New Approach for the Submanifolds of the Euclidean Space

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Abstract

In this paper are introduced principal directions and principal normal vectors of arbitrary n-dimensional submanifolds of the Euclidean space R^{n+k}. The canonical normal vectors and curvatures for the submanifolds of the Euclidean spaces are introduced as for the curves.

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1 Introduction

In the theory of the hypersurfaces are introduced the principal directions and principal curvatures [2]. It is natural to ask why it is not done for n dimensional surfaces imbedded in R^{n+k} for $k > 1$? In special case, when $n = 1$, it is introduced canonical orthonormal frame of the curve in the space and also the corresponding scalar curvature, connected with the Frenet equations [1]. It is also natural to ask why it is not done for the manifolds of arbitrary dimension n? Indeed we have two special cases: (i) $k = 1$ and (ii) $n = 1$. In this paper is considered the general case, introducing the principal directions and so called principal normal vectors of any manifold imbedded in R^{n+k}, and also are introduced canonical k normal vector fields and the corresponding curvatures, which do not depend on the choice of the parametrization of the surface.

2 Notations and basic results

Let M be an n-dimensional submanifold of R^{n+k} and let P be an arbitrary point of M. Because the examination of the principal curvatures and directions has a local character, we will consider only a neighborhood U of the point P instead of the whole manifold M. The Euclidean norm of $\mathbf{z}$ will be denoted by $||\mathbf{z}||$. Let $\epsilon > 0$. For each

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point $S \in U$ determined by a radius-vector $\mathbf{x}$, let N_S be the following subset of the orthogonal subspace of the tangent space at S,

$$N_S = \{\mathbf{y} \in R^{n+k} \mid \mathbf{y} - \mathbf{x} \text{ is vector orthogonal to } T_S(M) \text{ and } \|\mathbf{y} - \mathbf{x}\| < \epsilon\}.$$

If ϵ is sufficiently small, then $S_1 \neq S_2$ if and only if N_{S_1} and N_{S_2} do not intersect and moreover $\cup_{S \in U} N_S$ is an open subset of R^{n+k} of dimension $n+k$. Since N_{S_1} and N_{S_2} do not intersect for $S_1 \neq S_2$, it allows us uniquely to transport parallel on N_S all the vectors defined at S, according to the Euclidean flat metric in R^{n+k}. Thus for each vector field defined on U we have uniquely defined vector field on $\cup_{S \in U} N_S$, and moreover

$$\nabla_{\mathbf{N}} \equiv 0, \tag{2.1}$$

for each orthogonal vector $\mathbf{N}$, where ∇ denotes the covariant differentiation with respect to the Euclidean metric. Usually, by $\mathbf{N}$ we will denote a unit vector (field) which is orthogonal to the basic manifold.

For arbitrary vector field $\mathbf{X}$, (2.1) implies

$$[\mathbf{N}, \mathbf{X}] = \nabla_{\mathbf{N}} \mathbf{X} - \nabla_{\mathbf{X}} \mathbf{N} = -\nabla_{\mathbf{X}} \mathbf{N} \tag{2.2}$$

and hence $\mathbf{N} \perp [\mathbf{N}, \mathbf{X}]$ because

$$\mathbf{N} \cdot [\mathbf{N}, \mathbf{X}] = 0, \tag{2.3}$$

since $\|\mathbf{N}\| = 1$. Let φ be the linear mapping defined by

$$\varphi(\mathbf{X}) = [\mathbf{N}, \mathbf{X}]. \tag{2.4}$$

Lemma 2.1. *For each tangent vector fields* $\mathbf{X}$ *and* $\mathbf{Y}$, *it holds*

$$\varphi(\mathbf{X}) \cdot \mathbf{Y} = \varphi(\mathbf{Y}) \cdot \mathbf{X}. \tag{2.5}$$

Proof. $\varphi(\mathbf{X}) \cdot \mathbf{Y} - \varphi(\mathbf{Y}) \cdot \mathbf{X} = [\mathbf{N}, \mathbf{X}] \cdot \mathbf{Y} - [\mathbf{N}, \mathbf{Y}] \cdot \mathbf{X} = -\mathbf{Y} \cdot \nabla_{\mathbf{X}} \mathbf{N} +$

$$+\mathbf{X} \cdot \nabla_{\mathbf{Y}} \mathbf{N} = \mathbf{N} \cdot \nabla_{\mathbf{X}} \mathbf{Y} - \mathbf{N} \cdot \nabla_{\mathbf{Y}} \mathbf{X} = \mathbf{N} \cdot (\nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X}) = \mathbf{N} \cdot [\mathbf{X}, \mathbf{Y}] = 0.$$

In the special case when $k = 1$, it follows from (2.3) that $[\mathbf{N}, \mathbf{X}]$ is a tangent vector on U, and according to (2.2) and (2.4) we have $\varphi(\mathbf{X}) = -\nabla_{\mathbf{X}} \mathbf{N}$, and φ coincides with the mapping $A : T(U) \to T(U)$ whose eigenvalues are principal curvatures and the eigenvectors determine the principal directions [2]. Moreover, (2.5) shows that $\varphi = A$ is symmetric operator, and hence the principal curvatures are real numbers and the principal directions are orthogonal.

