Electromagnetic Dynamical Systems

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Abstract

§1 describes the dynamics induced by the Biot-Savart-Laplace vector field using Hamiltonians and new variants of Lorentz world-force laws on Riemann-Jacobi or Riemann-Jacobi-Lagrange manifolds and points out some open problems. §2 transcribes the Lorentz world-force laws in the first paragraph in the Hamiltonian language using suitable symplectic forms. §3 presents the classical theory of motion of a charged particle in the electromagnetic field in order to show that the classical Lorentz world-force law is different from those introduced in the first paragraph. §4 proves that the dynamics induced by the electric field $\vec{E}$ or the magnetic field $\vec{H}$ can be described by Hamiltonians and symplectic forms intrisecally connected to the field, obeying to some Lorentz world-force laws on Riemann-Jacobi-Lagrange manifolds whose structure is imposed just by the vector field and by Maxwell equations. §5 analyses the electromagnetic dynamical systems appearing in the relativistic model. The results can be extended to any C^∞ vector field on a Riemannian manifold.

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1 Biot-Savart-Laplace dynamical systems

Let U be an open and connected set (domain) of R^3, with a piecewise smooth boundary ∂U, and $\vec{J}$ be a C^∞ vector field on $\bar{U} = U \cup \partial U$. The Biot-Savart-Laplace formula

$$\vec{H}(m) = \frac{1}{4\pi}\int_U \frac{\vec{J} \times \overrightarrow{pm}}{pm^3} dv_p$$

defines on R^3 a vector field $\vec{H}$ which is C^∞ on $R^3 - \partial U$ and of class C^0 on the boundary ∂U.

Suppose $\vec{J}$ is solenoidal, and ∂U is a field surface of $\vec{J}$. If $\vec{J}$ is a stationary electrokinetic field (conduction current density), then $\vec{H}$ is an approximation of the magnetic field generated by $\vec{J}$. The magnetic field $\vec{H}$ satisfies the relations

$$\text{div}\vec{H} = 0$$

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$$\text{rot}\vec{H}(m) = \begin{cases} 0 & \text{for} \quad m \in R^3 \setminus \bar{U} \\ \vec{J}(m) & \text{for} \quad m \in \bar{U}. \end{cases}$$

Obviously the vector field $\vec{J}$ can have zeros on $\bar{U}$. Also the domain U can be replaced by a certain surface or a certain curve. In the case of a curve, $\vec{J}$ must be nonzero everywhere.

Let $m(x^1, x^2, x^3)$ be a point of R^3 and $\{\vec{i}_1, \vec{i}_2, \vec{i}_3\}$ be the Cartesian frame. The Biot-Savart-Laplace vector field can be express in the form $\vec{H} = H_1\vec{i}_1 + H_2\vec{i}_2 + H_3\vec{i}_3$. The magnetic line α which starts from the point $m_0(x_0^1, x_0^2, x_0^3)$ at the moment $t = 0$ is the oriented curve

$$\alpha : (-a, a) \to R^3, \quad \alpha(t) = (x^1(t), x^2(t), x^3(t))$$

which satisfies the Cauchy problem

$$\frac{dx^i}{dt} = H_i, \quad x^i(0) = x_0^i, \quad i = 1, 2, 3.$$

The set of all images of the maximal magnetic lines is called the *phase portrait* of the magnetic field $\vec{H}$.

Let $f : R^3 \to R, \quad f = \frac{1}{2}(H_1^2 + H_2^2 + H_3^2)$ be the energy of the magnetic field $\vec{H}$, leaving aside the multiplicative factor μ. The following theorems are true [7]-[11].

