Regularity of the solutions of elliptic systems in polyhedral domains

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Abstract

The solution of the Dirichlet problem relative to an elliptic system in a polyhedron has a complex singular behaviour near edges and vertices. Here, we show that this solution has a global regularity in appropriate weighted Sobolev spaces. Some useful embeddings of these spaces into classical Sobolev spaces are also established. As applications, we consider the Lamé, Stokes and Navier-Stokes systems. The present results will be applied in a forthcoming work to the constructive treatment of these problems by optimal convergent finite element method.

1 Preliminaries

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain whose boundary Γ is a straight polyhedron. On Ω and Γ, we shall consider the usual Sobolev spaces $H^s(\Omega)$ and $H^s(\Gamma)$, $s \in \mathbb{R}$, with respective norms and semi-norms denoted by $\|\cdot\|_{s,\Omega}$ or $|\cdot|_{s,\Omega}$ and $\|\cdot\|_{s,\Gamma}$ or $|\cdot|_{s,\Gamma}$ (see [5] for the precise definition). $\overset{\circ}{H}{}^s(\Omega)$ is the closure in $H^s(\Omega)$ of $\mathcal{D}(\Omega)$, the space of C^∞ functions with compact support in Ω.

We take as interior operators ADN-elliptic systems of multi-degree $\mathbf{m} = (m_1, \cdots, m_N)$, homogeneous with constant coefficients as explained below, with Dirichlet boundary conditions.

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For the system $\mathbf{L} = (L^{ij}(D))_{1\le i,j\le N}$ of partial differential operators, we make the following assumptions: there exists $\mathbf{m} = (m_1, \cdots, m_N) \in \mathbb{N}^N$ such that $\mathbf{L}$ is ADN-elliptic of multi-degree $(m_1, \cdots, m_N)$ and homogeneous with constant coefficients. This means that $L^{ij}(D)$ is a homogeneous operator with constant coefficients of order $m_i + m_j$ (obviously, L^{ij} could be equal to 0) and for all $\xi \in \mathbb{R}^n \setminus \{0\}$, we have

$$l(\xi) := \det(L^{ij}(\xi))_{1\le i,j\le N} \neq 0.$$

We moreover assume that the system $\mathbf{L}$ is properly elliptic [1], i.e., for every pair of linearly independent vectors $\xi, \xi' \in \mathbb{R}^3$, the polynomial $l(\xi + \tau\xi')$ in the complex variable τ has exactly $\sum_{i=1}^N m_i$ roots with positive imaginary parts. For such operators, the homogeneous Dirichlet conditions are written as

$$u_j \in \overset{\circ}{H}{}^{m_j}(\Omega), 1 \le j \le N,$$

which are complementing boundary conditions (in the sense of [1]).

Let us now denote by $\mathbf{b}$ a vector $(b_1, \cdots, b_N)$ of $\mathbb{R}^N$. Then the natural spaces associated with the above system are:

$$\mathbf{H}^{\mathbf{b}}(\Omega) := \prod_{i=1}^N H^{b_i}(\Omega), \overset{\circ}{\mathbf{H}}{}^{\mathbf{b}}(\Omega) := \prod_{i=1}^N \overset{\circ}{H}{}^{b_i}(\Omega),$$

with the product norm (resp. semi-norm) denoted by $\|\cdot\|_{\mathbf{b},\Omega}$ (resp. $|\cdot|_{\mathbf{b},\Omega}$).

Thus the operator $\mathbf{L}$ is continuous from $\mathbf{H}^{s+\mathbf{m}}(\Omega)$ to $\mathbf{H}^{s-\mathbf{m}}(\Omega)$, for all $s \ge 0$ (from now on, we make the convention that for any $s \in \mathbb{R}$ and $\mathbf{b} = (b_1, \cdots, b_N) \in \mathbb{R}^N$, we set $s + \mathbf{b} = (s + b_1, \cdots, s + b_N)$).

Therefore the boundary value problem we have in mind is the following one: Given $\mathbf{f} \in \mathbf{H}^{k-\mathbf{m}}(\Omega)$, with a fixed $k \in \mathbb{N}$, we are interested in $\mathbf{u} = (u_1, \cdots, u_N) \in \overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}(\Omega)$, the variational solution of

$$\mathbf{L}\mathbf{u} = \mathbf{f} \text{ in } \Omega, \tag{1}$$

or equivalently,

$$\mathbf{a}(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx, \forall\, \mathbf{v} \in \overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}(\Omega), \tag{2}$$

where the bilinear form $\mathbf{a}$ is defined by

$$\mathbf{a}(\mathbf{v}, \mathbf{w}) = \langle \mathbf{L}\mathbf{v}, \mathbf{w}\rangle, \forall\, \mathbf{v}, \mathbf{w} \in \overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}(\Omega). \tag{3}$$

Since we do not suppose that the form $\mathbf{a}$ is strongly coercive on $\overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}(\Omega)$, the existence of a solution to (3) is not guaranteed; therefore as in [3, §7], we assume that

$$\mathbf{L} \text{ is a Fredholm operator from } \overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}(\Omega) \text{ into } \mathbf{H}^{-\mathbf{m}}(\Omega). \tag{4}$$

Let us also notice this condition holds for strongly elliptic systems as stated in [3, §7].

The two examples that we have in mind are the Lamé system and the Stokes system:

Example 1.1 The *Lamé system* in $\mathbb{R}^3$ is defined by

$$L^{ij} = -\delta_{ij}\triangle - (1-2\nu)^{-1}\partial^2_{ij}, 1 \leq i,j \leq 3,$$

where $\nu \in]0, 1/2[$ is the Poisson ratio. It is strongly elliptic with multi-degree $(1,1,1)$ and its associated bilinear form is strongly coercive on $(\overset{\circ}{H}{}^1(\Omega))^3$.

Example 1.2 If $\mathbf{u} \in (\overset{\circ}{H}{}^1(\Omega))^3$ is the velocity and $p \in L^2(\Omega)$ the pressure, the *Stokes system* is defined by

$$\begin{aligned}\mathbf{S}(\mathbf{u},p) &= (-\triangle\mathbf{u} + \nabla p,\ \operatorname{div}\mathbf{u}) \\ &= \left((-\triangle u_i + \partial_i p)_{1\leq i\leq 3}, \sum_{i=1}^{3}\partial_i u_i\right).\end{aligned}$$

It is a properly elliptic system with multi-degree $(1,1,1,0)$, but not a strongly elliptic system, nevertheless it satisfies (4), since it is a isomorphism from $(\overset{\circ}{H}{}^1(\Omega))^3 \times (L^2(\Omega)/\mathbb{R})$ into its dual $(H^{-1}(\Omega))^3 \times (L^2_0(\Omega))$ [26, Th.I.2.4], where $L^2_0(\Omega)$ is the subspace of $L^2(\Omega)$ of functions q such that

$$\int_\Omega q(x)\, dx = 0.$$

It is well known [8, 16, 17, 18, 3, 5, 4, 7, 23] that a solution $\mathbf{u}$ of (1) presents edge and/or vertex singularities. To describe them we need some notation (cf. section 7.B of [3]): Firstly, we fix S in the set $\mathcal{S}(\Omega)$ of vertices of Ω. Let C_S be the infinite polyhedral cone of $\mathbb{R}^3$ which coïncides with Ω in a neighbourhood of S; we set $G_S = C_S \cap S^2(S)$, the intersection of C_S with the unit sphere centred at S. We denote by (r_S, ω_S) the spherical coordinates in C_S. With respect to these coordinates, the expression of the operator L^{ij} is