3 Introduction of principal directions and principal normal vectors in general case

Let $\mathbf{N}_1, \ldots, \mathbf{N}_k$ be arbitrary k orthonormal vector fields on U, i.e. $\mathbf{N}_\alpha \cdot \mathbf{N}_\beta = \delta_{\alpha\beta}$ for $\alpha, \beta \in \{1, \ldots, k\}$. For each tangent vector fields $\mathbf{X}_1$ and $\mathbf{X}_2$ we define a normal vector field $\mathbf{N}(\mathbf{X}_1, \mathbf{X}_2)$ as follows

$$\mathbf{N}(\mathbf{X}_1,\mathbf{X}_2)=\sum_{\alpha=1}^{k}\mathbf{N}_\alpha(\mathbf{X}_1\cdot[\mathbf{N}_\alpha,\mathbf{X}_2]). \tag{3.1}$$

First we prove that $\mathbf{N}(\mathbf{X}_1,\mathbf{X}_2)$ is well defined. Let $\mathbf{N}'_\alpha=\sum\limits_{\beta=1}^{k}P_{\alpha\beta}\mathbf{N}_\beta$ be another system of orthonormal vectors, such that P is orthogonal $k\times k$ matrix. Then

$$\mathbf{X}_1\cdot[\mathbf{N}'_\alpha,\mathbf{X}_2]=\mathbf{X}_1\cdot[\sum_{\beta=1}^{k}P_{\alpha\beta}\mathbf{N}_\beta,\mathbf{X}_2]=\sum_{\beta=1}^{k}P_{\alpha\beta}\mathbf{X}_1\cdot[\mathbf{N}_\beta,\mathbf{X}_2]-$$

$$-\sum_{\beta=1}^{k}\mathbf{X}_1\cdot(\mathbf{N}_\beta\cdot\mathbf{X}_2(P_{\alpha\beta}))=\sum_{\beta=1}^{k}P_{\alpha\beta}\mathbf{X}_1\cdot[\mathbf{N}_\beta,\mathbf{X}_2],$$

and

$$\mathbf{N}'(\mathbf{X}_1,\mathbf{X}_2)=\sum_{\alpha=1}^{k}\sum_{\delta=1}^{k}P_{\alpha\delta}\mathbf{N}_\delta\cdot\sum_{\beta=1}^{k}P_{\alpha\beta}(\mathbf{X}_1\cdot[\mathbf{N}_\beta,\mathbf{X}_2])=$$

$$=\sum_{\alpha=1}^{k}\mathbf{N}_\alpha(\mathbf{X}_1\cdot[\mathbf{N}_\alpha,\mathbf{X}_2])=\mathbf{N}(\mathbf{X}_1,\mathbf{X}_2).$$

The vector field $\mathbf{N}(\mathbf{X}_1,\mathbf{X}_2)$ has the following properties:

i) $\mathbf{N}(\mathbf{X}_1,\mathbf{X}_2)=\mathbf{N}(\mathbf{X}_2,\mathbf{X}_1)$,

ii) $\mathbf{N}(\mathbf{X}'_1+\mathbf{X}''_1,\mathbf{X}_2)=\mathbf{N}(\mathbf{X}'_1,\mathbf{X}_2)+\mathbf{N}(\mathbf{X}''{}_1,\mathbf{X}_2)$,
$\mathbf{N}(\mathbf{X}_1,\mathbf{X}'_2+\mathbf{X}''_2)=\mathbf{N}(\mathbf{X}_1,\mathbf{X}'_2)+\mathbf{N}(\mathbf{X}_1,\mathbf{X}''_2)$,

iii) $\mathbf{N}(f\mathbf{X}_1,\mathbf{X}_2)=f\mathbf{N}(\mathbf{X}_1,\mathbf{X}_2)$
$\mathbf{N}(\mathbf{X}_1,f\mathbf{X}_2)=f\mathbf{N}(\mathbf{X}_1,\mathbf{X}_2)$,

where $f\in C^1(U)$. The proof is trivial.

