow \mathrm{Aut}(\Sigma_+) = W$ is surjective by Definition 3. Its kernel is equal to $N \cap B_+ =: H$, and hence we may identify N/H with $W = \langle S \rangle$.

iii) It follows from (TA0) and Definition 3 that G acts strongly transitively on $(\Delta_\varepsilon, \mathcal{A}_\varepsilon)$ in the sense of [Br], ch. V. Therefore, (G, B_ε, N, S) is a Tits system. (We may replace $\mathrm{Stab}_G(\Sigma_\varepsilon)$ by N here, because the latter group is still transitive on $\mathrm{Ch}(\Sigma_\varepsilon)$.)

The two axioms of Definition 2 are most conveniently verified by applying the function δ^* of Remark 1. We first establish the following equivalence:

(1) $\delta^*(c_\varepsilon, gc_{-\varepsilon}) = w \iff g \in B_\varepsilon w B_{-\varepsilon} \quad (g \in G,\ w \in W, \varepsilon \in \{+,-\})$

The implication "$\Leftarrow$' follows directly from the definitions and the fact that δ^* is preserved by the G–action. Conversely, for given $w = \delta^*(c_\varepsilon, gc_{-\varepsilon})$, we choose an $n \in \nu^{-1}(w)$. Then $\delta^*(c_\varepsilon, nc_{-\varepsilon}) = w$ and Lemma 1 imply the existence of a $b_\varepsilon \in B_\varepsilon$ such that $b_\varepsilon n c_{-\varepsilon} = gc_{-\varepsilon}$. Hence $g \in b_\varepsilon n B_{-\varepsilon} \subseteq B_\varepsilon w B_{-\varepsilon}$.

(TBN1)$'$ Assume $g \in B_\varepsilon w B_{-\varepsilon}$ and $h \in B_{-\varepsilon} s B_{-\varepsilon}$, where $w \in W$ and $s \in S$. Then $\delta^*(c_\varepsilon, gc_{-\varepsilon}) = w$ and $\delta_{-\varepsilon}(gc_{-\varepsilon}, ghc_{-\varepsilon}) = s$. Therefore, by (Tw2)$'$, $\delta^*(c_\varepsilon, ghc_{-\varepsilon}) \in \{w, ws\}$ and by (1), $gh \in B_\varepsilon\{w, ws\}B_{-\varepsilon}$.

(TBN2)$'$ Let $n \in N,\ b_+ \in B_+,\ b_- \in B_-$ be given such that $b_+ n = b_-$ and set $w := \nu(n)$. Then $w = \delta^*(c_+, b_+ n c_-) = \delta^*(c_+, b_- c_-) = \delta^*(c_+, c_-) = 1$.

Let $\widetilde{\Delta} = (\widetilde{\Delta}_+, \widetilde{\Delta}_-, \widetilde{\mathcal{A}}, \widetilde{\mathrm{op}})$ be the pre–twin building associated to (G, B_+, B_-, N, S). Recall that $\widetilde{\Delta}_\varepsilon = \Delta(G, B_\varepsilon)$ for $\varepsilon \in \{+,-\}$, $\widetilde{\mathcal{A}} = \{g\widetilde{\Sigma}^0 \mid g \in G\}$ with $\widetilde{\Sigma}^0_\varepsilon = \{nP^\varepsilon_J \mid n \in N, J \subseteq I\}$ and $gB_+ \ \widetilde{\mathrm{op}}\ hB_- \iff gB_+ \cap hB_- \neq \emptyset$. We know from ordinary building theory that there are G–equivariant isomorphisms

$$\varphi_\varepsilon : \widetilde{\Delta}_\varepsilon \xrightarrow{\sim} \Delta_\varepsilon \text{ induced by } gB_\varepsilon \longmapsto gc_\varepsilon \text{ for } \varepsilon \in \{+,-\}\ .$$

In order to complete the proof of the second assertion, we have to show

(2) $\varphi(\widetilde{\mathcal{A}}) = \mathcal{A}$, where $\varphi = (\varphi_+, \varphi_-)$ and

(3) $gB_+ \ \widetilde{\mathrm{op}}\ hB_- \iff \varphi_+(gB_+) \ \mathrm{op}\ \varphi_-(hB_-) \quad \forall g, h \in G$.

Now (2) follows from $\varphi(\widetilde{\Sigma}^0) = \Sigma$ and the transitivity of G on $\mathcal{A}$. And using (1), we obtain

$$\begin{aligned} gc_+ \,\mathrm{op}\, hc_- \quad &\iff\ c_+ \,\mathrm{op}\, g^{-1}hc_- \iff \delta^*(c_+, g^{-1}hc_-) = 1 \iff g^{-1}h \in B_+B_- \\ &\iff\ h \in gB_+B_- \iff hB_- \cap gB_+ \neq \emptyset \iff gB_+ \ \widetilde{\mathrm{op}}\ hB_- \end{aligned}$$

hence (3).