$$L^{ij}(D_x) = r_S^{-m-m_i}\mathcal{L}^{ij}_S(\omega_S, r_S\partial_{r_S}, D_{\omega_S})r_S^{m-m_j},$$

where $m = \sup_i m_i$. Then for a parameter $\lambda \in \mathbb{C}$, we introduce the operator $\mathcal{L}_S(\lambda) \equiv (\mathcal{L}^{ij}_S(\omega_S, \lambda, D_{\omega_S}))_{1\leq i,j\leq N}$ acting from $\overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}(G_S)$ into $\mathbf{H}^{-\mathbf{m}}(G_S)$. The ellipticity assumption and (4) insures that $\mathcal{L}_S(\lambda)^{-1}$ is meromorphic on $\mathbb{C}$ and its poles generate some vertex singularities [3, §7]. More precisely, denoting by Λ'_S, the set of these poles, we associate to any $\lambda \in \Lambda'_S$ a Jordan chain $\varphi_S^{\lambda,\nu,q} \in \overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}(G_S), \nu = 1, \ldots, M(\lambda), q = 1, \ldots, \kappa(\lambda,\nu)$, of $\mathcal{L}_S(\lambda)$ satisfying (see [15, 2] for more details)

$$\sum_{q=1}^{l}\mathcal{L}_S^{(l-q)}(\lambda)\frac{\varphi_S^{\lambda,\nu,q}}{(l-q)!} = 0, \forall\, l = 1, \ldots, \kappa(\lambda,\nu). \tag{5}$$

The singular functions relative to λ are then given by

$$\sigma_S^{\lambda,\nu,l}(r_S,\omega_S) = \left(r_S^{\lambda-m+m_j}\sum_{q=1}^{l}\frac{(\log r_S)^{l-q}}{(l-q)!}\varphi_{S,j}^{\lambda,\nu,q}(\omega_S)\right)_{1\leq j\leq N}. \tag{6}$$

Since we are working in usual Sobolev spaces, the polynomial resolution [2] may provide extra singularities $\mathbf{e}^{\lambda,\nu}$, in the particular case when $\lambda \in \mathbb{N}$. Let us set

$$
\begin{aligned}
S^\lambda(C_S) \quad &= \{\mathbf{u} = (u_k)_{1\le k\le N} \text{ with} \\
&\quad u_k = r_S^{\lambda-m+m_k}\sum_{q=0}^{Q}(\log r_S)^q u_k^q(\omega_S); Q\in\mathbb{N} \text{ and } \mathbf{u}^q \in \overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}(G_S)\},\\
P^\lambda(C_S) \quad &= \{\mathbf{u}\in S^\lambda(C_S) : \mathbf{u} \text{ is a polynomial}\},\\
E^\lambda(C_S,\mathbf{L}) \quad &= \{\mathbf{u}\in S^\lambda(C_S) : \mathbf{Lu} \text{ is a polynomial}\},
\end{aligned}
$$

with the convention that $\mathbf{p} = (p_1,\cdots,p_N)$ is called a polynomial if each of p_k is a polynomial. Then we define $\{\mathbf{e}_S^{\lambda,\nu}\}_{\nu=1}^{N(\lambda)}$ a basis of $E^\lambda(C_S,L)/P^\lambda(C_S)$ (if its dimension is ≥ 1). From the definition of $S^\lambda(C_S)$, the $\mathbf{e}_S^{\lambda,\nu}$'s have clearly an expansion similar to (6) and have therefore the same regularity as the σ's. If $\Lambda"_S = \{\lambda \in \mathbb{N} : \dim E^\lambda(C_S,L)/P^\lambda(C_S) \ge 1\}$, then the set of singular exponents is $\Lambda_S = \Lambda'_S \cup \Lambda"_S$. Finally, we put

$$\bar\Lambda_S(k) = \{\lambda\in\Lambda_S : \Re\lambda \in]m-\frac{3}{2}, k+m-\frac{3}{2}]\}.$$

Regarding the edge-vertex singularities, we proceed as follows. To each edge $A_{S,j}$, $1\le j\le J_S$, adjacent to the vertex S, corresponds a singular point of G_S, still denoted by $A_{S,j}$. In this way, there exists a local chart sending a neighbourhood of the point $A_{S,j}$ on G_S into a neighbourhood of 0 in the infinite sector $C_{S,j}$ of $\mathbb{R}^2$ of opening $\omega_{S,j} \in]0,2\pi[$, which can be written in polar coordinates as

$$C_{S,j} = \{(\theta_{S,j},\varphi_{S,j}) : \theta_{S,j} > 0, 0 < \varphi_{S,j} < \omega_{S,j}\}.$$

Notice that for an arbitrary point $M \in C_S$, $\theta_{S,j}$ represents the angle between the edge $A_{S,j}$ and the vector $\vec{SM}$; while $\varphi_{S,j}$ determines the position of M (see Figure 1 of [24]).

We denote by $z_{S,j}$, the cartesian coordinates in $C_{S,j}$ and by $\tilde{\mathbf{L}}(z_{S,j}, D_{z_{S,j}})$ the system obtained from $\mathcal{L}_S(\omega_S, 0, D_{\omega_S})$ via the local chart. We then set $\mathbf{L}_{S,j}(D_{z_{S,j}}) = pp(\tilde{\mathbf{L}}(0, D_{z_{S,j}}))$. As previously, writing $\mathbf{L}_{S,j}$ in the polar coordinates $(\theta_{S,j},\varphi_{S,j})$, we obtain in the usual way the operator $\mathcal{L}_{S,j}(\lambda)$, defined from $\overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}(]0,\omega_{S,j}[)$ into $\mathbf{H}^{-\mathbf{m}}(]0,\omega_{S,j}[)$, with poles μ in the set $\Lambda'_{S,j}$ and generating singular functions $\sigma_{S,j}^{\mu,\nu,l}$ of $\mathbf{L}_{S,j}$ similar to those in (6). Given $t\ge 0$, the analogues of the set $\bar\Lambda_S(k)$ is (according to Corollary 5.16 of [3], which still holds for systems as explained in §7.D od [3], the set $\Lambda"_{S,j}$ is included in $\Lambda'_{S,j}$)

$$\bar\Lambda_{S,j}(t) = \{\mu\in\Lambda'_{S,j} : \Re\mu \in]m-1, t+m-1]\}.$$

Finally, we shall use two types of cut-off functions: $\chi_S = \chi_S(r_S) \in \mathcal{D}(\bar{\mathbb{R}}_+)$ and $\chi_{S,j} = \chi_{S,j}(\theta_{S,j}) \in \mathcal{D}([0,2\pi])$: χ_S (resp. $\chi_{S,j}$) is equal to 1 in a neighbourhood of S (resp. $A_{S,j}$) and has a support concentrated only near the vertex S (resp. the edge $A_{S,j}$). Moreover, the support of $\chi_{S,j}$ is chosen sufficiently small so that the functions $\theta_{S,j}$ and $\sin\theta_{S,j}$ are equivalent on this support.

The schedule of our paper is the following one: In section 2, we collect the main results. Sections 3 and 4 are devoted to edge and global regularity results respectively. In section 5, we establish different continuous embeddings, which are necessary for our future numerical goals. Finally, the last section deals with the applications of the former results to the Lamé, Stokes and Navier-Stokes systems.

2 Main results

The functional framework for global regularity results is provided in the next definition (see definition 2.1 of [14]).