Further, for each vector field $\mathbf{N}$ we define a mapping $h_\mathbf{N}:\chi(U)\to\chi(U)$ by

$$h_\mathbf{N}(\mathbf{X})=\sum_{i=1}^{n}\mathbf{Y}_i(\mathbf{Y}_i\cdot[\mathbf{N},\mathbf{X}]) \tag{3.2}$$

where $\mathbf{Y}_1,\dots,\mathbf{Y}_n$ is orthonormal basis of tangent vectors. Obviously, $h_\mathbf{N}(\mathbf{X})$ is the projection of the vector $[\mathbf{N},\mathbf{X}]$ on the tangent space. Hence for arbitrary tangent vector field $\mathbf{Z}$, we have

$$h_\mathbf{N}(\mathbf{X})\cdot\mathbf{Z}=\mathbf{Z}\cdot[\mathbf{N},\mathbf{X}].$$

Similarly, it holds

$$h_\mathbf{N}(\mathbf{Z})\cdot\mathbf{X}=\mathbf{X}\cdot[\mathbf{N},\mathbf{Z}].$$

Hence,

$$h_\mathbf{N}(\mathbf{X})\cdot\mathbf{Z}=h_\mathbf{N}(\mathbf{Z})\cdot\mathbf{X} \tag{3.3}$$

which means that $h_\mathbf{N}$ is a symmetric tensor filed on U. Thus all eigenvalues are real numbers and the eigenvectors are orthogonal.

Further, the vector $\mathbf{N}=\mathbf{N}(\mathbf{X}_1,\mathbf{X}_2)$ can be substituted in (3.2). Hence we obtain a mapping $H(\mathbf{X}_1,\mathbf{X}_2):\chi(U)\to\chi(U)$, defined by

$$H(\mathbf{X}_1,\mathbf{X}_2)=h_{\mathbf{N}(\mathbf{X}_1,\mathbf{X}_2)},$$

i.e.,

$$H(\mathbf{X}_1,\mathbf{X}_2)\mathbf{X}=\sum_{i=1}^{n}\mathbf{Y}_i(\mathbf{Y}_i\cdot[\sum_{\alpha=1}^{k}\mathbf{N}_\alpha(\mathbf{X}_1\cdot[\mathbf{N}_\alpha,\mathbf{X}_2]),\mathbf{X}]). \tag{3.4}$$

Since $\mathbf{N}_\alpha\cdot\mathbf{X}(\mathbf{X}_1\cdot[\mathbf{N}_\alpha,\mathbf{X}_2])$ is orthogonal to $\mathbf{Y}_i$, from (3.4) we obtain

$$H(\mathbf{X}_1,\mathbf{X}_2)\mathbf{X}=\sum_{\alpha=1}^{k}\sum_{i=1}^{n}\mathbf{Y}_i\{(\mathbf{Y}_i\cdot[\mathbf{N}_\alpha,\mathbf{X}])(\mathbf{X}_1\cdot[\mathbf{N}_\alpha,\mathbf{X}_2])\}. \tag{3.5}$$

Now let us put $H(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4)=\mathbf{X}_4\cdot H(\mathbf{X}_1,\mathbf{X}_2)\mathbf{X}_3$. Then

$$H(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4)=\sum_{\alpha=1}^{k}\sum_{i=1}^{n}(\mathbf{Y}_i\cdot\mathbf{X}_4)(\mathbf{Y}_i\cdot[\mathbf{N}_\alpha,\mathbf{X}_3])(\mathbf{X}_1\cdot[\mathbf{N}_\alpha,\mathbf{X}_2]).$$

Since the left side of this equality is independent of the choice of the orthonormal basis $\mathbf{Y}_1,\cdots,\mathbf{Y}_n$, we can assume that $\mathbf{Y}_1=\mathbf{X}_4/\|\mathbf{X}_4\|$. Then $\mathbf{Y}_i\cdot\mathbf{X}_4=0$ for $i>1$, and we obtain the identity

$$H(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4)=\sum_{\alpha=1}^{k}(\mathbf{X}_4\cdot[\mathbf{N}_\alpha,\mathbf{X}_3])(\mathbf{X}_1\cdot[\mathbf{N}_\alpha,\mathbf{X}_2]), \tag{3.6}$$

and the following properties follow immediately.

$$i)\qquad H(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4)=H(\mathbf{X}_2,\mathbf{X}_1,\mathbf{X}_3,\mathbf{X}_4),$$

$$ii)\qquad H(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4)=H(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_4,\mathbf{X}_3),$$

$$iii)\qquad H(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4)=H(\mathbf{X}_3,\mathbf{X}_4,\mathbf{X}_1,\mathbf{X}_2).$$

Further we will prove another property. Let $\mathbf{X}_3$ and $\mathbf{X}_4$ be fixed vectors,

$$S_{ij}=H(\mathbf{Y}_i,\mathbf{Y}_j,\mathbf{X}_3,\mathbf{X}_4)=\sum_{\alpha=1}^{k}(\mathbf{Y}_i\cdot[\mathbf{N}_\alpha,\mathbf{Y}_j])(\mathbf{X}_3\cdot[\mathbf{N}_\alpha,\mathbf{X}_4]),$$

and let $\mathbf{Y}'_i=\sum_{r=1}^{k}\alpha_{ir}\mathbf{Y}_r$ be another orthonormal basis. Then

$$\mathbf{Y}'_i\cdot[\mathbf{N}_\alpha,\mathbf{Y}'_j]=\sum_{r=1}^{k}\alpha_{ir}\mathbf{Y}_r\cdot[\mathbf{N}_\alpha,\sum_{s=1}^{k}\alpha_{js}\mathbf{Y}_s]=\sum_{r=1}^{k}\sum_{s=1}^{k}\alpha_{ir}\mathbf{Y}_r\cdot[\mathbf{N}_\alpha,\mathbf{Y}_s]\alpha^t_{sj}.$$