The last claim of the proposition is a direct consequence of the second and of Proposition 1. ∎

Next we are going to study a "fundamental domain" for the action of G on Δ. It is not difficult to find a subcomplex F of $\Delta_+ \times \Delta_-$ such that every G–orbit in $\mathcal{C}_+ \times \mathcal{C}_-$ contains exactly one element of $\mathrm{Ch}(F_+) \times \mathrm{Ch}(F_-)$ (cf. Proposition 3). But in order to characterize the "identifications on the boundary of F" induced by G, we need the following

Lemma 2: *Let G act strongly transitively on Δ, assume as before $\Sigma \in \mathcal{A}$, $N = \mathrm{Stab}_G(\Sigma)$, let $a = (a_+, a_-) \in \Sigma$ be given and set $P_\varepsilon := \mathrm{Stab}_G(a_\varepsilon)$ for $\varepsilon \in \{+,-\}$. Then it holds*

(4) $N \cap P_-P_+ = (N \cap P_-)(N \cap P_+)$.

Proof: For a given element $n = p_- p_+ \in N \cap P_- P_+$, we define $e_+ := p_- a_+ = n a_+ \in \Sigma_+$. Consider the following isomorphisms of twin apartments:

$$\begin{array}{ccccc} \Sigma & \xrightarrow{p_-} & p_-\Sigma & \xrightarrow{\alpha} & \Sigma \\ (a_+, a_-) & \longmapsto & (e_+, a_-) & \longmapsto & (e_+, a_-) \end{array}$$

Here, the first is given by multiplication with p_- and the second, α, fixes (e_+, a_-) and exists according to (TA3). Since G acts strongly transitively on Δ, α is also given by multiplication with an element of G which we call g. In particular, it follows

$$g \in \mathrm{Stab}_G(e_+),\ g \in \mathrm{Stab}_G(a_-) \text{ and } n_- := g p_- \in \mathrm{Stab}_G(\Sigma) \cap P_- = N \cap P_- .$$

On the other hand, $n_- a_+ = g e_+ = e_+ = n a_+$. Therefore

$$n \in n_- P_+ \cap N \subseteq (N \cap P_-) P_+ \cap N = (N \cap P_-)(N \cap P_+) .$$

■

Proposition 3: *Assume* $\Sigma = (\Sigma_+, \Sigma_-) \in \mathcal{A}$, $c_- \in \mathrm{Ch}(\Sigma_-)$ *and set* $F := \{(a_+, a_-) \in \Delta_+ \times \Delta_- \mid a_+ \in \Sigma_+$ *and* $a_- \subseteq c_-\}$. *If* G *acts strongly transitively on* Δ *and* $N := \mathrm{Stab}_G(\Sigma)$, *then one obtains:*

i) $GF = \Delta_+ \times \Delta_-$.

ii) $a = (a_+, a_-)$, $a' = (a'_+, a'_-) \in F$ *lie in the same* G*–orbit if and only if* $a_- = a'_-$ *and there exists an* $n_- \in N \cap \mathrm{Stab}_G(a_-)$ *such that* $n_- a_+ = a'_+$.

Proof:

i) This is a direct consequence of Definition 3 and (TA2): Given $a' = (a'_+, a'_-) \in \Delta_+ \times \Delta_-$, we choose a $\Sigma' \in \mathcal{A}$ such that $a' \in \Sigma'$, a $g_1 \in G$ mapping Σ' onto Σ and a $g_2 \in \mathrm{Stab}_G(\Sigma) = N$ satisfying $g_2(g_1 a'_-) \subseteq c_-$. Hence $(g_2 g_1) a' \in F$.

ii) is trivial if one replaces "$n_- \in N \cap \mathrm{Stab}_G(a_-)$" by "$n_- \in \mathrm{Stab}_G(a_-)$". So let us assume $a'_+ = p_- a_+$ with $p_- \in P_- := \mathrm{Stab}_G(a_-)$, where $a_- \subseteq c_- \in \Sigma_-$ and $a'_+, a_+ \in \Sigma_+$. We have to show $a'_+ \in (N \cap P_-) a_+$. Since $a'_+ = p_- a_+$ implies type $(a'_+) =$ type (a_+), there exists an $n \in N$ with $a'_+ = n a_+$. Furthermore, $n a_+ = p_- a_+$ yields $n \in p_- P_+$, where $P_+ := \mathrm{Stab}_G(a_+)$. Applying Lemma 2, we obtain $n \in N \cap P_- P_+ = (N \cap P_-)(N \cap P_+)$ and, in particular, $a'_+ = n a_+ \in (N \cap P_-) a_+$. ■

In the following corollary, the notion "simplicial fundamental domain" is used in the strictest sense, i.e. it denotes a subcomplex containing exactly one simplex of every orbit with respect to the group action in question.