1.1. Theorem. *Every magnetic line in $R^3 - \bar{U}$ is a trajectory of a potential dynamical system with 3 degrees of freedom associated to the potential $V = -f$, namely*

$$\frac{d^2x^i}{dt^2} = \frac{\partial f}{\partial x^i}, \quad i = 1, 2, 3. \tag{1}$$

1.2. Theorem. *Every magnetic line in U is the trajectory of a nonpotential dynamical system with 3 degrees of freedom determined by the potential $V = -f$ and by rot $\vec{H}$, namely*

$$\frac{d^2x^i}{dt^2} = \frac{\partial f}{\partial x^i} + \left(\frac{\partial H_i}{\partial x^j} - \frac{\partial H_j}{\partial x^i}\right)\frac{dx^j}{dt}. \tag{2}$$

1.3. Theorem.*1) The trajectories of the dynamical system (1) are the extremals of the Lagrangian*

$$L = \frac{1}{2}\delta_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} + f(x^1, x^2, x^3).$$

2) The trajectories of the dynamical system (2) are the extremal of the Lagrangian

$$L = \frac{1}{2}\delta_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} - H_j\frac{dx^j}{dt} + f(x^1, x^2, x^3).$$

3) The dynamical systems (1) and (2) are conservative, the total energy (Hamiltonian) being in both cases

$$\mathcal{H}\left(\S^{\infty}, \S^{\in}, \S^{\ni}, \frac{\lceil\S^{\infty}}{\lceil\sqcup}, \frac{\lceil\S^{\in}}{\lceil\sqcup}\frac{\lceil\S^{\ni}}{\lceil\sqcup}\right) = \frac{\infty}{\in}\delta_{\rangle|}\frac{\lceil\S^{\rangle}}{\lceil\sqcup}\frac{\lceil\S^{|}}{\lceil\sqcup} - \{(\S^{\infty}, \S^{\in}, \S^{\ni}), \tag{3}$$

where δ_{ij} is the Kronecker symbol.

1.4. Theorem (new variant of Lorentz world-force law). *Every nonconstant trajectory of the dynamical system (1) which has the total energy $\mathcal{H}$ is a reparametrized geodesic of the Riemann-Jacobi manifold*

$$(ext(\bar{U}) \setminus \mathcal{E},\ \}_{\rangle|} = (\mathcal{H} + \{)\delta_{\rangle|}, \quad \rangle, | = \infty, \in, \ni),$$

where $\mathcal{E}$ is the set of zeros of $\vec{H}$.

1.5. Theorem (new variant of Lorentz world-force law). *Every nonconstant trajectory of the dynamical system (2) which has the total energy $\mathcal{H}$ is a reparametrized horizontal geodesic of the Riemann-Jacobi-Lagrange manifold*

$$(U \setminus \mathcal{E},\ \}_{\rangle|} = (\mathcal{H} + \{)\delta_{\rangle|},\ \mathcal{N}^{\rangle}_{|} = -{}^{\rangle}_{||}\dagger^{\|} + \mathcal{F}^{\rangle}{}_{|}, \quad \rangle, | = \infty, \in, \ni),$$

where

$$\Gamma^i_{jk} = \textit{Riemannian connection induced by } g_{ij},$$

$$F_{ij} = \frac{\partial H_j}{\partial x^i} - \frac{\partial H_i}{\partial x^j} = H_{j,i} - H_{i,j}, \quad F^i{}_j = g^{ih} F_{hj}.$$

Open problems. 1) Obviously the differential systems (1) and (2), which describe the dynamics of some particles sensible to the magnetic field $\vec{H}$, have also solutions that are not field lines of $\vec{H}$. Till the present time for us is not clear which is the physical meaning of these trajectories though the preceding theorems show that they are not only mathematical entities.

2) The Lorentz law of Theorem 1.5 was obtained using an idea of Kawaguchi-Miron [4]. New variants of Lorentz world-force law associated to the dynamical system (2) can be obtained via the interesting and original ideas of Beil [1], [2], of Crampin [3], or of Obădeanu-Vernic [6].

3) The preceding theory can be extend to any vector field on a Riemannian manifold. In other words every dynamical system of order one can be prolonged to a suitable dynamical system of order two whose trajectories are geodesics of a Lagrangian defined by the velocity vector field (Lagrange structure of first order). In a similar way every dynamical system of order two can be prolonged to a suitable dynamical system of order three whose trajectories are geodesics of a Lagrangian defined by velocity and acceleration vector fields (Lagrange structure of order two). This point of view can create better examples for higher order Lagrange spaces [5].

4) The preceding Jacobi-Riemann-Lagrange structure blows up at equilibrium points. Is it possible to eliminate this deficiency ?