But the matrix $A=(\alpha_{ij})$ is an orthogonal matrix, and hence we obtain the following transformation law

$$S'=A\cdot S\cdot A^{-1}.$$

Thus the eigenvalues of the matrix S_{ij} are invariant of the basis $\mathbf{Y}_1,\ldots,\mathbf{Y}_n$. Also, if $(\lambda_1,\ldots,\lambda_n)$ is an arbitrary eigenvector of the matrix S_{ij}, then the vector $\lambda_1\mathbf{Y}_1+\ldots+\lambda_n\mathbf{Y}_n$ is eigenvector for the corresponding (1.1) tensor field S^i_j, and it does not depend on the basis $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$. These eigenvalues and eigenvectors depend on the choice of the vectors $\mathbf{X}_3$ and $\mathbf{X}_4$.

In order to introduce principal directions which are independent from any choice of vectors (like $\mathbf{X}_3$ and $\mathbf{X}_4$), analogously to S_{ij} we define tensor field T_{ij} as follows

$$T_{ij} = \sum_{\alpha=1}^{k} \sum_{r=1}^{n} (\mathbf{Y}_i \cdot [\mathbf{N}_\alpha, \mathbf{Y}_r])(\mathbf{Y}_r \cdot [\mathbf{N}_\alpha, \mathbf{Y}_j]). \tag{3.7}$$

Analogously as for S_{ij} this tensor is also symmetric tensor field invariant of the orthonormal basis $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$. Also, if $(\lambda_1, \ldots, \lambda_n)$ is an arbitrary eigenvector of the matrix T_{ij}, then the vector $\lambda_1 \mathbf{Y}_1 + \ldots + \lambda_n \mathbf{Y}_n$ is eigenvector for the corresponding (1.1) tensor field T^i_j, and it does not depend on the basis $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ or $\mathbf{N}_1, \cdots, \mathbf{N}_k$. The eigenvectors are orthogonal and they determine the *principal directions*. Its eigenvalues are real numbers and moreover they are non-negative according to the following lemma.

Lemma 3.1. *Let $(A_1, A_2, \ldots, A_n), (B_1, B_2, \ldots, B_n), \ldots, (C_1, C_2, \ldots, C_n)$ are arbitrary vectors. Then the eigenvalues of the following matrix*

$$S_{ij} = A_i A_j + B_i B_j + \ldots + C_i C_j \qquad (1 \le i, j \le n)$$

are non-negative real numbers.

Indeed these eigenvalues are squares of the "principal curvatures" which we would like to define. They are determined up to the sign. The following discussion shows why it is impossible to determine the n-typle $(\lambda_1, \cdots, \lambda_n)$ of "principal curvatures" up to the sign, as it is done for the special case $k = 1$. If it is possible, then it is natural to suppose that λ_i and λ_j have the same signs if and only if the sectional curvature of the plane determined by the corresponding eigenvectors (i.e. principal directions) is positive. This may not be satisfied always because if $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ are three eigenvectors, then $K(\mathbf{X}, \mathbf{Y}) \cdot K(\mathbf{Y}, \mathbf{Z}) \cdot K(\mathbf{Z}, \mathbf{X})$ is not always positive for $k > 1$, where $K(\mathbf{A}, \mathbf{B})$ denotes the sectional curvature for the plane generated by $\mathbf{A}$ and $\mathbf{B}$. This problem will be solved by introducing principal normal vectors instead of the principal curvatures.

Suppose that $\mathbf{Z}_1, \cdots, \mathbf{Z}_n$ are orthonormal vectors which determine the principal directions. Then we define the corresponding *principal normal vectors* $\mathbf{U}_i (1 \le i \le n)$ as

$$\mathbf{U}_i = \sum_{\alpha=1}^{k} (\mathbf{Z}_i \cdot [\mathbf{N}_\alpha, \mathbf{Z}_i]) \mathbf{N}_\alpha. \tag{3.8}$$

and also define vectors

$$\mathbf{U}_{ij} = \sum_{\alpha=1}^{k} (\mathbf{Z}_i \cdot [\mathbf{N}_\alpha, \mathbf{Z}_j]) \mathbf{N}_\alpha \tag{3.9}$$

such that $\mathbf{U}_i = \mathbf{U}_{ii}$. From the definition (3.7) it follows that these vectors are canonical. Indeed, it verifies easily that they do not depend on the choice of the orthonormal system $\{\mathbf{N}_\alpha\}$ and do not depend on the signs of the vectors $\mathbf{Z}_i$. Now we have the following proposition.