Corollary 1: *Retain the assumptions and notations of Proposition 3. Let* $a_- \subseteq c_-$ *be of type* $I \backslash J$ *and set* $B_- := \mathrm{Stab}_G(c_-)$, $P_- := \mathrm{Stab}_G(a_-) = B_-W_JB_- = P_J^-$. *Then every simplicial fundamental domain* D_+ *for the action of* W_J *on* $\Sigma(W,S) = \Sigma_+$ *is also a simplicial fundamental domain for the action of* P_- *on* Δ_+.
Proof:

i) By assumption, $W_JD_+ = \Sigma_+$, and Proposition 3 i) implies $B_-\Sigma_+ = \Delta_+$. Hence $P_-D_+ = \Delta_+$.

ii) Assume $a_+, a'_+ \in D_+$ and $g \in P_-$ such that $a'_+ = ga_+$. According to Proposition 3 ii), there exists an $n \in N \cap P_- = N \cap B_-W_JB_- = \nu^{-1}(W_J)$ (ν is defined as in step ii) of the proof of Propostion 2) satisfying $\nu(n)a_+ = na_+ = a'_+$. Therefore, our assumption on D_+ implies $a'_+ = a_+$ here. ∎

The advantage of Corollary 1 consists in the fact that it is quite easy to give simplicial fundamental domains for the actions of parabolic subgroups of W on $\Sigma(W,S)$. For the convenience of the reader and for lack of a precise reference, I recall the following statement which should be well known:

Lemma 3: *Let* $J \subseteq I$ *be given and set* $W^J := \{w \in W \mid w$ *is of minimal length in* $W_Jw\}$. *Then the subcomplex* $\Sigma^J := \{wW_K \mid w \in W^J,\ K \subseteq I\}$ *of* $\Sigma = \Sigma(W,S)$ *is a simplicial fundamental domain for the action of* W_J *on* Σ.
Proof: Note that this lemma is essentially a reformulation of Exercise 3 in [Bo], ch. IV, §1.

i) $\Sigma = W_J\Sigma^J$ follows immediately from $W = W_JW^J$.

ii) Let $v_1, v_2 \in W_J$, $w_1, w_2 \in W^J$ and $K \subseteq I$ be given such that $v_1w_1W_K = v_2w_2W_K$. We have to show $w_1W_K = w_2W_K$. Consider the element x_i of shortest length in w_iW_K ($i = 1,2$). The exercise quoted above implies in particular $\ell(w_i) = \ell(x_i) + \ell(x_i^{-1}w_i)$. Consequently, x_i is of minimal length in W_Jx_i, since w_i is already of minimal length in W_Jw_i. Therefore, in the language of that exercise, x_i is $(J,\emptyset)$–reduced as well as $(\emptyset,K)$–reduced. Hence it is also (J,K)–reduced. This means that x_i is the unique element of minimal length in $W_Jw_iW_K$. Therefore, $W_Jw_1W_K = W_Jw_2W_K$ implies $x_1 = x_2$ and hence $w_1W_K = x_1W_K = x_2W_K = w_2W_K$. ∎

Remark 5: Here is a more geometric description of Σ^J: Denote by $\alpha_i (i \in I)$ the root of $\Sigma(W,S)$ containing 1 but not s_i . Since

$$\mathrm{Ch}(\Sigma^J) = W^J = \{w \in W \mid \ell(w) < \ell(s_jw)\ \forall j \in J\} = \bigcap_{j \in J} \mathrm{Ch}(\alpha_j)$$

one obtains $\Sigma^J = \bigcap_{j \in J} \alpha_j$.

Note that $\{\alpha_j \mid j \in J\}$ is the set of all roots α posessing a boundary $\partial\alpha$ which contains a codimension–1–face of the chamber 1 such that $1 \in \alpha$ and $a := W_J \in \partial\alpha$. That the intersection of all these roots yields a simplicial fundamental domain for the action of $\mathrm{Stab}_W(a)$ on Σ may also be derived more geometrically by using projections and Proposition 12.5 of [T1].

An immediate consequence of Corallary 1 and Lemma 3 is the following

Corollary 2: *Let c_+ be the chamber of Σ_+ opposite c_- and assume that $a_- \subseteq c_-$ is of type $I \setminus J$. Then $\Sigma^J = \{we_+ \mid w \in W^J,\ e_+ \subseteq c_+\}$ is a simplicial fundamental domain for the action of $P_- = \mathrm{Stab}_G(a_-)$ on Δ_+.* ∎

Remark 6: If the pre–twin building Δ results from a pre–twin BN–pair (G, B_+, B_-, N, S) as described in Proposition 1 and if $P_- = B_-W_JB_- = P_J^-$,Σ^J takes the form $\Sigma^J = \{wP_K^+ \mid w \in W^J, K \subseteq I\}$. Then the statement of Corollary 2 is almost contained in §15.5 of [T2] (Tits' requirements (BNU1) and (BNU2) may be replaced by (TBN1)$'$ and (TBN2)$'$ in this context) and derived purely group theoretically there, without mentioning twin buildings or pre–twin buildings.