2 Hamiltonian formulation of the Biot-Savart-Laplace dynamical systems

Let us show that the differential systems (1) and (2) can be described like Hamiltonian systems.

Let M be a differentiable manifold and Ω be a 2-form on M. The pair (M, Ω) is called a *symplectic* manifold if Ω satisfies

1) $d\Omega = 0$ (i.e., Ω is closed),

2) Ω is nondegenerate.

Let (M, Ω) be a symplectic manifold and let $\varphi \in \mathcal{F}(\mathcal{M})$. Let X_φ be the unique vector field on M satisfying

$$\Omega_p(X_\varphi(p), v) = d\varphi(p) \cdot v, \quad \forall v \in T_pM.$$

We call X_φ the *Hamiltonian vector field* of φ. Hamiltonian equations are the differential equations on M given by

$$\frac{dp}{dt} = X_\varphi(p).$$

Let T_t be the flow of the Hamilton equations, i.e., $T_t(p)$ is a field line of X_φ starting at p. Then the energy φ is conserved, i.e., $\varphi \circ T_t = \varphi$ (*conservation of the energy*).

2.1. Theorem. *The dot "·" will stand for the derivative with respect to the parameter t. On $TR^3 \simeq R^6$, the equations of motion of a particle sensible to the Biot-Savart-Laplace magnetic field are Hamiltonian with respect to the total energy (3) and the following symplectic form:*

1) If the particle belong to int$(R^3 - U)$, then the symplectic form is

$$\Omega = \delta_{ij} dx^i \wedge d\dot{x}^j.$$

2) If the particle belong to U, then the symplectic form is

$$\Omega = \delta_{ij} dx^i \wedge d\dot{x}^j + J,$$

where the current density J is viewed as a closed 2-form

$$J = J_1 dx^2 \wedge dx^3 + J_2 dx^3 \wedge dx^1 + J_3 dx^1 \wedge dx^2$$

to whom is associated the solenoidal vector field

$$\vec{J} = J_1 \vec{i}_1 + J_2 \vec{i}_2 + J_3 \vec{i}_3.$$

Proof. Let $\mathcal{H}$ be the function (3) and

$$\Omega = \begin{cases} \delta_{ij} dx^i \wedge d\dot{x}^j + J & \text{on} \quad U \times R^3 \\ \delta_{ij} dx^i \wedge d\dot{x}^j & \text{on} \quad (\text{ext}\bar{U}) \times R^3. \end{cases}$$

Denote $X_{\mathcal{H}} = (u^1, u^2, u^3, \dot{u}^1, \dot{u}^2, \dot{u}^3)$ and let us discuss the case $U \times R^3$. The condition

$$i_{X_{\mathcal{H}}} \Omega = d\mathcal{H},$$

which defines $X_{\mathcal{H}}$, may be written as

$$\begin{aligned} &u^1 d\dot{x}^1 - \dot{u}^1 dx^1 + u^2 d\dot{x}^2 - \dot{u}^2 dx^2 + u^3 d\dot{x}^3 - \dot{u}^3 dx^3 + \\ &+ J_1 u^2 dx^3 - J_1 u^3 dx^2 + J_2 u^3 dx^1 - J_2 u^1 dx^3 + J_3 u^1 dx^2 - J_3 u^2 dx^1 = \\ &= \dot{x}^1 d\dot{x}^1 + \dot{x}^2 d\dot{x}^2 + \dot{x}^3 d\dot{x}^3 - \left(\tfrac{\partial f}{\partial x^1} dx^1 + \tfrac{\partial f}{\partial x^2} dx^2 + \tfrac{\partial f}{\partial x^3} dx^3 \right). \end{aligned}$$

By identification we find

$$u^1 = \dot{x}^1, \quad u^2 = \dot{x}^2, \quad u^3 = \dot{x}^3$$

$$\dot{u}^1 = \frac{\partial f}{\partial x^1} + J_2u^3 - J_3u^2, \quad \dot{u}^2 = \frac{\partial f}{\partial x^2} + J_3u^1 - J_1u^3, \quad \dot{u}^3 = \frac{\partial f}{\partial x^3} + J_1u^2 - J_2u^1,$$

i.e.,

$$\ddot{x}^1 = \frac{\partial f}{\partial x^1} + J_2\dot{x}^3 - J_3\dot{x}^2, \quad \ddot{x}^2 = \frac{\partial f}{\partial x^2} + J_3\dot{x}^1 - J_1\dot{x}^3, \quad \ddot{x}^3 = \frac{\partial f}{\partial x^3} + J_1\dot{x}^2 - J_2\dot{x}^1,$$

which is the same with the differential system (2).

