Centroaffine Surfaces with parallel traceless Cubic Form

Huili Liu* Changping Wang †

Abstract

In this paper, we classify the centroaffine surfaces with parallel cubic Simon form and the centroaffine minimal surfaces with complete positive definite flat metric.

1 Introduction.

Let $x : \mathbf{M} \to \mathbb{R}^3$ be a nondegenerate centroaffine surface. Then x induces a centroaffinely invariant metric g and a so-called induced connection ∇. The difference of the Levi-Civita connection $\widehat{\nabla}$ of g and the induced connection ∇ is a (1,2)-tensor C on $\mathbf{M}$ with the property that its associated cubic form $\widehat{C}$, defined by

$$\widehat{C}(u,v,w) = g(C(u,v),w)), \ \ u,v,w \in TM, \tag{1.1}$$

is totally symmetric. The so-called Tchebychev form is defined by

$$\widehat{T} = \frac{1}{2}\mathrm{trace}_g(\widehat{C}). \tag{1.2}$$

Using $\widehat{C}$ and $\widehat{T}$ one can define a traceless symmetric cubic form $\widetilde{C}$ by

$$\widetilde{C}(u,v,w) = \widehat{C}(u,v,w) - \frac{1}{2}(\widehat{T}(u)g(v,w) + \widehat{T}(v)g(u,w) + \widehat{T}(w)g(u,v)), \tag{1.3}$$

*Supported by the DFG-project "Affine Differential Geometry" at the TU Berlin.
†Partially supported by the DFG-project "Affine Differential Geometry" at the TU Berlin.

Received by the editors March 1996.
Communicated by M. De Wilde.
1991 *Mathematics Subject Classification* : 53 A 15.
Key words and phrases : Tchebychev form, Simon's cubic form, centroaffine minimal surface.

where $u, v, w \in TM$. This cubic form $\widetilde{C}$ was introduced and studied by U. Simon (cf. [15] and [16]) in relative geometry; it extends the Pick form and, in particular, plays an important role in centroaffine geometry. In fact, $\widetilde{C}$ is an analogue of the cubic form in equiaffine geometry: it is totally symmetric and satisfies an apolarity condition. Furthermore, in relative geometry it is independent of the choice of the relative normalizations (cf. [16]). In the case of the equiaffine normalization $\widetilde{C}$ coincides with the cubic form in the equiaffine geometry. For further interesting properties of $\widetilde{C}$ we refer to [16], [10], [11], [9] and [6]. We will call $\widetilde{C}$ cubic Simon form.

Affine hypersurfaces with parallel cubic Pick forms have been intensively studied by Dillen, Li, Magid, Nomizu, Pinkall, Vrancken, Wang and other authors (cf. [12], [13], [1], [2], [3], [17], [4], [18] and [8]). In this paper, we classify all surfaces with parallel cubic Simon form $\widetilde{C}$. We will prove the following theorem in $\mathbb{R}^3$.

Theorem 1: *Let $x : \mathbf{M} \to \mathbb{R}^3$ be a nondegenerate centroaffine surface with the $\widehat{\nabla}$-parallel cubic Simon form. Then x is centroaffinely equivalent to an open part of one of the following surfaces:*

(i) quadrics;

(ii) $x_1^\alpha x_2^\beta x_3^\gamma = 1,\ \alpha\beta\gamma(\alpha+\beta+\gamma) \neq 0;$

(iii) $[\exp(\alpha \arctan \frac{x_1}{x_2})](x_1^2 + x_2^2)^\beta x_3^\gamma = 1,\ \gamma(\gamma + 2\beta)(\alpha^2 + \beta^2) \neq 0;$

(iv) $x_3 = x_1(\alpha \log x_1 + \beta \log x_2),\ \beta(\alpha + \beta) \neq 0;$

where α, β and γ are constants.

Let T be the Tchebychev vector field om $\mathbf{M}$ defined by the equation

$$g(T, v) = \widehat{T}(v),\ v \in TM. \tag{1.4}$$

Then a centroaffine surface $x : \mathbf{M} \to \mathbb{R}^3$ is called centroaffine Tchebychev if

$$\widehat{\nabla} T = \lambda\, \mathrm{id}, \tag{1.5}$$

where λ is a function on $\mathbf{M}$; a centroaffine surface $x : \mathbf{M} \to \mathbb{R}^3$ is called centroaffine minimal if

$$\mathrm{trace}_g(\widehat{\nabla} T) = 0. \tag{1.6}$$

It is proved by the second author in [19] that x is minimal if and only if x is a critical surface of the volume functional of the centroaffine metric g. For a locally strongly convex surface the centroaffine metric is definite. It is positive (or negative) definite if the position vector x points outward (or inward) (cf. [19]). For centroaffine minimal surfaces, we will prove:

Theorem 2: *Let x be a centroaffine minimal surface with complete positive definite flat centroaffine metric g. Then, up to centroaffine transformations in $\mathbb{R}^3$, x is an open part of one of the following surfaces*

(i) $$x_3 = x_1^\alpha x_2^\beta,$$

where (α, β) *is constant in* $\mathbb{R}^+ \times \mathbb{R}$ *or* $\mathbb{R} \times \mathbb{R}^+$ *with* $\alpha\beta(\alpha + \beta - 1) < 0$;

(ii) $$x_3 = [\exp(-\alpha \arctan \frac{x_1}{x_2})](x_1^2 + x_2^2)^\beta,$$

where α *and* β *are constants with* $2\beta > 1$;

(iii) $$x_3 = -x_1(\alpha \log x_1 + \beta \log x_2),$$

where α *and* β *are constants in* $\mathbb{R}$ *with* $\beta(\alpha + \beta) < 0$.

Our main tool is a PDE for the square of the norm of $\widetilde{C}$ which we recently derived in [6].

This paper is organized as follows: In section 2, we prove Theorem 1; in section 3, we prove Theorem 2.

2 Proof of Theorem 1.

Let x be a nondegenerate centroaffine surface with $\widehat{\nabla}\widetilde{C} = 0$. Then by Proposition 4.2.1 of [9] we know that x is a Tchebychev surface. From $\widehat{\nabla}\widetilde{C} = 0$ we get $\|\widetilde{C}\|^2 =$ constant. By [6], 5.2.1.1, we have

(2.1) $$\Delta\|\widetilde{C}\|^2 = 2\|\widehat{\nabla}\widetilde{C}\|^2 + 6\kappa\|\widetilde{C}\|^2.$$

$\widehat{\nabla}\widetilde{C} = 0$ and (2.1) yield $\kappa\|\widetilde{C}\|^2 = 0$. Thus we get either (i) $\widetilde{C} \equiv 0$; or (ii) $\widetilde{C} \not\equiv 0$ but $\|\widetilde{C}\|^2 = 0$; or (iii) $\kappa \equiv 0$.

If (i) is true, we know that x is an open part of a quadric (cf. [16], 7.11, pp. 117).

Next we consider case (ii). In this case, the centroaffine metric g has to be indefinite. So we choose local asymptotic coordinates (u, v) of g with

(2.2) $$g = e^{2\omega}(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u)$$

for some local function ω. We define

(2.3) $$E_1 = e^{-\omega}\frac{\partial}{\partial u},\ E_2 = e^{-\omega}\frac{\partial}{\partial v},\ \theta_1 = e^{\omega}\mathrm{d}u,\ \theta_2 = e^{\omega}\mathrm{d}v.$$

Then for the basis $\{E_1, E_2\}$, the local functions $g_{ij} := g(E_i, E_j)$ are given by

(2.4) $$g_{11} = g_{22} = 0,\ g_{12} = g_{21} = 1.$$

Let $\{\hat{\theta}_{ij}\}$ be the Levi-Civita connection forms of g with respect to $\{E_1, E_2\}$, then

(2.5) $$\mathrm{d}\theta_i = \Sigma_j \hat{\theta}_{ij} \wedge \theta_j,\ \mathrm{d}g_{ij} = g_{ik}\hat{\theta}_{kj} + g_{jk}\hat{\theta}_{ki}.$$

From (2.4) and (2.5) we get

(2.6) $$\hat{\theta}_{12} = \hat{\theta}_{21} = 0,\ \hat{\theta}_{11} = -\hat{\theta}_{22} = \omega_u \mathrm{d}u - \omega_v \mathrm{d}v.$$

Since $\text{trace}_g \widetilde{C} = 0$ and $\widetilde{C}_{ijk} = \widetilde{C}^l_{ij} g_{lk}$ are totally symmetric, we have

$$\widetilde{C}^1_{1j} + \widetilde{C}^2_{2j} = \widetilde{C}_{12j} + \widetilde{C}_{12j} = 2\widetilde{C}_{12j} = 0, \ \ j = 1, 2. \tag{2.7}$$

Therefore

$$\|\widetilde{C}\|^2 = 2\widetilde{C}_{111}\widetilde{C}_{222}.$$

Since $\widetilde{C} \not\equiv 0$ and $\|\widetilde{C}\|^2 = 0$, we may assume that $\widetilde{C}_{111} = 0$ and $\widetilde{C}_{222} \neq 0$. From the fact that $\widehat{\nabla}\widetilde{C} = 0$ we get

$$\mathrm{d}\widetilde{C}_{222} + 3\widetilde{C}_{222}\hat{\theta}_{22} = \Sigma_i \widetilde{C}_{222,i}\theta^i = 0. \tag{2.8}$$

We define

$$\psi := e^{3\omega}\widetilde{C}_{222},$$

then (2.8) is equivalent to

$$\psi_u = 6\omega_u\psi, \ \ \psi_v = 0. \tag{2.9}$$

Since

$$\psi = e^{3\omega}\widetilde{C}_{222} \neq 0,$$

we get from (2.9) that

$$6\omega_{uv} = (\log|\psi|)_{uv} = 0,$$

which implies that the Gauss curvature $\kappa = 0$. Thus case (ii) reduces to case (iii).