Proposition 3.2. *The sums* $\sum_{j=1}^{n} \|\mathbf{U}_{ij}\|^2$ *for* $i \in \{1, \cdots, n\}$ *are the eigenvalues of* T.

Proof. If we put $\mathbf{Y}_i = \mathbf{Z}_i$ for $i \in \{1, \cdots, n\}$, then the following matrix

$$\sum_{\alpha=1}^{k} \sum_{r=1}^{n} (\mathbf{Z}_i \cdot [\mathbf{N}_\alpha, \mathbf{Z}_r])(\mathbf{Z}_r \cdot [\mathbf{N}_\alpha, \mathbf{Z}_j]) \tag{3.10}$$

similar to T, has the same eigenvalues. This is a diagonal matrix by construction, and hence the eigenvalues of T are

$$\sum_{\alpha=1}^{k} \sum_{r=1}^{n} (\mathbf{Z}_i \cdot [\mathbf{N}_\alpha, \mathbf{Z}_r])(\mathbf{Z}_r \cdot [\mathbf{N}_\alpha, \mathbf{Z}_i]) = \sum_{r=1}^{n} \|\mathbf{U}_{ir}\|^2 \qquad (1 \le i \le n).$$

Note that the matrix (3.10) is diagonal and hence

$$\sum_{\alpha=1}^{k} \sum_{r=1}^{n} (\mathbf{Z}_i \cdot [\mathbf{N}_\alpha, \mathbf{Z}_r])(\mathbf{Z}_r \cdot [\mathbf{N}_\alpha, \mathbf{Z}_j]) = 0 \qquad (i \ne j). \tag{3.11}$$

It is of interest to consider surfaces such that $(\mathbf{Z}_i \cdot [\mathbf{N}_\alpha, \mathbf{Z}_j]) = 0$ for $i \ne$j. We will consider the special case $k = 1$. Let denote $\mathbf{N}_1 = \mathbf{N}$. Now

$$T(\mathbf{X}, \mathbf{Y}) = (\mathbf{X} \cdot [\mathbf{N}, \mathbf{Y}])(\mathbf{X} \cdot [\mathbf{N}, \mathbf{Y}]) = A(\mathbf{X}, \mathbf{Y}) A(\mathbf{X}, \mathbf{Y}), \tag{3.12}$$

where $A(\mathbf{X}, \mathbf{Y}) = (\mathbf{X} \cdot [\mathbf{N}, \mathbf{Y}])$ and it is symmetric tensor field. The eigenvalues of the tensor A are principal curvatures according to the classical definition. Let us suppose that the symmetric matrix A has different eigenvalues. Indeed, it is sufficient the eigenvalues of T to be different. Then $A(\mathbf{Z}_i, \mathbf{Z}_j)$ must be diagonal matrix, because $T(\mathbf{Z}_i, \mathbf{Z}_j)$ is a diagonal matrix. Hence

$$\mathbf{Z}_i \cdot [\mathbf{N}, \mathbf{Z}_j] = 0 \qquad (i \ne j). \tag{3.13}$$

The principal directions defined by A and by T are the same while the eigenvalues of T are squares of the eigenvalues of A. Note that the classical definition of the principal curvatures determines them up to the sign, i.e. it depends of the sign of the unit vector $\mathbf{N}$, but now the principal vectors

$$\mathbf{U}_i = (\mathbf{Z}_i \cdot [\mathbf{N}, \mathbf{Z}_i])\mathbf{N} \qquad (1 \le i \le n)$$

are uniquely defined, $\mathbf{U}_{ij} = 0$ according to (3.13) and $\|\mathbf{U}\|^2$ $(1 \le i \le n)$ are the squares of the principal curvatures.

Some further research about principal directions will be done in another paper.

4 Introduction of canonical normal vectors and normal curvatures

Let us define the following $k \times k$ matrix

$$P^{(1)}_{\alpha\beta} = \sum_{i=1}^{n}\sum_{j=1}^{n}(\mathbf{Y}_i \cdot [\mathbf{N}_\alpha, \mathbf{Y}_j])(\mathbf{Y}_i \cdot [\mathbf{N}_\beta, \mathbf{Y}_j]). \tag{4.1}$$

It verifies that $P^{(1)}_{\alpha\beta}$ do not depend on the choice of the basis $\{\mathbf{Y}_i\}$, such that the matrix $P^{(1)}$ is well defined. It depends on the choice of the orthonormal basis $\{\mathbf{N}_\alpha\}$ and the matrix $P^{(1)}$ is symmetric. Its eigenvalues are non-negative (see the lemma in section 3) real numbers and the eigenvectors are orthogonal. Let $k_1 = rank(P^{(1)}_{\alpha\beta})$, such that that there exist k_1 non-zero eigenvectors $\mathbf{N}^{(1)}_1, \cdots, \mathbf{N}^{(1)}_{k_1}$ and corresponding k_1 positive eigenvalues $\lambda^{(1)}_1, \cdots, \lambda^{(1)}_{k_1}$. Indeed, if $(\lambda_1, \cdots, \lambda_k)$ is an eigenvector, then indeed it is the vector $\lambda_1\mathbf{N}_1 + \cdots + \lambda_k\mathbf{N}_k$. According to this identification similarly as in the section 3, it verifies that these eigenvectors and eigenvalues do not depend on the choice of the basis $\{\mathbf{N}_\alpha\}$.