Obviously $\mathrm{div}X_{\mathcal{H}} = 0$ and hence the flow generated by $X_{\mathcal{H}}$ preserves the volume.

3 Classical equations of motion for a charged particle in a stationary electromagnetic field

To avoid some misunderstandings, we recall some well known facts. Let

$$B = B_1dx^2 \wedge dx^3 + B_2dx^3 \wedge dx^1 + B_3dx^1 \wedge dx^2$$

be a closed 2-form on R^3 and

$$\vec{B} = B_1\vec{i}_1 + B_2\vec{i}_2 + B_3\vec{i}_3$$

the associated divergence free vector field. The connection between the magnetic induction $\vec{B}$ and the magnetic vector field $\vec{H}$ is $\vec{B} = \mu_0\vec{H}$. Thinking of $\vec{B}$ as a magnetic field, and taking the eletromagnetic field on R^3 given by the electric field $\vec{E}$ and the magnetic field $\vec{B}$, the equations of motion for a particle with charge e and mass m in the electromagnetic field are given by the Lorentz force-law

$$m\frac{d\vec{v}}{dt} = e(\vec{E} + \vec{v} \times \vec{B}),$$

where $\vec{v} = \dot{x}^1\vec{i}_1 + \dot{x}^2\vec{i}_2 + \dot{x}^3\vec{i}_3$ is the field of velocities and the dot "·" denote the derivative with respect to the parameter t.

Since rot $\vec{E} = 0$ $(\partial_t\vec{B} = 0)$, we can write (locally) $\vec{E} = \mathrm{grad}\varphi$.

On $R^3 \times R^3$, i.e., on $(x^1, x^2, x^3, \dot{x}^1, \dot{x}^2, \dot{x}^3)$ -space, we consider the symplectic form

$$\Omega_B = m\delta_{ij}dx^i \wedge d\dot{x}^j - eB$$

and the Hamiltonian (total energy)

$$\mathcal{H} = \frac{\Updownarrow}{\in}\delta_{\rangle|}\S^{\rangle}\S^{|} + \rceil\varphi(\S).$$

Denoting $X_{\mathcal{H}}(u^1, u^2, u^3) = (u^1, u^2, u^3, \dot{u}^1, \dot{u}^2, \dot{u}^3)$ the condition of defining $X_{\mathcal{H}}$, i.e.,

$$i_{X_{\mathcal{H}}}\Omega_B = d\mathcal{H}$$

becomes

$$\begin{aligned} &m(u^1d\dot{x}^1 - \dot{u}^1dx^1 + u^2d\dot{x}^2 - \dot{u}^2dx^2 + u^3d\dot{x}^3 - \dot{u}^3dx^3) - \\ &-e(B_1u^2dx^3 - B_1u^3dx^2 + B_2u^3dx^1 - B_2u^1dx^3 + B_3u^1dx^2 - B_3u^2dx^1) = \\ &= m(\dot{x}^1d\dot{x}^1 + \dot{x}^2d\dot{x}^2 + \dot{x}^3d\dot{x}^3) + e\left(\frac{\partial f}{\partial x^1}dx^1 + \frac{\partial f}{\partial x^2}dx^2 + \frac{\partial f}{\partial x^3}dx^3\right). \end{aligned}$$

Consequently

$$u^1 = \dot{x}^1, \quad u^2 = \dot{x}^2, \quad u^3 = \dot{x}^3,$$

$$m\dot{u}^1 = e(E_1 + B_3 u^2 - B_2 u^3), \quad m\dot{u}^2 = e(E_2 + B_1 u^3 - B_3 u^1),$$

$$m\dot{u}^3 = e(E_3 + B_2 u^1 - B_1 u^2)$$

or

$$m\ddot{x}^1 = e(E_1 + B_3\dot{x}^2 - B_2\dot{x}^3)$$

$$m\ddot{x}^2 = e(E_2 + B_1\dot{x}^3 - B_3\dot{x}^1)$$

$$m\ddot{x}^3 = e(E_3 + B_2\dot{x}^1 - B_1\dot{x}^2),$$

which are the same with classical Lorentz equations. Thus the equations of motion for a charged particle in an electromagnetic field are Hamiltonian, with total energy $\mathcal{H}$ and with the symplectic form Ω_B.