Definition 2.1 *For two real numbers α, β, and two nonnegative integers l, k such that $k > 0$, we set*

$$H^{l,k;\alpha,\beta}(\Omega) := \{v \in H^l(\Omega) : \ r^\alpha \theta^\beta D^\gamma v \in L^2(\Omega), \forall\, \gamma \in \mathbb{N}^3 : l < |\gamma| \le k + l\},$$

where $r(x)$ is the distance from x to the vertices of Ω and

$$\theta := \sum_{S \in \mathcal{S}(\Omega)} \chi_S [1 + \sum_{j=1}^{J_S} \chi_{S,j}(\theta_{S,j})(\theta_{S,j} - 1)] + (1 - \sum_{S \in \mathcal{S}(\Omega)} \chi_S)\delta,$$

δ being the distance to the edges of Ω. It is a Hilbert space for the norm:

$$\|v\|_{H^{l,k;\alpha,\beta}(\Omega)} := \left\{ \|v\|^2_{l,\Omega} + \sum_{l < |\gamma| \le l+k} \|r^\alpha \theta^\beta D^\gamma v\|^2_{0,\Omega} \right\}^{1/2}. \tag{7}$$

Further, for a multi-degree $\mathbf{l} \in \mathbb{N}^N$, we set

$$\mathbf{H}^{\mathbf{l},k;\alpha,\beta}(\Omega) = \prod_{i=1}^{N} H^{l_i,k;\alpha,\beta}(\Omega),$$

with the product norm.

Let us notice that the weight θ has different behaviours in different parts of the domain Ω: In a sufficiently small neighbourhood of a vertex S, θ coincides with the angular distance $\theta_{S,j}$ to the edge $A_{S,j}$ in a neighbourhood of this edge, while $\theta = 1$ far from the edges. On the contrary, far from the vertices, θ behaves like δ, the distance to the edges. Note also that $\delta \sim r\theta$.

Theorem 2.2 *Let $k \ge m$ and suppose that α, β are two real numbers satisfying (H_V) and (H_E) hereafter:*

$$(\mathbf{H_V}) \left\{ \begin{array}{l} \alpha = 0 \text{ or } \alpha \notin \mathbb{N}^*, \alpha > k + m - \frac{3}{2} - \Re\lambda, \forall\, \lambda \in \bar{\Lambda}_S(k), \\ \text{according as the set } \bar{\Lambda}_S(k) \text{ is empty or not.} \end{array} \right.$$

$$(\mathbf{H_E}) \left\{ \begin{array}{l} \beta = 0 \text{ or } \beta \notin \mathbb{N}^*, \beta > k + m - 1 - \Re\mu, \forall\, \mu \in \bar{\Lambda}_{S,j}(k), \\ \text{according as the sets } \bar{\Lambda}_{S,j}(k) \text{ are all empty or not.} \end{array} \right.$$

Then the solution $\mathbf{u} \in \overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}(\Omega)$ of (1), with $\mathbf{f} \in \mathbf{H}^{k-\mathbf{m}}(\Omega)$, belongs to $\mathbf{H}^{\mathbf{m},k,\alpha,\beta}(\Omega)$.

The proof of that theorem is quite similar to the proof of Theorem 2.3 of [14]. The two main steps are edge regularity for elliptic systems in a dihedral cone (§3 hereafter) and global regularity using Mellin transformation (§4). For the sake of brevity, we do not give the details and simply explained the differences with the method in [14].

For the treatment of problem (1) with data in weighted Sobolev spaces, we may refer to [17, 18, 22, 23] leading to similar results than ours (but with smoother data).

Finally, the results about the embeddings that we have in mind are summarized in the next Theorem.

Theorem 2.3 *Let us fix two nonnegative integers l, k such that $k > 0$. Then for all $\gamma \in [0, k[$, we have the continuous embedding*

$$H^{l,k;\gamma,\gamma}(\Omega) \hookrightarrow H^{l+k-\gamma-\varepsilon}(\Omega), \tag{8}$$

for any $\varepsilon \in]0, k-\gamma]$ if $\gamma \notin \mathbb{Z}$ and $\varepsilon = 0$, if $\gamma \in \mathbb{Z}$.

3 Edge regularity

Here we analyze the edge behaviour of a solution $\mathbf{v}$ of a problem similar to (1) in a dihedral cone

$$D = \mathbb{R} \times C,$$

where C is an infinite cone of $\mathbb{R}^2$. We then denote by $x = (y, z)$ the cartesian coordinates in D, with $y \in \mathbb{R}$ and $z \in D$. For our considerations below, we recall the Hilbert weighted Sobolev space of Kondratiev type:

Definition 3.1 [8] *For $\beta \in \mathbb{R}$, and $k \in \mathbb{N}$, $H^k_\beta(C)$ denotes the set of $w \in \mathcal{D}'(C)$ satisfying*

$$\|w\|_{H^k_\beta(C)} := \left\{ \sum_{|\gamma| \leq k} \int_C \left| |z|^{\beta+|\gamma|-k} D^\gamma w \right|^2 \, dz \right\}^{1/2} < +\infty, \tag{9}$$

endowed with the natural norm $\|\cdot\|_{H^k_\beta(C)}$. As usual, for a multi-degree $\mathbf{l} \in \mathbb{N}^N$, we set

$$\mathbf{H}^{\mathbf{l}}_\beta(C) = \prod_{i=1}^N H^{l_i}_\beta(C),$$

with the product norm. Moreover, $\overset{\circ}{\mathbf{H}}{}^{\mathbf{l}}_\beta(C)$ will be the closure of $(\mathcal{D}(C))^N$ in $\mathbf{H}^{\mathbf{l}}_\beta(C)$, and $\mathbf{H}^{-\mathbf{l}}_{-\beta}(C)$ its dual.

Let $\mathbf{L} = \mathbf{L}(z, D_y, D_z)$ be a properly elliptic system in the sense of Agmon-Douglis-Nirenberg of multi-degree $\mathbf{m}$ with $C^\infty(\bar{D})$ coefficients. Contrary to [14], where the authors assumed that the principal part

$$P(D_y, D_z) = pp\, L(0, D_y, D_z)$$

of L frozen at 0, is strongly elliptic, we here suppose that

$$pp\mathbf{L}(0, 0, D_z) \text{ is an isomorphism from } \overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}_0(C) \text{ onto } \mathbf{H}^{-\mathbf{m}}_0(C). \tag{10}$$

Let us remark that for strongly elliptic operators as treated in [14], this condition holds as a consequence of Theorem 1.1 of [8].

The problem in question is the edge regularity of the solution $\mathbf{v} \in \overset{\circ}{\mathbf{H}}^{\mathbf{m}}(D)$ of

$$\mathbf{L}\mathbf{v} = \mathbf{g} \in \mathbf{H}^{k-\mathbf{m}}(D). \tag{11}$$

The hypothesis (10) allows to adapt the method of section 16 of [3] in order to prove that the solution of (11) admits a decomposition into a regular part and a singular one, the singular functions being generated by the set of poles of $\mathcal{L}(\mu)^{-1}$ related to $pp\mathbf{L}(0,0,D_z)$ in the cone C in the usual way. Indeed the technique of section 16 of [3] consists in applying partial Fourier transform (with respect to y) leading to the problem

$$\mathbf{L}(z,\xi,D_z)\hat{\mathbf{v}}(\xi) = \hat{\mathbf{g}}(\xi) \text{ in } C, \text{ for a.e. } \xi \in \mathbb{R}. \tag{12}$$

The hypothesis (10) permits the application of Theorems 12.14 and 7.16 of [3] to the problem (12) leading to a decomposition into a regular part and a singular one for $\hat{\mathbf{v}}(\xi)$. The homogeneousness method of Dauge [3, p.134-136] and again (10) applied to $(pp\mathbf{L})(0,\omega,D_z)$, with $\omega = \pm 1$ yield a decomposition into a regular part and a singular one for $\hat{\mathbf{v}}(\xi)$, for large value of ξ, with a uniform estimate. We conclude by inverse Fourier transform.