For the case (iii), the surface x is flat and Tchebychev. Thus we know by the proof of Theorem 4.2 in [10] that $\widehat{\nabla}T = 0$. By choosing special asymptotic coordinates (u, v) of g we have $\omega = 0$. Then (2.6) implies that $\hat{\theta}_{ij} = 0$. From the fact that $\widehat{\nabla}\widetilde{C} = 0$ we get

$$\mathrm{d}\widetilde{C}_{111} = 0, \ \ \mathrm{d}\widetilde{C}_{222} = 0, \ \ \text{i.e. } \widetilde{C}_{ijk} = \text{constant}. \tag{2.10}$$

Moreover, $\widehat{\nabla}T = 0$, thus we obtain that T_i = constant. From (1.3) we know that $\widehat{C}_{ijk}$ = constant and therefore x is the so-called canonical surface classified in [8]. Thus Theorem 1 follows from [8], Theorem 1.3.

3 Proof of Theorem 2.

Let $x : \mathbf{M} \to \mathbb{R}^3$ be a centroaffine surface with positive definte centroaffine metric g. We introduce a local complex coordinate $z = u + iv$ with respect to g. Then

$$g = \frac{1}{2}e^{2\omega}(\mathrm{d}z \otimes \mathrm{d}\bar{z} + \mathrm{d}\bar{z} \otimes \mathrm{d}z), \tag{3.1}$$

for some local function ω. We define

$$\mathbf{E} = \frac{[x, x_z, x_{z\bar{z}}]}{[x, x_z, x_{\bar{z}}]}\mathrm{d}z := E\mathrm{d}z; \tag{3.2}$$

$$\mathbf{U} = e^{2\omega}\frac{[x, x_z, x_{zz}]}{[x, x_z, x_{\bar{z}}]}\mathrm{d}z^3 := U\mathrm{d}z^3. \tag{3.3}$$

It follows from [10] that $\mathbf{E}$ and $\mathbf{U}$ are globally defined centroaffine invariants. Moreover, $\{g, \mathbf{E}, \mathbf{U}\}$ form a complete system of centroaffine invariants which determines the surface up to centroaffine transformations in $\mathbb{R}^3$. The relations between g, $\mathbf{E}$ and $\mathbf{U}$ are given by (cf. [10], pp. 82-83)

$$2\omega_{z\bar{z}} - |E|^2 + e^{-4\omega}|U|^2 + \frac{1}{2} = 0; \tag{3.4}$$

$$E_{\bar{z}} = \bar{E}_z; \tag{3.5}$$

$$U_{\bar{z}} = e^{2\omega}(E_z - 2\omega_z E). \tag{3.6}$$

Furthermore, let $\{E_1, E_2\}$ be theorthonormal basis for g defined by

$$E_1 = e^{-\omega}\frac{\partial}{\partial u}, \;\; E_2 = e^{-\omega}\frac{\partial}{\partial v} \tag{3.7}$$

and

$$T = T_1 E_1 + T_2 E_2,$$

then

$$e^{-2\omega} E_{\bar{z}} = \frac{1}{4}\mathrm{trace}_g \widehat{\nabla} T. \tag{3.8}$$

Now if $y : \mathbf{M} \to \mathbb{R}^3$ be a centroaffine minimal surface with complete and flat centroaffine metric g_y, then we have a universal Riemannian covering $\pi : \mathbf{C} \to \mathbf{M}$ such that

$$g = \pi^* g_y = \frac{1}{2}(\mathrm{d}z \otimes \mathrm{d}\bar{z} + \mathrm{d}\bar{z} \otimes \mathrm{d}z) \tag{3.9}$$