4 Electromagnetic dynamical systems

The physical-mathematical objects of the electromagnetism are:
$U \subset R^3$ = domain of linear homogeneous isotropic media,
t = the time,
$\vec{E}$ = the electric vector field (electric intensity),
$\vec{H}$ = the magnetic vector field (magnetizing force),
$\vec{B}$ = the magnetic flux density (magnetic induction),
$\vec{D}$ = the electric displacement (electric induction),
$\vec{J}$ = the electric current density (conduction current density),
ρ = the electric charge density,
∂_t = the time derivative operator,
μ = the scalar permeability,
ε = the permitivity.
The previous fields are defined on $U \times R$ and satisfy the Maxwell equations

$$\text{div}\, \vec{D} = \rho, \quad \text{rot}\, \vec{H} = \vec{J} + \partial_t \vec{D}$$

$$\text{div}\, \vec{B} = 0, \quad \text{rot}\, \vec{E} = -\partial_t \vec{B},$$

the associated constitutive equations relating the fields being

$$\vec{B} = \mu\vec{H}, \quad \vec{D} = \varepsilon\vec{E}.$$

Let $\vec{E} = E_1\vec{i}_1 + E_2\vec{i}_2 + E_3\vec{i}_3$ be the electric vector field on the domain $U \times R$, i.e., $\vec{E} = \vec{E}(x,t)$. The electric line α which starts at the moment $s = 0$ from the point $m_0(x_0^1, x_0^2, x_0^3)$ is the oriented curve

$$\alpha : (-a, a) \to U, \quad \alpha(s) = (x^1(s), x^2(s), x^3(s)),$$

the solution of the Cauchy problem

$$\frac{dx^i}{ds} = E_i, \quad x^i(0) = x_0^i, \quad i = 1, 2, 3.$$

The set of all images of maximal electric lines is called the *phase portrait of the electric field* $\vec{E}$. Obviously the parameter s is different from the time parameter t. The time parameter t can produce bifurcation in the equilibrium set of $\vec{E}$ or Hopf bifurcation of the flow of $\vec{E}$. The coincidence between the parameters t and s remains open though we can use the ideas of the paper [6] in order to study the case $s = t$.

Let $f : U \to R, \quad f = \frac{1}{2}(E_1^2 + E_2^2 + E_3^2)$ be the energy of $\vec{E}$, leaving aside the multiplicative factor ε.

4.1. Theorem. *Every electric line is the trajectory of a nonpotential dynamical system with 3 degrees of freedom determined by the potential* $V = -f$ *and by* rot $\vec{E}$, *namely*

$$\frac{d^2x^i}{ds^2} = \frac{\partial f}{\partial x^i} + \partial_t B_k \frac{dx^j}{ds} - \partial_t B_j \frac{dx^k}{ds}, \tag{4}$$

$\{i, j, k\}$ *being a cyclic permutation of* $\{1, 2, 3\}$.

Proof. Deriving $\dfrac{dx^i}{ds} = E_i$ along a solution α, using rot $\vec{E} = -\partial_t \vec{B}$ and replacing $\dfrac{dx^i}{ds}$ only in terms which permit to recover ∇f we find the prolongation

$$\frac{d^2x^i}{ds^2} = \frac{\partial f}{\partial x^i} + \left(\frac{\partial E_i}{\partial x^j} - \frac{\partial E_j}{\partial x^i}\right)\frac{dx^j}{ds},$$

which is the same with (4).

In this context we can prove the following propositions.