In view to the proof of Theorem 3.6 of [14], we can then obtain a similar result for systems satisfying (10). More precisely, we show that $\mathbf{v}$ belongs to an appropriate weighted Sobolev space recalled hereafter.

Definition 3.2 [20] *For $\beta \in \mathbb{R}$, $l \in \mathbb{N}$ and $\delta(y,z) = |z|$ the distance of (y,z) to the edge of D, $W^l_\beta(D)$ denotes the set of $w \in \mathcal{D}'(D)$ satisfying*

$$\|w\|_{W^l_\beta(D)} := \left\{ \sum_{|\gamma|\le l} \int_D |\delta^\beta D^\gamma w|^2 \, dx \right\}^{1/2} < +\infty. \tag{13}$$

It is a Hilbert space for the natural norm $\|\cdot\|_{W^l_\beta(D)}$. Again, for a multi-degree $\mathbf{l} \in \mathbb{N}^N$, we define

$$\mathbf{W}^{\mathbf{l}}_\beta(D) = \prod_{i=1}^N W^{l_i}_\beta(D).$$

We are now ready to state the global edge regularity result whose proof is similar to those of Theorem 3.6 of [14], taking into account the modifications explained above.

Theorem 3.3 *Let $\mathbf{v} \in \overset{\circ}{\mathbf{H}}^{\mathbf{m}}(D)$ satisfying*

$$\mathbf{v} = 0 \textit{ if } |z| \ge 1, \tag{14}$$

be a solution of

$$\mathbf{L}\mathbf{v} = \mathbf{g} \in \mathbf{W}^{k-\mathbf{m}}_\beta(D), \tag{15}$$

with $k > \beta \geq 0$. If the condition
$(\mathbf{H})$ $\beta \notin \mathbb{N}$ *and* $\mathcal{L}(\mu)$ *is invertible for* $\Re\mu = k + m - 1 - \beta$
holds, then

$$\mathbf{v} \in \mathbf{W}_{\beta'}^{k+\mathbf{m}}(D), \tag{16}$$

for any β' such that

$$\left\{ \begin{array}{l} \beta' = \beta \text{ or } \beta' \notin \mathbb{N}^*, \beta' > k + m - 1 - \Re\mu, \forall\, \mu \in \bar{\Lambda}(k-\beta), \\ \text{according as the set } \bar{\Lambda}(k-\beta) \text{ is empty or not.} \end{array} \right. \tag{17}$$

As a consequence of that Theorem, like in [14, §3], we obtain the local edge behaviour of the solution of (1) in the framework of the weighted Sobolev spaces $\mathbf{H}^{\mathbf{m},k;\alpha,\beta}(\Omega)$.

Theorem 3.4 *Let $\eta \in \mathcal{D}(\mathbb{R}^3)$ be a cut-off function such that $\eta \equiv 0$ in a neighbourhood of the vertices of Ω. Then the solution $\mathbf{u} \in \overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}(\Omega)$ of (1) with a datum in $H^{k-\mathbf{m}}(\Omega)$ satisfies*

$$\eta u \in \mathbf{H}^{\mathbf{m},k;\alpha,\beta}(\Omega), \tag{18}$$

for any α and any β satisfying (H_E).

4 Global regularity

The aim of this section is to prove our fundamental Theorem 2.2. In order to control the edge-vertex singularities, we need again new weighted Sobolev spaces:

Definition 4.1 [14] *For $\alpha, \beta \in \mathbb{R}$, $k \in \mathbb{N}$, and a fixed vertex S of Ω, $M^k_{\alpha,\beta}(C_S)$ is the space of $v \in \mathcal{D}'(C_S)$ such that*

$$r_S^{-k+3/2+\alpha} \|(r_S \frac{\partial}{\partial r_S})^l v\|_{W_\beta^{k-l}(G_S)} \in L^2(\mathbb{R}^+, \frac{dr_S}{r_S}), \forall\, l = 0, \ldots, k,$$

which is a Hilbert space for the norm

$$\|v\|_{M^k_{\alpha,\beta}(C_S)} = \left(\sum_{l=0}^{k} \int_0^{+\infty} r_S^{-2k+3+2\alpha} \|(r_S \frac{\partial}{\partial r_S})^l v\|^2_{W_\beta^{k-l}(G_S)} \frac{dr_S}{r_S} \right)^{1/2}.$$

We define, in the usual way, $\mathbf{M}^{\mathbf{k}}_{\alpha,\beta}(C_S)$, for a multi-degree $\mathbf{k} \in \mathbb{N}^N$.

Remark 4.2 The letter M in $M^k_{\alpha,\beta}(C_S)$ expresses the mixed nature of the weights in this space, since we have a H^k_α-type weight (in r_S) in the vertex direction, while a W^k_α-type weight (in θ) occurs in the edge direction (see Remark 4.2 of [14]). Note also that the above space is denoted by $V^{k,2}_{\alpha,\beta}(C_S)$ in [17, 22].

Theorem 4.3 *If $\mathcal{L}_S(\lambda)$ is invertible, for $\Re\lambda = k+m-\frac{3}{2}$, then a solution $\mathbf{u} \in \overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}(\Omega)$ of (1) with a datum $\mathbf{f} \in \mathbf{H}^{k-\mathbf{m}}(\Omega)$, with $k \geq m$ admits the decomposition:*

$$\chi_S \mathbf{u} = \mathbf{u}_S + \chi_S \mathbf{u}_P + \sum_{\lambda \in \Lambda_S(k)} \mathbf{u}^{S,\lambda}, \tag{19}$$

where $\mathbf{u}_S \in \mathbf{H}^{\mathbf{m}}_{-k}(C_S) \cap \overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}(\Omega)$, *with* $\mathbf{L}\mathbf{u}_S \in \mathbf{M}^{k-\mathbf{m}}_{0,\beta}(C_S)$ *for any* β *satisfying* (H_E), $\mathbf{u}_P$ *is a polynomial and* $\mathbf{u}^{S,\lambda}$ *is the vertex singular part relative to* S *and* λ*; it takes the form:*

$$\mathbf{u}^{S,\lambda} = \begin{cases} \sum_{\nu=1}^{M(\lambda)} \sum_{l=1}^{\kappa(\lambda,\nu)} c_S^{\lambda,\nu,l} \sigma_S^{\lambda,\nu,l} \text{ if } \lambda \notin \mathbb{Z}, \\ \sum_{\nu=1}^{N(\lambda)} d_S^{\lambda,\nu} \mathbf{e}_S^{\lambda,\nu}, \text{ if } \lambda \in \mathbb{Z}, \end{cases}$$

where $c_S^{\lambda,\nu,l}, d_S^{\lambda,\nu} \in \mathbb{C}$. *If* $\mathcal{L}_S(\lambda)$ *is not invertible, for some* λ *such that* $\Re\lambda = k+m-\frac{3}{2}$, *then the same conclusion holds, but with* $\mathbf{u}_S \in \mathbf{H}^{\mathbf{m}}_{-k+\epsilon}(C_S) \cap \overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}(\Omega)$ *and* $\mathbf{L}\mathbf{u}_S \in \mathbf{M}^{k-\mathbf{m}}_{\epsilon,\beta}(C_S)$, $\epsilon > 0$ *sufficiently small and* β *as above.*

Proof: It follows those of Theorem 4.4 of [14], since, as explained in section 12.C of [3], the condition (4) implies that $\mathcal{L}(\lambda)$ satisfies the properties from Proposition 8.4 of [3]. ∎

This theorem describes explicitly the vertex singularities, while the edge-vertex singularities are hidden in $\mathbf{u}_S$, as explained hereafter.