Corollary 2 admits applications in situations where a subgroup X of G acts on a single building $\Delta_+ = \Delta(G, B_+)$ but where the corresponding Tits system (G, B_+, N, S) may be completed to a pre–twin BN–pair (G, B_+, B_-, N, S) such that X is parabolic with respect to (G, B_-, N, S) (for a concrete example, see Proposition 6 below).

3 Applications

First I wish to specialize two results of Section 2 to ordinary BN–pairs and buildings. Both statements can be proved more directly and are well known, surely. Lemma 4 occurs here because the way I proved it originally led me to Lemma 2, Proposition 3 and the axioms of pre–twin buildings. As for Proposition 4, I think it is interesting that this elementary statement admits the same proof as the seemingly much deeper Proposition 6 (compare the proof of Theorem 1 in [So]).

Lemma 4: *Let (G, B, N, S) be a Tits system with Weyl group $W = \langle s_i \mid i \in I\rangle$. Assume $J, K \subseteq I,\ w \in W$ and set $P_J = BW_JB,\ P_K = BW_KB$. Then it holds*

(5) $\quad W \cap P_JwP_Kw^{-1} = W_JwW_Kw^{-1}$

Proof: Apply Lemma 2 to the following situation: $\Delta = (\Delta', \Delta', \mathcal{A}, \mathrm{op})$ as described in Example 1 with $\Delta' = \Delta(G, B),\ \Sigma = (\Sigma', \Sigma')$ with $\Sigma' = \{vP_L \mid v \in W, L \subseteq I\}$ and $a_- = P_J,\ a_+ = wP_K \in \Sigma'$. Note that $\mathrm{Stab}_G(\Sigma)$ may be strictly bigger than N here. But reading (4) modulo $\mathrm{Fix}_G(\Sigma) = \bigcap_{c\in\Sigma'} \mathrm{Stab}_G(c)$, one obtains (5) nevertheless. ∎

Proposition 4: *If (G, B, N, S) is a Tits system with Weyl group W and W^J is defined as in Lemma 3, then $\Sigma^J = \{wP_K \mid w \in W^J,\ K \subseteq I\}$ is a simplicial fundamental domain for the action of P_J on $\Delta(G, B)$.*

Proof: Apply Corollary 2 to the pre–twin building $\Delta = (\Delta', \Delta', \mathcal{A}, \mathrm{op})$ associated to $\Delta' = \Delta(G, B)$ by setting $a_- = P_J$ and $c_- = B = c_+$. ∎

Finally, I want to show how Proposition 3 and its corollaries can be applied to certain very concrete groups, namely to **Chevalley groups over Laurent polynomial rings.** We need some more notations in this context:

Let Ψ be a reduced and irreducible root system in the Euclidean space $V = \mathbb{R}^n$, $\Pi = \{a_1, \ldots, a_n\}$ a base of Ψ, Ψ_+ the corresponding system of positive roots,

$\Psi_- = -\Psi_+$ and a_0 the root of maximal height in Ψ_+. Denote by Φ the set of "affine roots" associated to Ψ, i.e.

$$\Phi = \{\alpha_{a,\ell} \mid a \in \Psi,\ \ell \in \mathbb{Z}\} \quad \text{with} \quad \alpha_{a,\ell} := \{v \in V \mid (a,v) + \ell \geq 0\}\ .$$

Let $W = W_{\text{aff}}(\Psi)$ be the affine Weyl group of Ψ, generated by the reflections $s_{a,\ell}$ with respect to the hyperplanes $\partial\alpha_{a,\ell} = \{v \in V \mid (a,v) + \ell = 0\}$ $(a \in \Psi, \ell \in \mathbb{Z})$. Set $s_0 := s_{-a_0,1},\ s_i := s_{a_i,0}$ for $1 \leq i \leq n$ and $S := \{s_0, s_1, \ldots, s_n\}$. It is well known that (W,S) is a Coxeter system and that the corresponding Coxeter complex may be identified with the simplicial complex obtained from the cell decomposition of V induced by $\{\partial\alpha \mid \alpha \in \Phi\}$ (cf. [Bo], ch. V, §3, and ch. VI, §2, or [Br], ch. VI, §1).

