Metric Semi-Symmetric N - Linear Connections in the Bundle of Accelerations

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Abstract

The study of the higher order Lagrange spaces, based on the notion of bundle of velocities of order k has been recently studied by Radu Miron and Gheorghe Atanasiu in [3] - [6]. The bundle of accelerations correspond in this study to $k = 2$ [1], [2].

In this paper we shall determine all metri semi-symmetric N - linear connections in the bundle of accelerations, and we study the group of transformations of these connections and their invariants.

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Key words: 2 - osculator bundle, connections, curvature, torsion.

1 Group of transformations of N - linear connections in the bundle of accelerations

Let M be an n - dimensional real C^∞- manifold and (Osc^2M, π, M) its 2-osculator bundle, or the bundle of accelerations. The local coordinates on the total space $E = Osc^2M$ are denoted by $(x^i, y^{(1)i}, y^{(2)i})$. If N is a nonlinear connection on E with the canonical coefficients $\underset{(1)}{N}{}^i{}_j$, $\underset{(2)}{N}{}^i{}_j$, then let be $D\Gamma(N) = (L^i_{jk}, \underset{(1)}{C}{}^i_{jk}, \underset{(2)}{C}{}^i_{jk})$ an N- linear connection D on $E = Osc^2M$. The elements of the abelian group $\mathcal{T}_N = \{t(0,0,B^i_{jk}, \underset{(1)}{D}{}^i_{jk}, \underset{(2)}{D}{}^i_{jk}) \in \mathcal{T}\}$ are the transformations $t(0,0,B^i_{jk}, \underset{(1)}{D}{}^i_{jk}, \underset{(2)}{D}{}^i_{jk})$: $D\Gamma(N) = (L^i_{jk}, \underset{(1)}{C}{}^i_{jk}, \underset{(2)}{C}{}^i_{jk}) \to D\overline{\Gamma}(N) = (\overline{L}^i_{jk}, \underset{(1)}{\overline{C}}{}^i_{jk}, \underset{(2)}{\overline{C}}{}^i_{jk})$ given by

$$\underset{(\alpha)}{\overline{N}}{}^i{}_j = \underset{(\alpha)}{N}{}^i_j,\ \overline{L}^i_{jk} = L^i_{jk} - B^i_{jk},\ \underset{(\alpha)}{\overline{C}}{}^i_{jk} = \underset{(\alpha)}{C}{}^i_{jk} - \underset{(\alpha)}{D}{}^i_{jk},\ (\alpha = 1,2). \tag{1.1}$$

Proposition 1.1. *The transformation $\mathcal{T}_N$, given by (1.1), leads to the transformation of the torsions and curvatures in the following way:*

$$\underset{(0\alpha)}{\overline{R}}{}^i{}_{jk} = \underset{(0\alpha)}{R}{}^i{}_{jk}, \quad (\alpha = 1,2), \tag{1.2}$$

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$$\overline{T}_{(0)}{}^{i}{}_{jk} = T_{(0)}{}^{i}{}_{jk} + (B^{i}_{kj} - B^{i}_{jk}), \tag{1.3}$$

$$\overline{S}_{(\alpha)}{}^{i}{}_{jk} = S_{(\alpha)}{}^{i}{}_{jk} + (D_{(\alpha)}{}^{i}_{kj} - D_{(\alpha)}{}^{i}_{jk}), \tag{1.4}$$

$$\overline{Q}_{(11)}{}^{i}{}_{jk} = Q_{(11)}{}^{i}{}_{jk} + (D_{(1)}{}^{i}_{kj} - D_{(1)}{}^{i}_{jk}), \tag{1.5}$$

$$\overline{Q}_{(12)}{}^{i}{}_{jk} = Q_{(12)}{}^{i}{}_{jk} - D_{(2)}{}^{i}_{jk}, \tag{1.6}$$

$$\overline{Q}_{(21)}{}^{i}{}_{jk} = Q_{(21)}{}^{i}{}_{jk}, \tag{1.7}$$

$$\overline{Q}_{(22)}{}^{i}{}_{jk} = Q_{(22)}{}^{i}{}_{jk} + D_{(1)}{}^{i}_{kj}, \tag{1.8}$$

$$\overline{P}_{(\alpha\alpha)}{}^{i}{}_{jk} = P_{(\alpha\alpha)}{}^{i}{}_{jk} + B^{i}_{kj}, \quad (\alpha = 1,2), \tag{1.9}$$