The geometrical interpretation of the vectors $\mathbf{N}^{(1)}_1, \ldots, \mathbf{N}^{(1)}_{k_1}$ follows. If $P^{(1)}$ is zero matrix at each point, then it is easy to verify that locally M is affine subspace of R^{n+k}. So suppose that $k_1 > 0$. Then the vectors $\mathbf{Y}_1, \cdots, \mathbf{Y}_n, \mathbf{N}_1, \cdots, \mathbf{N}_{k_1}$ generate the osculating space at the considered point. The positive scalars $\lambda^{(1)}_1, \cdots, \lambda^{(1)}_{k_1}$ we define to be the squares of the *first normal curvatures* and the corresponding eigenvectors $\mathbf{N}^{(1)}_1, \cdots, \mathbf{N}^{(1)}_{k_1}$ we define to be the *first normal vectors.*

Since the matrix $P^{(1)}$ is a sum of n^2 matrices of the special form "$C_\alpha C_\beta$", it follows that

$$k_1 \le \min(n^2, k). \tag{4.2}$$

Specially, if $n = 1$ then $k_1 = 1$ or $k_1 = 0$, the eigenvector is

$$\mathbf{N}^{(1)}_1 = (\mathbf{Y}\cdot[\mathbf{N}_1, \mathbf{Y}])\mathbf{N}_1 + \cdots + (\mathbf{Y}\cdot[\mathbf{N}_k, \mathbf{Y}])\mathbf{N}_k \tag{4.3}$$

and $(\mathbf{Y}\cdot[\mathbf{N}^{(1)}_1, \mathbf{Y}])(\mathbf{Y}\cdot[\mathbf{N}^{(1)}_1, \mathbf{Y}])$ is the square of the first curvature.

Note that the vectors $\mathbf{N}^{(1)}_1, \cdots, \mathbf{N}^{(1)}_{k_1}$ are uniquely determined (up to permutation) if and only if the eigenvalues $\lambda^{(1)}_1, \cdots, \lambda^{(1)}_{k_1}$ are different numbers. If some of the eigenvalues are equal, then instead of normal vectors we have normal subspace. For example let us consider a 2-dimensional Euclidean subspace of R^4. In this case it is not possible to distinguished two normal vectors, but we have only normal space.

Now we are going to give the second step. Without loss of generality we suppose that $\mathbf{N}_1 = \mathbf{N}^{(1)}_1, \cdots, \mathbf{N}_{k_1} = \mathbf{N}^{(1)}_{k_1}$. Hence $\mathbf{N}_i \perp \mathbf{N}^{(1)}_j$ for $i > k_1$ and $j \le k_1$. Let us write temporary $\mathbf{Y}_{n+1} = \mathbf{N}_1, \cdots, \mathbf{Y}_{n+k_1} = \mathbf{N}_{k_1}$. Now we define the second $(k-k_1)\times(k-k_1)$ matrix

$$P^{(2)}_{\alpha\beta} = \sum_{i=1}^{n+k_1}\sum_{j=1}^{n+k_1}(\mathbf{Y}_i \cdot [\mathbf{N}_\alpha, \mathbf{Y}_j])(\mathbf{Y}_i \cdot [\mathbf{N}_\beta, \mathbf{Y}_j]). \tag{4.4}$$

for $\alpha, \beta \in \{k_1 + 1, \ldots, k\}$. Since $[\mathbf{N}_\alpha, \mathbf{N}_\beta] = 0$ and according to the choice of the vectors $\mathbf{N}^{(1)}_1, \cdots, \mathbf{N}^{(1)}_{k_1}$, (4.4) implies

$$P^{(2)}_{\alpha\beta} = \sum_{i=1}^{k_1}\sum_{j=1}^{n}(\mathbf{N}^{(1)}_i \cdot [\mathbf{N}_\alpha, \mathbf{Y}_j])(\mathbf{N}^{(1)}_i \cdot [\mathbf{N}_\beta, \mathbf{Y}_j]). \tag{4.5}$$

Similarly to (4.2) now we have

$$k_2 = \operatorname{rank}(P^{(2)}_{\alpha\beta}) \leq \min(nk_1, k-k_1). \tag{4.6}$$