Let $\mathcal{G}$ be a simply connected Chevalley group (scheme) of type Ψ, $\mathcal{T}$ a maximal torus of $\mathcal{G}$ and $\mathcal{N}$ its normalizer in $\mathcal{G}$. Identify Ψ with the root system of $\mathcal{T}$. Denote by $\mathcal{U}_a$ the 1–dimensional unipotent subgroup of $\mathcal{G}$ associated to $a \in \Psi$ and set $\mathcal{U}_\varepsilon := \langle \mathcal{U}_a \mid a \in \Psi_\varepsilon \rangle$, $\mathcal{B}_\varepsilon := \mathcal{T}\,\mathcal{U}_\varepsilon$ for $\varepsilon \in \{+,-\}$. Select isomorphisms $x_a : \text{Add} \xrightarrow{\sim} \mathcal{U}_a$ $(a \in \Psi,\ \text{Add} := \text{additive group})$ such that the constants in Chevalley's commutator formulas are integers and such that there exist homomorphisms $\varphi_a\ :\ SL_2 \longrightarrow \mathcal{G}$ satisfying $\varphi_a\left(\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\right) = x_a(\lambda)$ and $\varphi_a\left(\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}\right) = x_{-a}(\lambda)$. The groups and the homomorphisms are assumed to be defined over $\mathbb{Z}$ here. Furthermore, there always exists a faithful representation $\mathcal{G} \hookrightarrow SL_r$ for some $r \in \mathbb{N}$ such that $\mathcal{T}$ becomes diagonal, $\mathcal{B}_+$ upper and $\mathcal{B}_-$ lower triangular (explicit constructions can be found in [St], §3 and §5).

For a given field of constants k, we consider the ring $R := k[t, t^{-1}]$ of Laurent polynomials over k and set $G := \mathcal{G}(R)$, $N := \mathcal{N}(R)$ and $H := \mathcal{T}(k)$. It can be shown that N/H is isomorphic to $W = \langle S \rangle$ (cf. for example the proof of the following lemma). Denote by $\rho_\varepsilon : \mathcal{G}(k[t^{-\varepsilon}]) \longrightarrow \mathcal{G}(k)$ the homomorphism induced by $k[t^{-\varepsilon}] \twoheadrightarrow k,\ t^{-\varepsilon} \longmapsto 0$ (where $t^+ := t$ and $t^- := t^{-1}$) and set $B_\varepsilon := \rho_\varepsilon^{-1}(\mathcal{B}_\varepsilon(k))$ for $\varepsilon \in \{+,-\}$. G being a "Kac–Moody group of minimal type", it is known since long that (G, B_+, N, S) and (G, B_-, N, S) are Tits systems; cf. [MT] or, more explicitly, [M]. With the same methods as used there, one can also show that (G, B_+, B_-, N, S) satisfies (TBN1) and (TBN2) and hence is a twin BN–pair. However, the most systematic approach to this question is, in my opinion, provided by the system of axioms introduced in [T3], §5.2, and [T5], §3.3, referring directly to "root groups" and not to Lie algebras. This is one of the reasons (another is mentioned in Remark 8) why I want to demonstrate how Tits' results can be applied in our situation.

Lemma 5: *With the above notations, (G, B_+, B_-, N, S) is a twin BN–pair with Weyl group $W = W_{\text{aff}}(\Psi)$.*

Proof: I shall use the notations introduced in [T3], §5.2, and [T5], §3.3. In particular, (W,S) is a Coxeter system, $S = \{s_i \,|\, i \in I\}$, Φ is the set of all half apartments of $\Sigma(W,S)$, Φ_+ the set of all elements of Φ containing the fundamental chamber 1 and $\Phi_- = \Phi \setminus \Phi_+$. For every $s_i \in S$, α_i denotes the unique half–apartment containing 1 but not s_i. We assume that we are given a group G, a family of subgroup $(U_\alpha)_{\alpha\in\Phi}$ and a subgroup $H \subseteq \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$ such that the following conditions hold:

(RGD0) $U_\alpha \neq \{1\} \quad \forall \alpha \in \Phi$.

(RGD1) For every prenilpotent pair $\{\alpha, \beta\} \subseteq \Phi$ with $\alpha \neq \beta$, the commutator $[U_\alpha, U_\beta]$ is contained in $\langle U_\gamma \mid \gamma \in [\alpha, \beta] \setminus \{\alpha, \beta\}\rangle$.

(RGD2) For all $s_i \in S$ and for all $u \in U_{\alpha_i} \setminus \{1\}$, there exists an $m(u) \in U_{s_i(\alpha_i)}\, u\, U_{s_i(\alpha_i)}$ satisfying $m(u) U_\alpha m(u)^{-1} = U_{s_i(\alpha)} \quad \forall \alpha \in \Phi$. Furthermore, $m(u)H = m(u')H$ for all $u, u' \in U_{\alpha_i} \setminus \{1\}$.

(RGD3)* $HU_+ \cap U_- = \{1\}$ if $U_\varepsilon := \langle U_\alpha \mid \alpha \in \Phi_\varepsilon\rangle$ for $\varepsilon \in \{+, -\}$.