Theorem 4.4 *Let* $\mathbf{v} \in \overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}_{-k+\epsilon}(C_S)$ *with a compact support be such that* $\mathbf{L}\mathbf{v} \in \mathbf{M}^{k-\mathbf{m}}_{\epsilon,\beta}(C_S)$, *with* $k \geq m, \beta \geq 0$ *satisfying* (H_E) *and* $\epsilon \geq 0$. *Then*

$$\mathbf{v} \in \mathbf{M}^{k+\mathbf{m}}_{\epsilon,\beta}(C_S). \tag{20}$$

Proof: We perform the usual change of variable

$$\mathbf{w}(t, \omega_S) = e^{\eta t}\mathbf{v}(e^t, \omega_S),$$
$$h_i(t, \omega_S) = e^{(\eta+2m_i)t} \sum_{j=1}^{N} (L^{ij} v_j)(e^t, \omega_S),$$

where $\eta = -(k + m - \frac{3}{2} - \epsilon)$. Then $\mathbf{w} \in \overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}(\mathbb{R} \times G_S)$ (cf. Theorem AA.3 in [3]) is a solution of

$$\mathcal{L}_S(\omega_S, \frac{\partial}{\partial t} - \eta I, D_{\omega_S})\mathbf{w} = \mathbf{h} \text{ in } \mathbb{R} \times G_S, \tag{21}$$

with, as one can show, $\mathbf{h} \in \mathbf{W}^{k-\mathbf{m}}_{\beta}(\mathbb{R} \times G_S)$. Since $\mathcal{L}_S(\omega_S, \lambda, D_{\omega_S})$ is a Fredholm operator (of index 0) from $\overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}(G_S)$ into $\mathbf{H}^{-\mathbf{m}}(G_S)$ (see §1), by section 12.C of [3], we deduce that its principal part *pp* $\mathcal{L}_S(A_{S,j}, \lambda, D_{\omega_S})$ frozen at the vertex $A_{S,j}$ is an isomorphism from $\overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}_0(C_{S,j})$ into $\mathbf{H}^{-\mathbf{m}}_{\mathbf{0}}(C_{S,j})$. We can then apply Theorem 3.3 to problem (21), which yields, since $\bar{\Lambda}_{S,j}(k - \beta) = \emptyset$:

$$\mathbf{w} \in \mathbf{W}^{k+\mathbf{m}}_{\beta}(\mathbb{R} \times G_S).$$

This last inclusion written in the initial coordinates yields the desired regularity (20). ∎

The proof of Theorem 2.2 is now a consequence of Theorems 3.4, 4.3 and 4.4, as explained, for scalar operators, at the end of section 4 of [14].

5 Embeddings

The proof of Theorem 2.3 requires the introduction of weighted Sobolev spaces of Kondratiev type, where the weight is here the distance δ to the edges. More precisely, let us take the

Definition 5.1 *For an arbitrary real number α and a nonnegative integer k, let us define*

$$V^k_\alpha(\Omega) := \{v \in \mathcal{D}'(\Omega) : \delta^{\alpha-k+|\gamma|} D^\gamma v \in L^2(\Omega), \forall\ |\gamma| \le k\}.$$

It is clearly a Hilbert space for the norm

$$\|v\|_{k;\alpha} := \left\{ \sum_{|\gamma|\le k} \int_\Omega |\delta^{\alpha-k+|\gamma|} D^\gamma v|^2 \, dx \right\}^{1/2}.$$

The first result that we need concerns the interpolation of those spaces (from now on, for two Hilbert spaces X, Y with $X \hookrightarrow Y$, $(X,Y)_{\theta,2}$ means the real interpolation space between X and Y, for $\theta \in [0,1]$, see [12, 27]).

Theorem 5.2 *Let us fix a nonnegative integer k and three real numbers α, β, γ such that $\alpha < \gamma < \beta$. Then we have the two continuous embeddings:*

$$(V^k_\alpha(\Omega), V^k_\beta(\Omega))_{\theta,2} \hookrightarrow V^k_\gamma(\Omega) \text{ for } \theta = \frac{\gamma-\alpha}{\beta-\alpha}, \tag{22}$$

$$V^k_\gamma(\Omega) \hookrightarrow (V^k_\alpha(\Omega), V^k_\beta(\Omega))_{\theta',2}, \text{ for any } \theta' \in]\theta, 1[. \tag{23}$$

Proof: To prove the first inclusion, for any $\eta \in \mathbb{R}$, let us introduce the mapping

$$\begin{aligned} A : V^k_\eta(\Omega) &\to \prod_{|\mu|\le k} V^0_{\eta-k+|\mu|}(\Omega) \\ u &\to (D^\mu u)_{|\mu|\le k}. \end{aligned}$$

Since A is a bounded operator, by interpolation (Theorem I.5.1 of [12]), it is also bounded from $(V^k_\alpha(\Omega), V^k_\beta(\Omega))_{\theta,2}$ into $\prod_{|\mu|\le k}(V^0_{\alpha-k+|\mu|}(\Omega), V^0_{\beta-k+|\mu|}(\Omega))_{\theta,2}$. But Theorem 1.18.5 of [27] yields $(V^0_{\alpha-k+|\mu|}(\Omega), V^0_{\beta-k+|\mu|}(\Omega))_{\theta,2} = V^0_{\gamma-k+|\mu|}(\Omega)$. In other words, A is bounded from $(V^k_\alpha(\Omega), V^k_\beta(\Omega))_{\theta,2}$ into $\prod_{|\mu|\le k} V^0_{\gamma-k+|\mu|}(\Omega)$, which simply means that the embedding (22) holds.

To establish the second embedding, we shall use the K-method of J. Peetre (cf. [27, §1.3]). Let us fix $v \in V^k_\gamma(\Omega)$. Then we shall estimate the quantity

$$K(v,t) := \inf_{v=v_0+v_1} (\|v_0\|_{k;\alpha} + t\|v_1\|_{k;\beta}), \forall\ t > 0.$$

We actually distinguish the case $t < 1$ from the case $t \ge 1$. In the first case, we take $v_0 = 0$ and $v_1 = v$ (recall that we have $V^k_\alpha(\Omega) \hookrightarrow V^k_\gamma(\Omega) \hookrightarrow V^k_\beta(\Omega)$, due to the condition $\alpha < \gamma < \beta$). Indeed, we then have

$$K(v,t) \le t\|v\|_{k;\beta} \le Ct\|v\|_{k;\gamma},$$

for some positive constant C (independent upon v). Consequently, we get

$$\int_0^1 |t^{-\theta'} K(v,t)|^2 \frac{dt}{t} \leq C \int_0^1 t^{-2\theta'+1}\, dt \|v\|_{k;\gamma}^2 \leq C\|v\|_{k;\gamma}^2, \tag{24}$$

because $\theta' < 1$.