It can be verified that if $k_2 \equiv 0$, i.e. $P^{(2)}_{\alpha\beta} \equiv 0$, then the manifold can locally be imbedded in $n+k_1$ dimensional affine subspace of R^{n+k}. So, suppose that $k_2 > 0$. Let $(\lambda_1, \cdots, \lambda_{k-k_1})$ be an eigenvector of the matrix $P^{(2)}_{\alpha\beta}$, then we consider the following vector $\lambda_1 \mathbf{N}_{k_1+1} + \lambda_2 \mathbf{N}_{k_1+2} + \cdots + \lambda_{k-k_1}\mathbf{N}_k$ as an eigenvector. With respect to this identification, all the eigenvectors of $P^{(2)}_{\alpha\beta}$ do not depend on the choice of the basis $\{N_\alpha\}$, and also the eigenvalues do not depend on the basis $\{N_\alpha\}$. The eigenvectors $\mathbf{N}^{(2)}_1, \cdots, \mathbf{N}^{(2)}_{k_2}$ of $P^{(2)}_{\alpha\beta}$ have the following geometrical meaning. The vectors $\mathbf{Y}_1, \cdots, \mathbf{Y}_n, \mathbf{N}^{(1)}_1, \cdots, \mathbf{N}^{(1)}_{k_1}, \mathbf{N}^{(2)}_1, \cdots, \mathbf{N}^{(2)}_{k_2}$ generate the osculated space of second order at the considered point. The eigenvalues $\lambda^{(2)}_1, \cdots, \lambda^{(2)}_{k_2}$ we define to be the squares of the *second normal curvatures* and the corresponding eigenvectors $\mathbf{N}^{(2)}_1, \cdots, \mathbf{N}^{(2)}_{k_2}$ we define to be the *second normal vectors.*

Specially, if $n = 1$ then $k_2 = 1$ or $k_2 = 0$, the eigenvector is

$$\mathbf{N}^{(2)}_1 = (\mathbf{N}^{(1)}_1 \cdot [\mathbf{N}_{k_1+1}, \mathbf{Y}])\mathbf{N}_{k_1+1} + \cdots + (\mathbf{N}^{(1)}_1 \cdot [\mathbf{N}_k, \mathbf{Y}])\mathbf{N}_k \tag{4.7}$$

and $(\mathbf{N}^{(1)}_1 \cdot [\mathbf{N}^{(2)}_1, \mathbf{Y}])(\mathbf{N}^{(1)}_1 \cdot [\mathbf{N}^{(2)}_1, \mathbf{Y}])$ is the square of the second curvature.

Note that the vectors $\mathbf{N}^{(2)}_1, \ldots, \mathbf{N}^{(2)}_{k_2}$ are uniquely determined (up to permutation) if and only if the eigenvalues $\lambda^{(2)}_1, \ldots, \lambda^{(2)}_{k_2}$ are different numbers.

In order to give the third step, without loss of generality we suppose that $\mathbf{N}_{k_1+1} = \mathbf{N}^{(2)}_1, \cdots, \mathbf{N}_{k_1+k_2} = \mathbf{N}^{(2)}_{k_2}$. Hence $\mathbf{N}_i \perp \mathbf{N}^{(2)}_j$ for $i > k_1 + k_2$ and $j \in \{1, \ldots, k_2\}$. Analogously to (4.5) the third $(k-k_1-k_2) \times (k-k_1-k_2)$ matrix can be formed and the procedure can be continued. We give only the recurrent formulas

$$P^{(s+1)}_{\alpha\beta} = \sum_{i=1}^{k_s} \sum_{j=1}^{n} (\mathbf{N}^{(s)}_i \cdot [\mathbf{N}_\alpha, \mathbf{Y}_j])(\mathbf{N}^{(s)}_i \cdot [\mathbf{N}_\beta, \mathbf{Y}_j]) \tag{4.8}$$

and

$$k_{s+1} = rank(P^{(s)}_{\alpha\beta}) \leq \min(nk_s, k - k_1 - \ldots - k_s), \tag{4.9}$$

which are analogous to (4.5) and (4.6). From (4.9) and (4.2) it follows the following inequality

$$k_s \leq n^{s+1}. \tag{4.10}$$

Thus finally we obtain canonical orthogonal vector fields

$$\mathbf{N}^{(1)}_1, \ldots, \mathbf{N}^{(1)}_{k_1}, \mathbf{N}^{(2)}_1, \ldots, \mathbf{N}^{(2)}_{k_2}, ..., \mathbf{N}^{(r)}_1, \ldots, \mathbf{N}^{(r)}_{k_r}$$

and canonical squares

$$\lambda^{(1)}_1, \ldots, \lambda^{(1)}_{k_1}, \lambda^{(2)}_1, \ldots, \lambda^{(2)}_{k_2}, ..., \lambda^{(r)}_1, \ldots, \lambda^{(r)}_{k_r}$$

of the normal curvatures, such that $k_1 + \cdots + k_r = k$. The normal curvatures are determined up to the sign. Indeed they are unique determined if we suppose that they are non-negative. The orthogonal vectors $\mathbf{N}_1^{(1)}, \cdots, \mathbf{N}_{k_r}^{(r)}$ can be chosen as unit vectors.

We conclude this section considering the special case $n = 1$ and $k = 2$. This example also shows the practical method of calculation.