(RGD4) $G = H\langle U_\alpha \mid \alpha \in \Phi\rangle$.

These axioms are equivalent to, though slightly different from those stated in [T5], §3.3 (see the hints at the end of the proof). Setting $B_\alpha := HU_\alpha$ for $\alpha \in \Phi$, it is a matter of routine to check that the system $(G, H, (B_\alpha)_{\alpha\in\Phi})$ satisfies the axioms (RD1) – (RD5) of [T3], Section 5. Now it is shown there that N'/H is isomorphic to $W = \langle S\rangle$ and that (G, B'_+, B'_-, N', S) is a twin BN–pair, if one defines $N' := \langle H, m(u) \mid u \in U_{\alpha_i} \setminus \{1\},\ i \in I\rangle$ and $B'_\varepsilon := HU_\varepsilon$ $(\varepsilon \in \{+, -\})$.

So we only need to demonstrate how the system of (RGD)–axioms can be satisfied in the situation we are interested in. Recall the following: $W = W_{\mathrm{aff}}(\Psi)$ is a Coxeter group with generating set $S = \{s_i \mid 0 \leq i \leq n\}$, the set of all half–apartments of $\Sigma(W, S)$ can be identified with the set of all affine roots, the "simple" roots being defined by $\alpha_0 := \alpha_{-a_0,1},\ \alpha_i := \alpha_{a_i,0}$ $(1 \leq i \leq n)$ and the fundamental chamber is the unique chamber contained in $\bigcap_{i=0}^{n} \alpha_i$, implying $\Phi_+ = \{\alpha_{a,\ell} \in \Phi \mid (a \in \Psi_+$ and $\ell \geq 0)$ or $(a \in \Psi_-$ and $\ell \geq 1)\}$. The pair $\{\alpha_{a,\ell}, \alpha_{b,m}\} \subseteq \Phi$ is prenilpotent if and only if $b \neq -a$, and $[\alpha_{a,\ell}, \alpha_{b,m}] = \{\alpha_{pa+qb,p\ell+qm} \in \Phi \mid p, q \geq 0\}$ in that case.

The groups G and H are equal to $\mathcal{G}(R)$ and $\mathcal{T}(k)$ respectively. Define

$$U_{\alpha_{a,\ell}} := \{x_a(ct^{-\ell}) \mid c \in k\} \textbf{ for } \alpha_{a,\ell} \in \Phi\ .$$

Then $N = \mathcal{N}(R)$ acts on $\{U_\alpha \mid \alpha \in \Phi\}$:
Setting $w_a(\lambda) := x_a(\lambda) x_{-a}(-\lambda^{-1}) x_a(\lambda)$ for $a \in \Psi$ and $\lambda \neq 0$, the usual relations in Chevalley groups (cf. for example [St], §3, p. 30) yield

(6) $w_a(ct^{-\ell})\, U_\alpha\, w_a(ct^{-\ell})^{-1} = U_{s_{a,\ell}(\alpha)} \quad \forall a \in \Psi,\ \ell \in \mathbb{Z},\ c \in k^*,\ \alpha \in \Phi$

Note that $\mathcal{T}(R) = \langle w_{a_i}(\lambda) w_{a_i}(1)^{-1} \,|\, 1 \leq i \leq n,\ \lambda \in R^*\rangle$ (cf. [St], Lemma 35) and hence $N = \langle w_a(\lambda) \,|\, a \in \Psi,\ \lambda \in R^*\rangle$, because $\mathcal{G}$ is simply connected. Therefore, there exists a homomorphism $\nu : N \twoheadrightarrow W$ satisfying $mU_\alpha m^{-1} = U_{\nu(m)(\alpha)} \quad \forall m \in N,\ \alpha \in \Phi$. In particular, $H \subseteq \ker \nu \subseteq \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$ (later we shall derive $H = \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$, cf. Hint 1).

The verification of the (RGD)–axioms is easy now: (RGD0) is trivial and (RGD1) a consequence of Chevalley's commutator formulas. (RGD2) follows from (6), the identity $w_a(ct^{-\ell}) = w_{-a}(-c^{-1}t^\ell) = x_{-a}(-c^{-1}t^\ell) x_a(ct^{-\ell}) x_{-a}(-c^{-1}t^\ell)$ and $w_a(ct^{-\ell}) \in w_a(t^{-\ell})H$ $(a \in \Psi,\ c \in k^*,\ \ell \in \mathbb{Z})$. The inclusions $U_+ \subseteq \rho_+^{-1}(\mathcal{U}_+(k))$ and $U_- \subseteq \rho_-^{-1}(\mathcal{U}_-(k))$ first imply $HU_+ \cap U_- \subseteq \mathcal{G}(k[t^{-1}]) \cap \mathcal{G}(k[t]) = \mathcal{G}(k)$ and then $HU_+ \cap U_- \subseteq \mathcal{B}_+(k) \cap \mathcal{U}_-(k)$, hence (RGD3)*. (RGD4) follows from $G = \langle \mathcal{U}_a(R) \mid a \in \Psi\rangle$ and the latter from the fact that $\mathcal{G}$ is simply connected and R is a Euclidean domain. Finally, we note that $N = N'$ and $B_\varepsilon = B'_\varepsilon$ for