$$\overline{P}_{(12)}{}^{i}{}_{jk} = P_{(12)}{}^{i}{}_{jk}, \tag{1.10}$$

$$\overline{P}_{(21)}{}^{i}{}_{jk} = P_{(21)}{}^{i}{}_{jk}, \tag{1.11}$$

$$\begin{aligned} \overline{R}_{h}{}^{i}{}_{jk} &= R_{h}{}^{i}{}_{jk} - D_{(1)}{}^{i}_{hs} R_{(01)}{}^{s}{}_{jk} - D_{(2)}{}^{i}_{hs} R_{(02)}{}^{s}{}_{jk} - B^{i}_{hs} T_{(0)}{}^{s}{}_{jk} + \\ &+ \mathcal{A}_{jk}\{-B^{i}_{hj|k} + B^{s}_{hj} B^{i}_{sk}\}, \end{aligned} \tag{1.12}$$

$$\begin{aligned} \overline{P}_{(\alpha)h}{}^{i}{}_{jk} &= P_{(\alpha)h}{}^{i}{}_{jk} - D_{(1)}{}^{i}_{hs} P_{(\alpha 1)}{}^{s}{}_{jk} - D_{(2)}{}^{i}_{hs} P_{(\alpha 2)}{}^{s}{}_{jk} - B^{i}_{hs} C_{(\alpha)}{}^{s}_{jk} - \\ &- B^{i}_{hj} \,|^{(\alpha)}_{k} + D_{(\alpha)}{}^{i}_{hk|j} + B^{s}_{hj} D_{(\alpha)}{}^{i}_{sk} - D_{(\alpha)}{}^{s}_{hk} B^{i}_{sj}, \end{aligned} \tag{1.13}$$

$$\begin{aligned} \overline{S}_{(21)h}{}^{i}{}_{jk} &= S_{(21)h}{}^{i}{}_{jk} - C_{(2)}{}^{s}_{jk} D_{(1)}{}^{i}_{hs} + C_{(1)}{}^{s}_{kj} D_{(2)}{}^{i}_{hs} - D_{(1)}{}^{i}_{hj} \,|^{(2)}_{k} + \\ &+ D_{(2)}{}^{i}_{hk} \,|^{(1)}_{j} + D_{(1)}{}^{s}_{hj} D_{(2)}{}^{i}_{sk} - D_{(2)}{}^{s}_{hk} D_{(1)}{}^{i}_{sj} - D_{(2)}{}^{i}_{hs} B_{(21)}{}^{s}{}_{jk}, \end{aligned} \tag{1.14}$$

$$\begin{aligned} \overline{S}_{(\alpha\alpha)h}{}^{i}{}_{jk} &= S_{(\alpha\alpha)h}{}^{i}{}_{jk} - D_{(\alpha)}{}^{i}_{hs} S_{(\alpha)}{}^{s}{}_{jk} + \mathcal{A}_{jk}\{-D_{(\alpha)}{}^{i}_{hj} \,|^{(\alpha)}_{k} + \\ &+ D_{(\alpha)}{}^{s}_{hj} D_{(\alpha)}{}^{i}_{sk}\} - D_{(\alpha)}{}^{i}_{hs} R_{(\alpha 2)}{}^{s}{}_{jk}, \quad (\alpha = 1,2). \end{aligned} \tag{1.15}$$

Consider the tensor fields

$$K_{h}{}^{i}{}_{jk} = R_{h}{}^{i}{}_{jk} - C_{(1)}{}^{i}_{hs} R_{(01)}{}^{s}{}_{jk} - C_{(2)}{}^{i}_{hs} R_{(02)}{}^{s}{}_{jk}, \tag{1.16}$$

$$\begin{aligned}\mathcal{P}_{(1)h\ jk}^{\ \ \ \ i} &= \mathcal{A}_{jk}\Big\{P_{(1)h\ jk}^{\ \ \ \ i} - C_{(1)hs}^{\ \ \ i}\frac{\delta N_{(1)\ j}^{\ \ \ s}}{\delta y^{(1)k}} - C_{(2)hs}^{\ \ \ i}(N_{(1)\ m}^{\ \ \ s}\frac{\delta N_{(1)\ j}^{\ \ \ m}}{\delta y^{(1)k}} + \\ &+ \frac{\delta N_{(2)\ j}^{\ \ \ s}}{\delta y^{(1)k}} - \frac{\delta N_{(2)\ k}^{\ \ \ s}}{\delta y^{(1)j}})\Big\},\end{aligned} \tag{1.17}$$

$$\begin{aligned}\mathcal{P}_{(2)h\ jk}^{\ \ \ \ i} &= \mathcal{A}_{jk}\Big\{P_{(2)h\ jk}^{\ \ \ \ i} - C_{(1)hs}^{\ \ \ i}\frac{\partial N_{(1)\ j}^{\ \ \ s}}{\delta y^{(2)k}} - C_{(2)hs}^{\ \ \ i}(N_{(1)\ m}^{\ \ \ s}\frac{\partial N_{(1)\ j}^{\ \ \ m}}{\delta y^{(2)k}} - \\ &- \frac{\partial N_{(2)\ j}^{\ \ \ s}}{\partial y^{(2)k}})\Big\}.\end{aligned} \tag{1.18}$$