In the case $t \geq 1$, we need more investigations: we introduce a cut-off $\chi \in \mathcal{D}(\mathbb{R})$ such that

$$\chi(t) = \begin{cases} 1 \text{ if } 0 < t < 1/2, \\ 0 \text{ if } t > 1. \end{cases}$$

For a parameter $s \leq 1$, we define $\chi_s(t) = \chi(t/s)$ and $v_s = v \cdot \chi_s(\delta)$. Remark that v_s coincides with v in a neighbourhood of the edges. We then take

$$v_0 = v - v_s, v_1 = v_s,$$

with $s = t^{\frac{1}{\alpha-\beta}}$ (note that v_0 really belongs to $V_\alpha^k(\Omega)$ since it is zero in a neighbourhood of the edges). By Leibniz's rule, it follows

$$\begin{aligned} \|v_0\|_{k;\alpha}^2 &\leq C \sum_{\eta \leq k} \int_\Omega |\delta^{\alpha-k+|\eta|}(1-\chi_s(\delta)) D^\eta v|^2 \, dx \\ &+ C \sum_{\eta \leq k} \sum_{0<\eta' \leq \eta} \int_\Omega |\delta^{\alpha-k+|\eta|} D^{\eta'}(\chi_s(\delta)) D^{\eta-\eta'} v|^2 \, dx \\ &\leq C \sum_{\eta \leq k} \int_{\delta > s/2} \delta^{2(\alpha-\gamma)} |\delta^{\gamma-k+|\eta|}(1-\chi_s(\delta)) D^\eta v|^2 \, dx \\ &+ C \sum_{\eta \leq k} \sum_{0<\eta' \leq \eta} \int_{s/2<\delta<s} |\delta^{\alpha-k+|\eta|} s^{-|\eta'|} D^{\eta-\eta'} v|^2 \, dx, \end{aligned}$$

taking into account the estimate $|D^\eta \delta| \leq C\delta^{1-|\eta|}$ and the equivalence between δ and s on the set $s/2 < \delta < s$. As $\alpha < \gamma$, this finally leads to the estimate

$$\|v_0\|_{k;\alpha} \leq C s^{\alpha-\gamma} \|v\|_{k;\gamma}. \tag{25}$$

Similarly, we show that

$$\|v_1\|_{k;\beta} \leq C s^{\beta-\gamma} \|v\|_{k;\gamma}. \tag{26}$$

The estimates (25), (26) and the choice $s = t^{\frac{1}{\alpha-\beta}}$ yield

$$K(v,t) \leq C t^{\frac{\alpha-\gamma}{\alpha-\beta}} \|v\|_{k;\gamma}, \forall\, t \geq 1.$$

Consequently, we get

$$\int_1^\infty |t^{-\theta'} K(v,t)|^2 \frac{dt}{t} \leq C\|v\|_{k;\gamma}^2 \int_1^\infty t^{-2\theta'+2\frac{\alpha-\gamma}{\alpha-\beta}-1}\, dt \leq C\|v\|_{k;\gamma}^2, \tag{27}$$

since $-2\theta' + 2\frac{\alpha-\gamma}{\alpha-\beta} < 0$.

In conclusion, Definition 1.3.2 of [27] and the estimates (24) and (27) imply that v belongs to $(V_\alpha^k(\Omega), V_\alpha^k(\Omega))_{\theta',2}$ and satisfies

$$\|v\|_{(V_\alpha^k(\Omega), V_\alpha^k(\Omega))_{\theta',2}} \leq C\|v\|_{k;\gamma}.$$

This proves the embedding (23). ∎

In the remainder of this section, we only need the second embedding given in the above Theorem. The first one was given to justify the conjecture that $V_\gamma^k(\Omega) = (V_\alpha^k(\Omega), V_\beta^k(\Omega))_{\theta,2}$.

The second step is to show that the spaces $V_\gamma^1(\Omega)$ are embedded into usual Sobolev spaces.

Theorem 5.3 *For $\gamma \in]0,1[$, we have*

$$V_\gamma^1(\Omega) \hookrightarrow H^{1-\theta'}(\Omega), \forall\ 1 > \theta' > \gamma. \tag{28}$$

Proof: The previous theorem shows that

$$V_\gamma^1(\Omega) \hookrightarrow (V_0^1(\Omega), V_1^1(\Omega))_{\theta',2}, \tag{29}$$

for $1 > \theta' > \gamma$. Moreover the definition of the spaces $V_\gamma^1(\Omega)$ directly leads to

$$\begin{array}{rcl} V_0^1(\Omega) & \hookrightarrow & H^1(\Omega), \\ V_1^1(\Omega) & \hookrightarrow & L^2(\Omega). \end{array}$$

By interpolation, and Theorem 4.3.1/2 of [27], we get

$$(V_0^1(\Omega), V_1^1(\Omega))_{\theta',2} \hookrightarrow (H^1(\Omega), L^2(\Omega))_{\theta',2} = H^{1-\theta'}(\Omega). \tag{30}$$

The composition of (29) with (30) yields (28). ■

We are now able to prove Theorem 2.3 in the case $k = 1$. The general case will follow by induction.

Proposition 5.4 *For $\gamma \in [0,1[$ and a nonnegative integer l, the next embedding holds:*

$$H^{l,1;\gamma,\gamma}(\Omega) \hookrightarrow H^{l+1-\gamma-\varepsilon}(\Omega), \tag{31}$$

for any $\varepsilon \in]0, 1-\gamma[$ if $\gamma > 0$ and $\varepsilon = 0$ if $\gamma = 0$.

Proof: The case $\gamma = 0$ is trivial. Suppose now that $\gamma > 0$. Define the Hilbert space

$$W_\gamma^1(\Omega) := \{v \in \mathcal{D}'(\Omega) : \delta^\gamma D^\beta v \in L^2(\Omega), \forall\ |\beta| \leq 1\},$$

equipped with its natural norm.

Let $u \in H^{l,1;\gamma,\gamma}(\Omega)$ be fixed. Then $u \in H^l(\Omega)$ and satisfies for all $|\alpha| = l$:

$$D^\alpha u \in W_\gamma^1(\Omega).$$

But Proposition 5.1 of [14] (based on Hardy's inequality) implies that

$$W_\gamma^1(\Omega) \hookrightarrow V_\gamma^1(\Omega), \tag{32}$$

because $\gamma > 0$ and δ is equivalent to $r\theta$. Therefore, we have

$$D^\alpha u \in V_\gamma^1(\Omega), \forall\ |\alpha| = l.$$

Owing to Theorem 5.3, we conclude that

$$D^\alpha u \in H^{1-\theta'}(\Omega), \forall\ 1 > \theta' > \gamma, |\alpha| = l.$$

■

Proof of Theorem 2.3: We may suppose that $\gamma > 0$. We use an iterative argument on k. The embedding (8) holds for $k = 1$ as showed in Proposition 5.4. Let us then show that if (8) holds for $k-1$, then it also holds for $k \geq 2$. As $\gamma > 0$, Proposition 5.1 of [14] yields

$$H^{l,k;\gamma,\gamma}(\Omega) \hookrightarrow H^{l,k-1;\gamma-1,\gamma-1}(\Omega). \tag{33}$$

i) If $\gamma \in [1, k[$, then by the induction hypothesis, we have

$$H^{l,k-1;\gamma-1,\gamma-1}(\Omega) \hookrightarrow H^{l+k-1-(\gamma-1)-\varepsilon}(\Omega); \tag{34}$$

and the composition of (33) with (34) leads to (8).
ii) If $0 < \gamma < 1$, then for all $\alpha \in \mathbb{N}^3$ such that $l \leq |\alpha| < l+k$, any $u \in H^{l,k;\gamma,\gamma}(\Omega)$ clearly fulfils

$$D^\alpha u \in W^1_\gamma(\Omega).$$

From (32) and Theorem 5.3, we deduce that

$$D^\alpha u \in H^{1-\gamma-\varepsilon}(\Omega), \forall\, l \leq |\alpha| < l+k.$$

This firstly implies that $u \in H^{l+k-1}(\Omega)$ (because $H^{1-\gamma-\varepsilon}(\Omega) \hookrightarrow L^2(\Omega)$) and secondly that $u \in H^{l+k-\gamma-\varepsilon}(\Omega)$. That is the conclusion. ∎

6 Applications

In this section, we first apply the previous results to the Lamé and Stokes systems. We secondly give a similar result for the solution of the Navier-Stokes equations.