$\varepsilon \in \{+,-\}$. The first equation follows from $\nu(N) = \nu(N') = W,\ H \subseteq N' \subseteq N$ and $H = \ker \nu$ (see below), the second from $B'_\varepsilon \subseteq B_\varepsilon,\ \nu^{-1}(S) \cap B_\varepsilon = \emptyset$ and the fact that $(G, B'_\varepsilon, N', S)$ is a Tits system.

Hint 1: Taking into account that $(G, B'_\varepsilon, N', S)$ is a Tits system for $\varepsilon \in \{+,-\}$, the (RGD)–axioms imply $\bigcap\limits_{\alpha\in\Phi} N_G(U_\alpha) \subseteq N_G(U_+) \cap N_G(U_-) = B'_+ \cap B'_- = H$.

Therefore, one obtains an equivalent set of axioms if one defines $H := \bigcap\limits_{\alpha\in\Phi} N_G(U_\alpha)$ from the outset and cancels the second sentence in (RGD2), as carried out in [T5], §3.3. The modification introduced here has the advantage that the verfication of $H \subseteq \bigcap\limits_{\alpha\in\Phi} N_G(U_\alpha)$ is easier than that of $H = \bigcap\limits_{\alpha\in\Phi} N_G(U_\alpha)$, in case one is dealing with concrete groups.

Hint 2: On the other hand, it was not difficult to establish (RGD3)* in our example instead of Tits' (seemingly) weaker axiom

(RGD3) $U_{s_i(\alpha_i)} \not\subseteq U_+ \quad \forall s_i \in S$.

Now (RGD3)* immediately implies $B'_+ \cap B'_- = H$ and hence (TBN2) (cf. [T3], §5.12), whereas the proof of this equality given in [T3] involves the trickiest part of Section 5, namely Theorem 2. So our second modification of the (RGD)–axioms allows to derive rather directly the fact that (G, B'_+, B'_-, N', S) is a twin BN–pair, using only the elementary results of [T3], Section 5, which are very similar to those of [BrT], §6.1.

Note, however, that the axioms (RGD0) – (RGD4) in fact imply (RGD3)* which I think is surprising (it is even not clear that they imply $U_{\alpha_i} \not\subseteq U_- \quad \forall i \in I$) as well as interesting. The demonstration of this statement depends on a careful analysis of Tits' arguments in [T3], Section 5, especially of his proof of Theorem 2. ∎

Remark 7: Let $K_\varepsilon := k((t^{-\varepsilon}))$ be the complete, discretely valuated field of Laurent series in $t^{-\varepsilon}$ with coefficients in k, and let $\mathcal{O}_\varepsilon := k\,[[t^{-\varepsilon}]]$ be the corresponding valuation ring ($\varepsilon \in \{+,-\}$). Set $\overline{G}_\varepsilon := \mathcal{G}(K_\varepsilon),\ \overline{B}_\varepsilon := \overline{\rho}_\varepsilon^{-1}\,(\mathcal{B}_\varepsilon(k))$, where $\overline{\rho}_\varepsilon : \mathcal{G}(\mathcal{O}_\varepsilon) \longrightarrow \mathcal{G}(k)$ is the obvious extension of $\rho_\varepsilon,\ \overline{N}_\varepsilon := \mathcal{N}(K_\varepsilon)$ and $\overline{H}_\varepsilon := \mathcal{T}(\mathcal{O}_\varepsilon)$. It follows $\overline{B}_\varepsilon \cap \overline{N}_\varepsilon = \overline{H}_\varepsilon,\ \overline{N}_\varepsilon = N\overline{H}_\varepsilon,\ N \cap \overline{H}_\varepsilon = H$ and hence $\overline{N}_\varepsilon/\overline{H}_\varepsilon \cong N/H \cong W = \langle S \rangle$.