on $\mathbf{C}$. We consider the centroaffine surface $x = y \circ \pi : \mathbf{C} \to \mathbb{R}^3$ with $x(\mathbf{C}) = y(\mathbf{M}) \in \mathbb{R}^3$. It is clear that x is again a centroaffine minimal surface with centroaffine metric g given by (3.9), i.e. $\omega = 0$. Since x is centroaffine minimal, we have $\mathrm{trace}_g \widehat{\nabla} T = 0$. By (3.8) we get $E_{\bar{z}} = 0$. Thus $E : \mathbf{C} \to \mathbf{C}$ is a holomorphic function. From (3.4) we know that $|E|^2 \geq \frac{1}{2}$. Thus it follows from Picard theorem (cf. [5], pp. 213, Theorem 27.13) that $E =$ constant. Therefore, (3.4) and (3.6) imply that U is holomorphic and $|U|^2 = |E|^2 - \frac{1}{2} =$ constant. So U must be constant. Since from (2.13) of [10] we know that

$$E = \frac{1}{2}(T_1 - iT_2), \;\; U = \frac{1}{4}(\widetilde{C}_{111} + i\widetilde{C}_{222}). \tag{3.10}$$

Hence we get that T_i and $\widetilde{C}_{ijk}$ are constants. Thus x is canonical in the sence of [8]. By the classification theorem 1.3 of [8] and the positive definiteness of the centroaffine metric g we obtain the surfaces in Theorem 2.

This complete the proof of the theorem 2.

Acknowledgements. We would like to thank Professor U. Simon for helpful discussions.

References

[1] F. Dillen and L. Vrancken: *Generalized Cayley surfaces*, Lecture Notes in Math. 1481, Springer, Berlin, 36-47(1991).

[2] F. Dillen and L. Vrancken: *3-dimensional affine hypersurfaces in* $\mathbb{R}^4$ *with parallel cubic form*, Nagoya Math. J., **124**(1991), 41-53.

[3] F. Dillen and L. Vrancken: *Hypersurfaces with parallel difference tensor*, Preprint, KU Leuven 1995.

[4] F. Dillen, L. Vrancken, S. Yaprak: *Affine hypersurfaces with parallel cubic forms*, Nagoya Math. J., **135**(1994), 153-164.

[5] O. Forster: *Lectures on Riemann Surfaces*, Graduate Texts in Mathematics, No. 81, Springer-Verlag, New York, Heidelberg, Berlin, 1981.

[6] A. M. Li, H. L. Liu, A. Schwenk-Schellschmidt, U. Simon, C. P. Wang: *Cubic form methods and relative Tchebychev hypersurfaces*, Geometriae, Dedicata, to appear.

[7] A. M. Li, U. Simon and G. Zhao: *Global Affine Differential Geometry of Hypersurfaces*, W. De Gruyter, Berlin-New York, 1993.

[8] A. M. Li and C. P. Wang: *Canonical centroaffine hypersurfaces in* R^{n+1}, Results in Mathematics, **20**(1991) 660-681.

[9] H. L. Liu, U. Simon, C. P. Wang: *Conformal structures in affine geometry: complete Tchebychev hypersurfaces*, Abh. Math. Sem. Hamburg, **66**(1996), 249-262.

[10] H. L. Liu and C. P. Wang: *The centroaffine Tchebychev operator*, Results in Mathematics, **27**(1995), 77-92.

[11] H. L. Liu and C. P. Wang: *Relative Tchebychev surfaces in* $\mathbb{R}^3$, Kyushu J. of Math., **50**(1996), 533-540.

[12] M. Magid, M. Nomizu: *On affine surfaces whose cubic forms are parallel relative to the affine metric*, Proc. Japan Acad. 65, Ser. A, 215-218(1989).

[13] K. Nomizu, U. Pinkall: *Cubic form theorem for affine immersions*, Results in Math., **13**(1988), 338-362.

[14] K. Nomizu and T. Sasaki: *Affine Differential Geometry*, Cambridge University Press, 1994.

[15] U. Simon: *Zur Relativgeometrie: Symmetrische Zusammenhänge auf Hyperflächen*, Math. Z., **106**(1968), 36-46.

[16] U. Simon, A. Schwenk-Schellschmidt and H. Viesel: *Introduction to the Affine Differential Geometry of Hypersurfaces*, Lecture Notes, Science University Tokyo, ISBN 3-7983-1529-9, 1991.

[17] L. Vrancken: *Affine higher order parallel hypersurfaces*, Ann. Fac. Sci. Toulouse, V. Ser., Math. 9, No. 3, 341-353(1988).

[18] C. P. Wang: *Canonical equiaffine hypersurfaces in* $\mathbf{R}^{n+1}$, Math. Z., **214**(1993), 579-592.

[19] C. P. Wang: *Centroaffine minimal hypersurfaces in* $\mathbb{R}^{n+1}$, Geom. Dedicata, **51**(1994), 63-74

Huili Liu
Department of Mathematics
Northeastern University
Shenyang 110006
P. R. China

Changping Wang
Department of Mathematics
Beijing University
Beijing 100871
P. R. China
E-mail: wang@sxx0.math.pku.edu.cn