4.2. Theorem. *The dynamical system (4) is conservative, the total energy being*

$$\mathcal{H} = \frac{\infty}{\in}\delta_{\rangle|}\frac{\lceil\S^{\rangle}}{\lceil\int}\frac{\lceil\S^{|}}{\lceil\int} - \{(\S^{\infty}, \S^{\in}, \S^{\ni}). \tag{5}$$

4.3. Theorem. *On* $TR^3 \simeq R^6$, *the equations (4) of motion of a particle sensible to the electric field* $\vec{E}$ *are Hamiltonian with respect to the total energy (5) and the symplectic form*

$$\Omega = \delta_{ij}dx^i \wedge d\dot{x}^j - \partial_t B,$$

where the magnetic induction is viewed as a closed 2-form

$$B = B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2$$

to whom is associated the solenoidal vector field $\vec{B} = B_1\vec{i}_1 + B_2\vec{i}_2 + B_3\vec{i}_3$, *and the dot "$\cdot$" denotes the derivative with respect to the parameter* s.

4.4. Theorem (a new version of the Lorentz world force-law). *Every non-constant trajectory of the dynamical system (4) which has the total energy* $\mathcal{H}$ *is a reparametrized horizontal geodesic of the Riemann-Jacobi-Lagrange manifold*

$$(U \setminus \mathcal{E},\ \}_{\rangle|} = (\mathcal{H} + \{)\delta_{\rangle|}, \quad \mathcal{N}^{\rangle}_{|} = -^{\rangle}_{|\|}\dagger^{\|} + \mathcal{F}^{\rangle}_{|}, \quad \rangle, |, \| = \infty, \in, \ni),$$

where

$$\Gamma^i_{jk} = \textit{Riemannian connection induced by } g_{ij},$$

$$F_{ij} = \frac{\partial E_j}{\partial x^i} - \frac{\partial E_i}{\partial x^j} = E_{j,i} - E_{i,j}, \quad F^i{}_j = g^{ih} F_{hj}.$$

Remarks. 1) The vector field $\frac{d\vec{\alpha}}{ds} \times \partial_t \vec{B}$ does not produce a dissipation of energy along the electric line α since it is orthogonal to α.

2) Other prolongation on U of the dynamical system $\frac{dx^i}{ds} = E_i, \quad i = 1,2,3$, is the nonconservative dynamical system of order two

$$\frac{d^2x^i}{ds^2} = \frac{\partial f}{\partial x^i} + E_j \partial_t B_k - E_k \partial_t B_j, \quad \{i,j,k\} = \text{permutation of } \{1,2,3\}.$$

3) The flow generated by $X_{\mathcal{H}}$ conserves the volume.

4) For the magnetic lines (the field lines of $\vec{H}$) one obtains similar results. The difference is that the associated symplectic form contains the closed 2-form $J + \partial_t D$ associated to the solenoidal vector field $\vec{J} + \partial_t \vec{D}$.

Open problem. Find the properties of the field lines of the Poynting vector field

$$\vec{S} = \vec{E} \times \vec{H}.$$

5 Electromagnetic dynamical systems in the relativistic model

Let M be a connected 4-dimensional differentiable manifold and g a Lorentz metric on M. The pair (M,g) is called *Lorentz manifold.*

5.1. Definition. A *spacetime* (M, g, ∇) is a connected 4-dimensional, oriented, and time-oriented Lorentz manifold (M,g) together with its Levi-Civita connection ∇.

Let F be the electromagnetic field like a 2-form on M, and J be the charge-current density of the matter model $\mathcal{M}$. The relativistic model will be denoted either by $(M, \mathcal{M}, \mathcal{F})$ or by (M, F, J).

5.2. Definition. $(M, \mathcal{M}, \mathcal{F})$ or (M, F, J) verifies *Maxwell equations* iff:

1) F is closed, i.e., $dF = 0$;

2) div $\hat{F} = J$, where $\hat{F}$ is the (1,1)-tensor field physically equivalent to F via the Lorentz metric g.

As a consequence of 1), locally, there exists a 1-form η such that $F = d\eta$. We denote by ξ the vector field physically equivalent to η via the Lorentz metric g.

Obviously, J is a solenoidal vector field, i.e.,

$$\text{div } J = \text{div div } \hat{F} = 0.$$

Usually, the authors study the influence of spacetime M and of the matter model $\mathcal{M}$ on the electromagnetic field F.