Proposition 1.2. *By the transformation (1.1) the tensor fields* $K_{h\ jk}^{\ i}$, $\mathcal{P}_{(1)h\ jk}^{\ \ \ \ i}$ *and* $\mathcal{P}_{(2)h\ jk}^{\ \ \ \ i}$ *are transformed according to the following laws*

$$\overline{K}_{h\ jk}^{\ i} = K_{h\ jk}^{\ i} - B^i_{hs}T_{(0)\ jk}^{\ \ \ s} + \mathcal{A}_{jk}\{-B^i_{hj|k} + B^s_{hj}B^i_{sk}\}, \tag{1.19}$$

$$\begin{aligned}\overline{\mathcal{P}}_{(\alpha)h\ jk}^{\ \ \ \ i} &= \mathcal{P}_{(\alpha)h\ jk}^{\ \ \ \ i} - D_{(\alpha)hs}^{\ \ \ i}T_{(0)\ jk}^{\ \ \ s} - B^i_{hs}S_{(\alpha)\ jk}^{\ \ \ s} + \mathcal{A}_{jk}\{-B^i_{hj}\ |^{(\alpha)}_k - \\ &- D_{(\alpha)hj|k}^{\ \ \ i} + B^s_{hj}D_{(\alpha)sk}^{\ \ \ i} + D_{(\alpha)hj}^{\ \ \ s}B^i_{sk}\}, \quad (\alpha = 1, 2).\end{aligned} \tag{1.20}$$

2 Metric semi-symmetric N- linear connections in the bundle of accelerations

Let N be the canonical nonlinear connection on $E = Osc^2M$.
Definition 2.1. An N-linear connection $D\Gamma(N) = (L^i_{jk}, C_{(1)jk}^{\ \ \ i}, C_{(2)jk}^{\ \ \ i})$ with the property

$$g_{ij|k} = 0, \quad g_{ij}|^{(\alpha)}_k = 0, \quad (\alpha = 1, 2) \tag{2.1}$$

is said to be a *metric N-linear connection.*

A class of metric N-linear connections, which have interesting properties is that of metric semi-symmetric $N-$ linear connections.
Definition 2.2. An $N-$ linear connection $D\Gamma(N) = (L^i_{jk}, C_{(1)jk}^{\ \ \ i}, C_{(2)jk}^{\ \ \ i})$ is called *semi-symmetric* if the torsion tensor fields $T_{(0)\ jk}^{\ \ \ i}, S_{(\alpha)\ jk}^{\ \ \ i}$, $(\alpha = 1, 2)$ have the form

$$\begin{cases} T_{(0)\ jk}^{\ \ \ i} = \frac{1}{n-1}(T_{(0)j}\delta^i_k - T_{(0)k}\delta^i_j) = \frac{1}{n-1}\mathcal{A}_{jk}\{T_{(0)j}\delta^i_k\}, \\ S_{(\alpha)\ jk}^{\ \ \ i} = \frac{1}{n-1}(S_{(\alpha)j}\delta^i_k - S_{(\alpha)k}\delta^i_j) = \frac{1}{n-1}\mathcal{A}_{jk}\{S_{(\alpha)j}\delta^i_k\}, \end{cases} \tag{2.2}$$

where $T_{(0)j} = T_{(0)\ ji}^{\ \ \ i}$, $S_{(\alpha)j} = S_{(\alpha)\ ji}^{\ \ \ i}$, $(\alpha = 1, 2)$ are called $h-$ and v_α - torsion vector fields.

The canonical metric N-linear connection $D\overset{c}{\Gamma}(N)$ will be considered semi-symmetric with the torsion vector fields $\overset{c}{T}_{(0)} = \overset{c}{S}_{(\alpha)} = 0$, $(\alpha = 1, 2)$.