$\overline{B}_\varepsilon$ is open in $\overline{G}_\varepsilon$ with respect to the topology induced by the discrete valuation, $G = \langle \mathcal{U}_a(R) \mid a \in \Psi \rangle$ is dense in $\overline{G}_\varepsilon = \langle \mathcal{U}_a(K_\varepsilon) \mid a \in \Psi \rangle$ and $G \cap \overline{B}_\varepsilon = B_\varepsilon$ is dense in $\overline{B}_\varepsilon$, therefore. From the fact that (G, B_ε, N, S) is a Tits system, the same now follows for $(\overline{G}_\varepsilon, \overline{B}_\varepsilon, \overline{N}_\varepsilon, S)$. Furthermore, the buildings $\Delta_\varepsilon := \Delta(G, B_\varepsilon)$ and $\overline{\Delta}_\varepsilon := \Delta(\overline{G}_\varepsilon, \overline{B}_\varepsilon)$ associated to these Tits systems are canonically isomorphic and will be identified in the following. Note that **$\boldsymbol{\Delta_\varepsilon}$ is the Bruhat–Tits building of $\boldsymbol{\mathcal{G}}$ over $\mathbf{K}_\varepsilon$** (cf. [BrT], Example 6.2.3 b), Theorem 6.5 and Definition 7.4.2) and that Lemma 5 yields its existence without using the results of [BrT].

Because G acts strongly transitively on the twin building associated to the twin BN–pair of Lemma 5, Propositions 3 immediately implies the following

Proposition 5: *Let Δ_ε be the Bruhat–Tits building of $\mathcal{G}\,(k((t^{-\varepsilon})))$ for $\varepsilon \in \{+,-\}$, Σ_+ the standard apartment of Δ_+ corresponding to $\mathcal{T}$ and c_- the chamber of Δ_- stabilized by $\overline{B}_-$. Then $G(\Sigma_+ \times c_-) = \Delta_+ \times \Delta_-$,* *and* $(a_+, a_-), (a'_+, a'_-) \in \Sigma_+ \times c_-$ *are equivalent under the action of* $G = \mathcal{G}(k[t, t^{-1}])$ *if and only if they are under that of* $N = \mathcal{N}(k[t, t^{-1}])$. ■

We conclude this section by giving a new proof for Theorem 1 in [So].

Proposition 6: *Denote by* D_+ *the closed Weyl chamber corresponding to the base* Π *of* Ψ, *i.e.* $D_+ = \bigcap_{i=1}^{n} \alpha_{a_i,0}$. *Then* D_+, *viewed as a subcomplex of* Σ_+, *is a simplicial fundamental domain for the action of* $\mathcal{G}(k[t])$ *on the Bruhat–Tits building* Δ_+ *of* $\mathcal{G}\,(k((t^{-1})))$.

Proof: Let 0 be the origin of the standard apartment Σ_- of Δ_-. Then $\mathrm{Stab}_{\mathcal{G}(K_-)}(0) = \mathcal{G}(\mathcal{O}_-)$ and $\mathrm{Stab}_G(0) = \mathcal{G}(k[t])$. Furthermore, $\mathrm{Stab}_W(0)$ is equal to the linear Weyl group $W(\Psi)$. It follows from Lemma 3 and Remark 5 (or from the classical theory of root systems) that D_+ is a simplicial fundamental domain for the action of $W(\Psi)$ on Σ_+. Hence our claim is a consequence of Corollary 1 of Proposition 3. ■

Remark 8: Assuming Lemma 5, Propositions 5 and 6 immediately follow from Proposition 3. Therefore, analogous statements are true for a reductive instead of a Chevalley group $\mathcal{G}$, whenever $G = \mathcal{G}(k[t, t^{-1}])$ admits a twin BN–pair. This is the case, for example, for every simply connected almost simple group $\mathcal{G}$ which is **defined and isotropic over *k*** (cf. [T5], Example of §3.2):

Denote by $\mathcal{S}$ a maximal k–split torus of $\mathcal{G}$, by $\mathcal{N}$ its normalizer, by $\mathcal{T}$ its centralizer, by Ψ the relative root system of $\mathcal{G}$ with respect to $\mathcal{S}$, let $(\mathcal{B}_+, \mathcal{B}_-)$ be a pair of opposite minimal parabolic k–subgroups containing $\mathcal{S}$ and repeat with these re–interpretations the definitions preceeding Lemma 5. Then (G, B_+, B_-, N, S) is again a twin BN–pair. In fact, it is again possible to define appropriate root groups U_α satisfying the (RGD)–axioms. The verfication of these axioms is technically more difficult than the proof of Lemma 5 — especially in case Ψ is not reduced — and uses the Borel–Tits theory of reductive groups (cf. [BoT]).

Twin BN–pairs may also be constructed if $\mathcal{G}$ is not simply connected. In that case, $\mathcal{G}(k[t, t^{-1}])$ has to be replaced by $G = \mathcal{G}(k[t, t^{-1}])^+$, the group generated by all "elementary matrices" with entries in $k[t, t^{-1}]$. Furthermore, if $\mathcal{G}$ is a classical group, the (RGD)–axioms for $\mathcal{G}(k[t, t^{-1}])^+$ can easily be verified without referring to the general theory of reductive groups by simply applying the relations stated in [BrT], §10.

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