Example. Let us consider the helix $x = a\cos t$, $y = a\sin t$, $z = bt$. In this case it is well known that $k_1 = a/(a^2+b^2)$ and $k_2 = b/(a^2+b^2)$. We use the orthonormal system of vectors

$$\mathbf{Y} = (-a\sin t, a\cos t, b)\cdot(a^2+b^2)^{-1/2},$$

$$\mathbf{N}_1 = (-\cos t, -\sin t, 0),$$

$$\mathbf{N}_2 = (b\sin t, -b\cos t, a)\cdot(a^2+b^2)^{-1/2}.$$

In general case, the normal plane at the point $(x_0, y_0, z_0) = (x(t_0), y(t_0), z(t_0))$ is

$$x_0'(X - x_0) + y_0'(Y - y_0) + z_0'(Z - z_0) = 0. \tag{4.11}$$

In this case it holds

$$\frac{\partial}{\partial X} = \frac{1}{\frac{dX}{dt}}\frac{d}{dt}, \qquad \frac{\partial}{\partial Y} = \frac{1}{\frac{dY}{dt}}\frac{d}{dt} \qquad \text{and} \qquad \frac{\partial}{\partial Z} = \frac{1}{\frac{dZ}{dt}}\frac{d}{dt},$$

where $\frac{dX}{dt}, \frac{dY}{dt}$ and $\frac{dZ}{dt}$ can be found from (4.11). Indeed,

$$X = x_0 - \frac{y_0'}{x_0'}(Y - y_0) - \frac{z_0'}{x_0'}(Z - z_0),$$

$$\frac{dX}{dt}(x_0, y_0, z_0) = x_0' - \frac{y_0'}{x_0'}(-y_0') - \frac{z_0'}{x_0'}(-z_0') = \frac{(x_0')^2 + (y_0')^2 + (z_0')^2}{x_0'}.$$

Similarly,

$$\frac{dY}{dt}(x_0, y_0, z_0) = \frac{(x_0')^2 + (y_0')^2 + (z_0')^2}{y_0'}$$

and

$$\frac{dZ}{dt}(x_0, y_0, z_0) = \frac{(x_0')^2 + (y_0')^2 + (z_0')^2}{z_0'}.$$

Hence

$$\frac{\partial}{\partial X} = \frac{x_0'}{(x_0')^2 + (y_0')^2 + (z_0')^2}\frac{d}{dt},$$

$$\frac{\partial}{\partial Y} = \frac{y_0'}{(x_0')^2 + (y_0')^2 + (z_0')^2}\frac{d}{dt} \qquad \text{and} \qquad \frac{\partial}{\partial Z} = \frac{z_0'}{(x_0')^2 + (y_0')^2 + (z_0')^2}\frac{d}{dt}.$$

In our case of the example of the helix, we obtain

$$\frac{\partial}{\partial X} = \frac{-a\sin t}{a^2+b^2}\frac{d}{dt}, \qquad \frac{\partial}{\partial Y} = \frac{a\cos t}{a^2+b^2}\frac{d}{dt} \qquad \text{and} \qquad \frac{\partial}{\partial Z} = \frac{b}{a^2+b^2}\frac{d}{dt}.$$

Now by direct calculation one obtains

$$[\mathbf{N}_1, \mathbf{Y}] = (a^2+b^2)^{-1/2}[-\cos t\frac{\partial}{\partial X} - \sin t\frac{\partial}{\partial Y}, -a\sin t\frac{\partial}{\partial X} + a\cos t\frac{\partial}{\partial Y} + b\frac{\partial}{\partial Z}] =$$

$$= (a^2+b^2)^{-1/2}(-\sin t\frac{\partial}{\partial X} + \cos t\frac{\partial}{\partial Y})$$

and

$$[\mathbf{N}_2, \mathbf{Y}] = (a^2+b^2)^{-1}[b\sin t\frac{\partial}{\partial X} - b\cos t\frac{\partial}{\partial Y} + a\frac{\partial}{\partial Z}, -a\sin t\frac{\partial}{\partial X} + a\cos t\frac{\partial}{\partial Y} + b\frac{\partial}{\partial Z}] =$$

$$= -b(a^2+b^2)^{-1}(\cos t\frac{\partial}{\partial X} + \sin t\frac{\partial}{\partial Y}).$$

Further,

$$P^{(1)} = \begin{bmatrix} \frac{a^2}{(a^2+b^2)^2} & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$$

and hence the eigenvector is (1,0), i.e. $\mathbf{N}_1^{(1)} = \mathbf{N}_1$, and $k_1^2 = a^2/(a^2+b^2)^2$.

$$P^{(2)} = \Big[\frac{b^2}{(a^2+b^2)^2}\Big]_{1\times 1}$$

and hence $\mathbf{N}_1^{(2)} = \mathbf{N}_2$ and $k_2^2 = b^2/(a^2+b^2)^2$.

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