Taking $\sigma_j = \frac{T_{(0)j}}{n-1}$, $\tau_{(\alpha)j} = \frac{S_{(\alpha)j}}{n-1}$, $(\alpha = 1, 2)$ and applying the Theorem 5.4.3, [8] we obtain:

Theorem 2.1. *The set of all metric semi-symmetric N-linear connections $D\Gamma(N)$ is given by*

$$(2.3)\qquad \begin{cases} L^i_{jk} = \overset{c}{L}{}^i_{jk} + \sigma_j\delta^i_k - g_{jk}g^{is}\sigma_s, \\ C_{(\alpha)}{}^i_{jk} = \overset{c}{C}_{(\alpha)}{}^i_{jk} + \tau_{(\alpha)j}\delta^i_k - g_{jk}g^{is}\tau_{(\alpha)s}, \quad (\alpha = 1,2), \end{cases}$$

where $D\overset{c}{\Gamma}(N) = (\overset{c}{L}{}^i_{jk}, \overset{c}{C}_{(1)}{}^i_{jk}, \overset{c}{C}_{(2)}{}^i_{jk})$ is the canonical metric N-linear connection on E and $\sigma_j, \tau_{(\alpha)j}, \quad (\alpha = 1,2)$ are arbitrary covariant vector fields.

One notices that (2.3) gives the transformations of the metric semi-symmetric N-linear connections. Let $t(\sigma_j, \tau_{(\alpha)j}) : D\Gamma(N) \to D\overline{\Gamma}(N), \quad (\alpha = 1,2)$ be a transformation of this form, i.e.,

$$(2.4)\qquad \begin{cases} \overline{L}^i_{jk} = L^i_{jk} + \sigma_j\delta^i_k - g_{jk}g^{is}\sigma_s, \\ \overline{C}_{(\alpha)}{}^i_{jk} = C_{(\alpha)}{}^i_{jk} + \tau_{(\alpha)j}\delta^i_k - g_{jk}g^{is}\tau_{(\alpha)s}, \quad (\alpha = 1,2). \end{cases}$$

Theorem 2.2. *The set $\overset{s}{\mathcal{T}}_N$ of the transformations $t(\sigma_j, \tau_{(1)j}, \tau_{(2)})$ of the metric semi-symmetric N-linear connections, given by (2.4), with the mapping product, is an abelian group. This group acts effectively on the set of all N- linear connections, and for each nonlinear connection N it acts transitively on the set of metrical semi-symmetric $N-$ linear connections.*

By applying the result from §1, one obtains

Theorem 2.3. *By means of transformation (2.4) the tensor fields $K_h{}^i{}_{jk}$, $\mathcal{P}_{(\alpha)h}{}^i{}_{jk}$, $S_{(\alpha\alpha)h}{}^i{}_{jk}$, $(\alpha = 1,2)$ are changed by the laws*

$$(2.5)\qquad \overline{K}_h{}^i{}_{jk} = K_h{}^i{}_{jk} + 2\mathcal{A}_{jk}\{\Omega^{ir}_{jh}\sigma_{rk}\},$$

$$(2.6)\qquad \overline{\mathcal{P}}_{(\alpha)h}{}^i{}_{jk} = \mathcal{P}_{(\alpha)h}{}^i{}_{jk} + 2\mathcal{A}_{jk}\{\Omega^{ir}_{jh}\rho_{(\alpha)rk}\}, \quad (\alpha = 1,2),$$

$$(2.7)\qquad \overline{\mathcal{S}}_{(\alpha\alpha)h}{}^i{}_{jk} = \mathcal{S}_{(\alpha\alpha)h}{}^i{}_{jk} + 2\mathcal{A}_{jk}\{\Omega^{ir}_{jh}\tau_{(\alpha)rk}\} + 2\Omega^{ir}_{sh}\tau_{(\alpha)r}R_{(\alpha 2)}{}^s{}_{jk},$$

$$(\alpha = 1,2) \text{ and } R_{(22)}{}^i{}_{jk} = 0,$$

where Ω^{ir}_{sj}, Ω^{*ir}_{sj} are the Obata's operators of the metric structure $g_{ij}(x, y^{(1)}, y^{(2)})$:

$$(2.8)\qquad \Omega^{ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j - g_{sj}g^{ir}),\ \Omega^{*ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j + g_{sj}g^{ir}),$$

and

$$(2.9)\qquad \sigma_{rk} = +\sigma_{r|k} - \sigma_r\sigma_k + \frac{1}{2}g_{rk}\sigma - \frac{\sigma_r T_{(0)k}}{n-1},\ (\sigma = g^{rs}\sigma_r\sigma_s),$$