Let F_{ij} be the componets of F. Then $dF = 0$ is equivalent to

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0$$

and

$$\text{div}\hat{F} = J$$

is equivalent to

$$F^j{}_{i,j} = -J_i.$$

If M and J are given ab initio, and the influence of F on M and on $\mathcal{M}$ is neglected, then the Maxwell equations become conditions determining F. Here we use Maxwell equations to obtain information about the dynamical systems generated by η and J.

Examples. 1) *Constant magnetic field.* Set $E = 0$, and $F = 2B dx^3 \wedge dx^1$ is an electromagnetic field on the Minkowski space (R^4, g); B is a scalar field on R^4, and the electric field E in covariant constant (parallel, inertial) reference frame ∂_4 is everywhere zero. The condition $dF = 0$ is equivalent to $\partial_4 B = 0 = \partial_2 B$. The condition $\text{div}\hat{F} = 0$ (zero source J) gives $\partial_3 B = 0 = \partial_1 B$. Consequently $B = \text{constant}$.

2) *Waves.* Let (R^4, g) be a Minkowski space. Near the origin of 3-space are some electric charges that move back and forth in the ∂_1 direction of 3-space. An electromagnetic field is generated. In the observation region ("wave zero"), this field can be described as

$$F = 2(f \circ \phi) d\phi \wedge dx^1,$$

where $f : R \to R$ is C^∞, and $\phi = (x^3 - x^4) : R^4 \to R$. The set $(R^4, F, 0)$ verifies the Maxwell equations

$$dF = f' \circ \phi \quad d\phi \wedge d\phi \wedge dx^1 = 0$$

$$\hat{F} = 2(f \circ \phi)(\partial_3 + \partial_4) \wedge \partial_1 =$$

= the (2,0)-tensor field physically equivalent to F via the Lorentz metric,

$$\text{div}\hat{F} = \partial_1(f \circ \phi) - (\partial_3 + \partial_4)(f \circ \phi) = 0.$$

F is called a *plane, linearly polarized electromagnetic wave* on Minkowski space.

The *stress-energy tensor* T of an electromagnetic field F on M is defined as a (0,2)-tensor field on M of components

$$T_{ij} = F_{im} F_j{}^m - \frac{1}{4} g_{ij} F^{mn} F_{mn}.$$

5.3. Theorem. *Let $\hat{T}$ be the (2,0)-tensor field physically equivalent to T via the Lorentz metric g.*

1) $\hat{T}$ is symmetric and trace $\hat{T} = 0$.

2) $\hat{T}(\omega, \omega) \geq 0$ for every causal 1-form ω.

3) If (M, F, J) verifies Maxwell equations, then div$\hat{T} = -\hat{F}J$.

Using the components F_{ij} of F, the condition $\text{div}\hat{T} = -\hat{F}J$ is equivalent to

$$T^{ij}{}_{,j} = -F^i{}_m J^m.$$

Remark. The stress-energy tensor T unifies and replaces the classical energy density $\frac{1}{2}(\varepsilon\|\vec{E}\|^2 + \mu\|\vec{H}\|^2)$, Poynting vector field $\vec{S} = \vec{E} \times \vec{H}$ and Maxwell stress tensor field of components

$$t^{\alpha\beta} = -(\varepsilon E^\alpha E^\beta + \mu H^\alpha H^\beta - \frac{1}{2}\delta^{\alpha\beta}(\varepsilon\|\vec{E}\|^2 + \mu\|\vec{H}\|^2)).$$

We consider the vector field ξ of components $\xi^i,\ i=1,2,3,4$, physically equivalent to the 1-form η via the Lorentz metric g. The energy associated to ξ is $f: M \to R,\quad f = \frac{1}{2}g(\xi,\xi)$. Obviously

$$f = \frac{1}{2}g_{ij}\xi^i\xi^j = \frac{1}{2}g^{ij}\eta_i\eta_j.$$

The field line α of ξ which starts from the point $(x_0^1, x_0^2, x_0^3, x_0^4)$ at the moment $s=0$ is the oriented curve $\alpha: (-a,a) \to M,\quad \alpha(s) = (x^1(s), x^2(s), x^3(s), x^4(s))$ which satisfies the Cauchy problem

$$\frac{dx^i}{ds} = \xi^i, \quad x^i(0) = x_0^i, \quad i = 1,2,3,4.$$

Since ξ is an irrotational vector field the following theorem is true.