6.1 The Lamé system

The vertex and edge singular exponents of the Lamé system were largely studied in [21, 9, 11]. In [21], it was shown that any $\lambda \in \Lambda'_S$ with $\Re\lambda > -1/2$ satisfies

$$\Re\lambda > \frac{(3-4\nu)\mu}{(\mu+6-4\nu)},$$

where $\mu > 0$ and $\mu(\mu+1)$ is the first eigenvalue of the Laplace-Beltrami operator on G_S with Dirichlet boundary conditions. Consequently, any $\lambda \in \Lambda_S(k)$ satisfies $\Re\lambda > 0$. Moreover, from Theorem 7.3 of [9], we know that if $G_S \ll S^2_+$ (included and different from the unit half-sphere S^2_+), then the strip $\Re\lambda \in [-1/2, 1]$ has no element of Λ'_S, while if $S^2_+ \ll G_S \ll S^2$, then the strip $\Re\lambda \in [-1/2, 1]$ has exactly 3 elements of Λ'_S (counted according to their multiplicity).

On the other hand, for a fixed edge $A_{S,j}$, the system $\mathbf{L}_{S,j}(D_{z_{S,j}})$ is

$$(\mathbf{L}_2(u_1, u_2), -\triangle u_3),$$

where $\mathbf{L}_2$ means the 2-dimensional Lamé system. Consequently, the edge singular exponents $\mu \in \Lambda_{S,j}$ are either the roots of

$$\sin^2(\mu\omega_{S,j}) = \kappa^2 \sin^2(\omega_{S,j})\mu^2, \tag{35}$$

where $\kappa = (3-4\nu)^{-1}$ or $l\pi/\omega_{S,j}$ with $l \in \mathbb{Z}$. By a careful study of the equation (35), we readily check that the exponent of smaller real part ξ_1 is real and solution of

$$\sin(\xi\omega_{S,j}) = \kappa|\sin(\omega_{S,j})|\xi.$$

It satisfies $1 < \xi_1 < \pi/\omega_{S,j}$ if $\omega_{S,j} < \pi$ and $\xi_1 < \pi/\omega_{S,j}$ if $\omega_{S,j} > \pi$; in both cases, $\xi_1 > 1/2$.

All these considerations lead to the following regularity result:

Theorem 6.1 *Let* $\mathbf{f} \in (H^{k-1}(\Omega))^3$, $k \in \mathbb{N}, k \geq 1$, *then the solution* $\mathbf{u} \in (\overset{\circ}{H}{}^1(\Omega))^3$ *of*

$$-\triangle \mathbf{u} - (1-2\nu)^{-1}\nabla\nabla\cdot\mathbf{u} = \mathbf{f},$$

belongs to $(H^{1,k;\alpha,\beta}(\Omega))^3$, *with* $0 \leq \alpha < k-1/2$ *satisfying* $(\mathbf{H_V})$ *and* $0 \leq \beta < k-1/2$ *satisfying* $(\mathbf{H_E})$ *(the sets* Λ_S *and* $\Lambda_{S,j}$ *being defined above). In particular, if* Ω *is convex and* $k = 1$, *then* α, β *can be chosen equal to 0.*

6.2 The Stokes system

The vertex eigenvalues were studied in [21, 4, 10, 11], where it is proved that any $\lambda \in \Lambda'_S$ with $\Re\lambda > -1/2$ satisfies

$$\Re\lambda > \frac{\mu}{(\mu+4)},$$

with μ as above, which yields that any $\lambda \in \Lambda_S(k)$ satisfies $\Re\lambda > 0$. From Theorem 6 of [10], we also know that if $G_S \ll S^2_+$, then the strip $\Re\lambda \in [-1/2, 1[$ has no element of Λ'_S, $\lambda = 1$ being a simple eigenvalue.

Further for a fixed edge $A_{S,j}$, the system $\mathbf{L}_{S,j}(D_{z_{S,j}})$ satisfies

$$\mathbf{L}_{S,j}(D_{z_{S,j}})(u_1, u_2, u_3, p) = (f_1, f_2, f_3, g),$$

if and only if

$$\mathcal{S}_2(u_1, u_2, p) = (f_1, f_2, g) \text{ and } -\triangle u_3 = f_3,$$

where $\mathcal{S}_2$ denotes the 2-dimensional Stokes system. Consequently, the edge singular exponents $\mu \in \Lambda_{S,j}$ are either the roots of

$$\sin^2(\mu\omega_{S,j}) = \sin^2(\omega_{S,j})\mu^2, \tag{36}$$

(corresponding to (35) with $\nu = 1/2$) or $l\pi/\omega_{S,j}$ with $l \in \mathbb{Z}$. The roots of (36) have been studied in [25, 13, 4], from which we deduce that the exponent of smaller real part ξ_1 is real and is solution of

$$\sin(\xi\omega_{S,j}) = -\sin(\omega_{S,j})\xi,$$

if $\omega_{S,j} > \pi$ and $\xi_1 = \pi/\omega_{S,j}$ if $\omega_{S,j} < \pi$. In the first case, it moreover satisfies $\sup(1/2, \omega_1/\omega_{S,j}) < \xi_1 < \pi/\omega_{S,j}$, where $\omega_1 \approx 0.812825\pi$ (see [4] for its exact definition). As for the Lamé system, this leads to the estimate $\xi_1 > 1/2$, in both cases.

As a consequence, the following regularity result holds.

Theorem 6.2 *Let* $(\mathbf{f}, g) \in H^{k-\mathbf{m}}(\Omega)$, *with* $k \in \mathbb{N}, k \geq 1$, $\mathbf{m} = (1,1,1,0)$, *then a solution* $(\mathbf{u}, p) \in \overset{\circ}{\mathbf{H}}{}^{\mathbf{m}}(\Omega) = (\overset{\circ}{H}{}^{1}(\Omega))^3 \times L^2(\Omega)$ *of the Stokes system*

$$-\triangle \mathbf{u} + \nabla p = \mathbf{f}, \operatorname{div} \mathbf{u} = g,$$

belongs to $(H^{1,k;\alpha,\beta}(\Omega))^3 \times H^{0,k;\alpha,\beta}(\Omega)$, *with* $0 \leq \alpha < k - 1/2$ *satisfying* $(\mathbf{H_V})$ *and* $0 \leq \beta < k - 1/2$ *satisfying* $(\mathbf{H_E})$ *(with the sets* Λ_S *and* $\Lambda_{S,j}$ *defined above). If, moreover,* Ω *is convex and* $k = 1$, *then* α, β *can be chosen equal to 0.*

Corollary 6.3 *Under the assumption of Theorem 6.2, the solution* $(\mathbf{u}, p)$ *of the Stokes system belongs to* $(H^{3/2+\varepsilon}(\Omega))^3 \times H^{1/2+\varepsilon}(\Omega)$, *for some* $\varepsilon > 0$ *small enough.*

Proof: For α, β from Theorem 6.2 with $k = 1$ and setting $\gamma = \max(\alpha, \beta)$, we clearly have

$$H^{l,1;\alpha,\beta}(\Omega) \hookrightarrow H^{l,1;\gamma,\gamma}(\Omega), l = 0, 1.$$

Moreover, by Theorem 2.3, the embedding

$$H^{l,1;\gamma,\gamma} \hookrightarrow H^{l+1-\gamma-\varepsilon}(\Omega),$$

holds, for any $\varepsilon > 0$. The conclusion follows from the composition of both embeddings and remarking that $\gamma < 1/2$. ■

6.3 The Navier-Stokes system

Here we investigate the regularity of a solution $(\mathbf{u}, p)$ in $(\overset{\circ}{H}{}^{1}(\Omega))^3 \times L^2(\Omega)$ of the stationary incompressible Navier-Stokes system:

$$\begin{cases} -\triangle \mathbf{u} + \nabla p + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{f} \text{ in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega, \end{cases} \tag{37}$$

where $\mathbf{f} \in (L^2(\Omega))^3$. The unknowns represent the velocity $\mathbf{u}$ and the pressure p. For the proof of the existence of a solution, we refer to [26, Th.II.1.2].