$$(2.10)\qquad \begin{aligned} \rho_{(\alpha)rk} &= +\sigma_r|^{(\alpha)}_k + \tau_{(\alpha)r|k} - (\sigma_r\tau_{(\alpha)k} + \sigma_k\tau_{(\alpha)r}) + g_{rk}\rho_{(\alpha)} - \\ &- \tfrac{\tau_{(\alpha)r}T_{(0)k} + \sigma_r S_{(\alpha)k}}{n-1}, \quad (\rho_{(\alpha)} = g^{rs}\tau_{(\alpha)r}\sigma_s), \quad (\alpha = 1,2), \end{aligned}$$

$$\begin{aligned}(2.11)\qquad &\tau_{(\alpha)rk} = +\tau_{(\alpha)r}|^{(\alpha)}_{k} - \tau_{(\alpha)r}\tau_{(\alpha)k} + \tfrac{1}{2}g_{rk}\tau_{(\alpha)} - \tfrac{\tau_{(\alpha)r}S_{(\alpha)k}}{n-1},\\ &(\tau_{(\alpha)} = g^{rs}\tau_{(\alpha)r}\tau_{(\alpha)s}), \quad (\alpha = 1,2).\end{aligned}$$

Using these results we can determine some invariants of the group $\overset{s}{\mathcal{T}}_N$. To this aim we eliminate σ_{ij}, $\rho_{(\alpha)ij}$, $\tau_{(2)ij}$ from (2.5), (2.6), (2.7).

Theorem 2.4. *Let $n > 2$. The metric semi-symmetric N-linear connection determines the following tensor fields*

$$(2.12)\qquad H_{h\ jk}^{\ i} = K_{h\ jk}^{\ i} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ir}_{jh}(K_{rk} - \frac{Kg_{rk}}{2(n-1)})\},$$

$$(2.13)\qquad N_{(\alpha)h\ jk}^{\ \ \ \ i} = \mathcal{P}_{(\alpha)h\ jk}^{\ \ \ \ i} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ir}_{jh}(\mathcal{P}_{(\alpha)rk} - \frac{\mathcal{P}_{(\alpha)}g_{rk}}{2(n-1)})\},\ (\alpha = 1,2),$$

$$(2.14)\qquad M_{(22)h\ jk}^{\ \ \ \ \ i} = \mathcal{S}_{(22)h\ jk}^{\ \ \ \ \ i} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ir}_{jh}(\mathcal{S}_{(22)rk} - \frac{\mathcal{S}_{(22)}g_{rk}}{2(n-1)})\},$$

where

$$\begin{aligned}&K_{hj} = K_{h\ ji}^{\ i},\ \mathcal{P}_{(\alpha)hj} = \mathcal{P}_{(\alpha)h\ ji}^{\ \ \ \ i},\ \mathcal{S}_{(22)hj} = \mathcal{S}_{(22)h\ ji}^{\ \ \ \ \ i},\ K = g^{hj}K_{hj},\\ &\mathcal{P}_{(\alpha)} = g^{hj}\mathcal{P}_{(\alpha)hj},\ \mathcal{S}_{(22)} = g^{hj}\mathcal{S}_{(22)hj},\ (\alpha = 1,2).\end{aligned}$$

These tensor fields are invariants of the group $\overset{s}{\mathcal{T}}_N$.

It follows to study same other invariants and their properties.

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References

[1] Gh.Atanasiu, *The Equations of Structure of an N- linear Connection in the Bundle of Accelerations*, Balkan Journal of Geometry and Its Applications,1, 1 (1996), 11-19.

[2] R.Miron and Gh.Atanasiu, *Lagrange Geometry of Second Order*, Math. Comput. Modelling, Pergamon, 20, 4/5(1994), 41-56.

[3] R.Miron and Gh.Atanasiu, *Differential geometry of the k-osculator bundle*, Rev. Roumaine Math. Pures Appl., 41, 3/4 (1996), 205-236.

[4] R.Miron and Gh.Atanasiu, *Prolongation of Riemannian, Finslerian and Lagrangian structures*, Rev. Roumaine Math. Pures Appl., 41, 3/4 (1996), 237-249.

[5] R.Miron and Gh.Atanasiu, *Higher-order Lagrange spaces*, Rev. Roumaine Math. Pures Appl., 41, 3/4 (1996), 251-262.

[6] R.Miron and Gh.Atanasiu, *Compendium sur les espaces Lagrange d'ordre supérior*, Seminarul de Mecanica 40, Universitatea din Timişoara, 1994.

[7] R.Miron and M.Hashiguchi, *Metrical Finsler Connections*, Rep. Fac. Sci., Kagoshima Univ (Math., Phys. & Chem.), 12 (1979), 21-35.

[8] R.Miron, *The Geometry of Higher-Order Lagrange Spaces*, Kluwer Academic Publishers, 1996.

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