5.4. Theorem. *Every field line of ξ is a trajectory of a potential dynamical system with 4 degrees of freedom associated to the potential $V = -f$.*

We can obtain automatically a new version of the Lorentz world-force law determined by ξ and g.

Now we consider the vector field J of components $J^i,\ i=1,2,3,4$. The energy associated to J is $\varphi: M \to R,\ \varphi = \frac{1}{2}g(J,J)$, and the field line of J which start from the point $(x_0^1, x_0^2, x_0^3, x_0^4)$ at the moment $s=0$ is the oriented curve $\alpha: (-a,a) \to M,\quad \alpha(s) = (x^1(s), x^2(s), x^3(s), x^4(s))$ which satisfies the Cauchy problem

$$\frac{dx^i}{ds} = J^i, \quad x^i(0) = x_0^i, \quad i = 1,2,3,4.$$

We can obtain easily the prolongation of this dynamical system to a conservative differential system of order two and hence a new variant of the Lorentz world- force law induced by J and g. The flow generated by J conserves the volume because J is a solenoidal vector field.

Suppose that J has no zero on M. Then $J = \|J\|J_0,\ \|J_0\| = 1$ and the restriction of the energy φ to a field line $\alpha(s),\ s \in I$ of J_0 (s being here the curvilinear abscissa) is well determined by the restriction of $\mathrm{div}J_0$ to that line. Indeed, denoting $l = \|J\| \circ \alpha,\ m = \mathrm{div}J_0$ and taking into account that

$$0 = \mathrm{div}J = D_{J_0}\|J\| + \|J\|\mathrm{div}J_0,$$

we find

$$\frac{dl}{ds} = -lm.$$

Consequently

$$l(s) = l_0 \exp\left(-\int_{s_0}^{s} m(t)dt\right), \quad l(s_0) = l_0.$$

If m is nowhere zero, the field line α cannot be closed (the field lines of J_0 are reparametrizations of the field lines of J).

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References

[1] R.G.Beil, *Comparison of unified field theories*, Tensor N.S., 56, 2(1995), 175-183.

[2] R.G.Beil, Notes on a new Finsler metric function, Balkan Journal of Geometry and Its Applications, 2, 1(1997), 1-6.

[3] F.M.Crampin, *A linear connection associated with any second order differential equation field*, In: Eds. L.Tamassy, J.Szenthe, New Developments in Geometry, 77-86, Kluwer Academic Publishers, 1996.

[4] T.Kawaguchi, R.Miron, *A Lagrangian model for gravitation and electromagnetism*, Tensor N.S., 48(1989), 153-168.

[5] R.Miron, *The geometry of higher-order Lagrange spaces*, Fundamental Theories of Physics, 82, Kluwer Academic Publishers, 1996.

[6] V.Obădeanu, C.Vernic, Systemes dynamiques sur des espaces de Riemann, Balkan Journal of Geometry and Its Applications, 2, 1 (1997), 73-82.

[7] S.Ştefănescu, C.Udrişte, *Magnetic field lines around filiform electrical circuits of right angle type*, Scientific Bulletin, Applied Mathematics and Physics, Politehnica University of Bucharest, 55, 1-2 (1993), 3-18.

[8] A.Udrişte, C.Udrişte, *Dynamics induced by a magnetic field*, Conference on Differential Geometry, Budapest, July 27-30, 1996 (will appear in Proceedings at Kluwer Academic Publishers).

[9] C.Udrişte, *Convex functions and optimization methods on Riemannian manifolds*, Mathematics and Its Applications, 297, Kluwer Academic Publishers, Dordrecht, Boston, London, 1994.

[10] C.Udrişte, *Electromagnetic dynamical systems as Hamilton-Poisson systems*, Workshop on Differential Geometry, Global Analysis, Lie Algebras, Aristotle University of Thessaloniki, June 25-29, 1997.

[11] C.Udrişte, S.Udrişte, *Biot-Savart-Laplace dynamical systems*, Balkan Journal of Geometry and Its Applications, 1,2 (1996), 125-136.

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