To study the regularity of the solution of (37), as usual, we send the nonlinearity in the right-hand side and consider $(\mathbf{u}, p)$ as solution of the Stokes system with right-hand side $(\mathbf{F}, 0)$, with $\mathbf{F} = \mathbf{f} - \mathbf{u} \cdot \nabla \mathbf{u}$, i.e.

$$\begin{cases} -\triangle \mathbf{u} + \nabla p = \mathbf{F} \text{ in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega. \end{cases} \tag{38}$$

As each u_i belongs to $H^1(\Omega)$ and $\frac{\partial \mathbf{u}}{\partial x_i}$ belongs to $(L^2(\Omega))^3$, Theorem 1.4.4.2 of [5] implies that the product $u_i \frac{\partial u_j}{\partial x_i}$ belongs to $H^{-1/2-\varepsilon}(\Omega)$, for all $i, j = 1, 2, 3$. Accordingly, for $\mathbf{f} \in (L^2(\Omega))^3$, $\mathbf{F}$ satisfies

$$\mathbf{F} \in (H^{-1/2-\varepsilon}(\Omega))^3, \forall\, \varepsilon > 0.$$

By Theorem 3.6 of [4] applied to the Stokes system (38) and the properties of the singular exponents of the Stokes system given in the previous subsection, we conclude that $(\mathbf{u}, p)$ belongs to $(H^{3/2-\varepsilon}(\Omega))^3 \times H^{1/2-\varepsilon}(\Omega)$. We now reiterate the process: with the help of Theorem 1.4.4.2 of [5], the product of a function in $H^{3/2-\varepsilon}(\Omega)$ with a element of $H^{1/2-\varepsilon}(\Omega)$ belongs to $L^2(\Omega)$. This leads to the regularity $\mathbf{F} \in (L^2(\Omega))^3$. Therefore Theorem 6.2 applied to (38) yields the

Theorem 6.4 *Let* $\mathbf{f} \in (L^2(\Omega))^3$*, then a solution* $(\mathbf{u}, p) \in (\overset{\circ}{H}{}^1(\Omega))^3 \times L^2(\Omega)$ *of the Navier-Stokes system (37) belongs to* $(H^{1,1;\alpha,\beta}(\Omega))^3 \times H^{0,1;\alpha,\beta}(\Omega)$*, with* $0 \le \alpha < 1/2$ *satisfying* $(\mathbf{H_V})$ *and* $0 \le \beta < 1/2$ *satisfying* $(\mathbf{H_E})$ *(with the sets* Λ_S *and* $\Lambda_{S,j}$ *defined in subsection 6.2). In particular, if* Ω *is convex, then* $\mathbf{u} \in (H^2(\Omega))^3$ *and* $p \in H^1(\Omega)$*.*

The first assertion of that theorem is also proved in Theorem 10.3 of [18], where a slightly different argument is used. The second assertion is stated in section 1.2 of [11]. We shall now improve these results for smoother data. Namely, we prove the

Theorem 6.5 *Let* $\mathbf{f} \in (H^{k-1}(\Omega))^3$*, with* $k \in \mathbb{N}, k \ge 1$*. Suppose that the sets* $\bar{\Lambda}_S(k-1)$ *and* $\bar{\Lambda}_{S,j}(k-1)$ *are empty, for all vertices* S *and all* $j = 1, \cdots, J_S$*. Then the solution* $(\mathbf{u}, p) \in (\overset{\circ}{H}{}^1(\Omega))^3 \times L^2(\Omega)$ *of the Navier-Stokes system (37) belongs to* $(H^k(\Omega))^3 \times H^{k-1}(\Omega)$ *as well as to* $(H^{1,k;\alpha,\beta}(\Omega))^3 \times H^{0,k;\alpha,\beta}(\Omega)$*, with* $0 \le \alpha < k - 1/2$ *satisfying* $(\mathbf{H_V})$ *and* $0 \le \beta < k - 1/2$ *satisfying* $(\mathbf{H_E})$*.*

Proof: We use an iterative argument. The case $k = 1$ was treated in Theorem 6.4 (remark that the assumptions on $\bar{\Lambda}_S(0)$ and $\bar{\Lambda}_{S,j}(0)$ always hold due to the properties of the singular exponents given in subsection 6.2). It remains to show that if the conclusion holds for $k-1$, then it also holds for $k \ge 2$. Since the set $\bar{\Lambda}_S(k-2)$ is included in $\bar{\Lambda}_S(k-1)$ (and similarly for $\bar{\Lambda}_{S,j}(k-2)$), the induction hypothesis yields the regularity

$$(\mathbf{u}, p) \in (H^{k-1}(\Omega))^3 \times H^{k-2}(\Omega). \tag{39}$$

This regularity for $\mathbf{u}$ and Theorem 1.4.4.2 imply that the product $u_i \cdot \frac{\partial u_j}{\partial x_i}$ belongs to $H^{k-2}(\Omega)$, if $k \ge 3$. The case $k = 2$ needs a special treatment: in that case, we remark that Theorem 6.4 and Corollary 6.3 yield the regularity

$$(\mathbf{u}, p) \in (H^{3/2+\varepsilon}(\Omega))^3 \times H^{1/2+\varepsilon}(\Omega),$$

for some $\varepsilon > 0$ small enough (which is better than (39) with $k = 2$). This permits the use of Theorem 1.4.4.2 and to conclude that the product $u_i \cdot \frac{\partial u_j}{\partial x_i}$ belongs to $L^2(\Omega)$. In other words, we have shown that

$$u_i \cdot \frac{\partial u_j}{\partial x_i} \in H^{k-2}(\Omega), \forall\, 1 \le i, j \le 3, \tag{40}$$

which implies that $\mathbf{F}$ belongs to $(H^{k-2}(\Omega))^3$.

Applying now Theorem 6.2 to the Stokes system (38) with $k-1$ instead of k, we get the regularity

$$(\mathbf{u}, p) \in (H^k(\Omega))^3 \times H^{k-1}(\Omega), \tag{41}$$

because the sets $\bar{\Lambda}_S(k-1)$ and $\bar{\Lambda}_{S,j}(k-1)$ are empty by assumption. This is the first part of the conclusion. To establish the second part, we need to reiterate the process. Indeed this new regularity (41) and Theorem 1.4.4.2 lead to

$$u_i \cdot \frac{\partial u_j}{\partial x_i} \in H^{k-1}(\Omega), \forall\, 1 \le i, j \le 3.$$

As before, we arrive at the property $\mathbf{F} \in (H^{k-1}(\Omega))^3$. It then remains to apply Theorem 6.2 to the system (38). ∎

Corollary 6.6 *Let* $\mathbf{f} \in (H^{k-1}(\Omega))^3$*, with* $k \in \mathbb{N}, k \geq 1$*. Suppose that the sets* $\bar{\Lambda}_S(k)$ *and* $\bar{\Lambda}_{S,j}(k)$ *are empty, for all vertices* S *and all* $j = 1, \cdots, J_S$*. Then the solution* $(\mathbf{u}, p) \in (\overset{\circ}{H}{}^1(\Omega))^3 \times L^2(\Omega)$ *of the Navier-Stokes system (37) belongs to* $(H^{k+1}(\Omega))^3 \times H^k(\Omega)$*.*

Proof: We simply remark that then α and β may be chosen equal to 0. ∎

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