On the Structure Equations of the Induced Linear Connections

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Abstract

The aim of this paper is to write the equations of Gauss-Weingarten and Gauss-Codazzi in the case of induced linear connections on two supplementary vector subbundles. Their forms are different from those given by V. Crucianu, since the exterior calculus formalism is used, but they are very closed to the similar equations deduced by R. Miron and M. Anastasiei for induced tangential and normal connections on vector subbundles, by d-connections.

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1 Linear connections which are induced on supplementary vector subbundles

Consider as in [1] a vector bundle $\xi = (E, \pi, M)$ and two supplementary vector subbundles $\xi' = (E', \pi', M)$ and $\xi'' = (E'', \pi'', M)$. Denote as $n = \dim M$, $m = m_1 + m_2$, where m, m_1 and m_2 are the dimensions of the fibres of ξ, ξ' and ξ''. Denote also by $P' : \xi \to \xi'$ and $P'' : \xi \to \xi''$ the canonical projections. Every connection ∇ on ξ induces by $\nabla^1 = P' \circ \nabla$ and $\nabla^2 = P'' \circ \nabla$ two linear connections on ξ', respectively ξ'', which we call in that follows as *induced connections* .

Consider now $B^1 : \mathcal{X}(M) \times S(\xi') \to S(\xi'')$ and $B^2 : \mathcal{X}(M) \times S(\xi'') \to S(\xi')$ defined by

(1) $$B^1(X, s') = \nabla_X s' - \nabla^1_X s' \ , \quad B^2(X, s'') = \nabla_X s'' - \nabla^2_X s'' \ .$$

The following two interpretations can be given for B^1 and B^2:

- as fundamental forms of second kind, of the supplementary vector subbundles ξ' and ξ'';
- as covariant derivatives of the inclusions $I' : \xi' \to \xi$ and $I'' : \xi'' \to \xi$, considering the pairs of linear connections (∇^1, ∇) respectively (∇^2, ∇).

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Generally, the above pairs of linear connections can define a covariant derivative of mixt tensors, which extends the case of [2, pg. 149], [3, pg. 74]. The *mixt tensors* are elements of the $\mathcal{F}(M)$-tensor products of the $\mathcal{F}(M)$-modules $\mathcal{X}(M)$, $\mathcal{X}^*(M)$, $S(\xi)$, $S(\xi^*)$, $S(\xi')$, $S(\xi'^*)$, $S(\xi'')$ and $S(\xi''^*)$, where ξ^* denote the dual vector bundle of ξ. A *covariant derivative* D of mixt tensors is an internal map of mixt tensors which is a local map, an $I\!R$-linear map, commutes with contractions and is a derivation related to the tensor product.

Proposition 1.1 *Let $\bar{\nabla}$ be a linear connection on M, and ξ, ξ', ξ'', ∇, P', P'', ∇^1 be ∇^2 as above. Then there is only one covariant derivative D of mixt tensors, which has the same action as $\bar{\nabla}$, ∇, ∇^1 and ∇^2 on the sections of τM, ξ, ξ' and ξ'' respectively.*

Proof. The covariant derivatives $\bar{\nabla}$, ∇, ∇^1 and ∇^2 define covariant derivatives on the sections of the dual vector bundles $\mathcal{X}^*(M)$, $S(\xi^*)$, $S(\xi'^*)$and $S(\xi''^*)$ respectively. For example for ξ: from the condition that ∇ commutes with contractions, if $\omega \in S(\xi^*)$ then $(\forall) X \in \mathcal{X}(M)$, $s \in S(\xi)$, we have

$$\left(\nabla_X \omega\right)(s) = X\left(\omega(s)\right) - \omega\left(\nabla_X s\right).$$

Let us impose the conditions on D to have the same action as $\bar{\nabla}$, ∇, ∇^1 and ∇^2 on the sections of τM, ξ, ξ' and ξ'' respectively. It follows that D has the same action $\bar{\nabla}$, ∇, ∇^1 and ∇^2 on the sections of $\tau^* M$, ξ^*, ξ'^* and ξ''^*, since it commutes with contractions. The action of D extends to every mixt tensor using the Leibniz condition on the tensor product. q.e.d.

An example is given by the covariant derivatives of the inclusions $I' : \xi' \to \xi$ and $I'' : \xi'' \to \xi$, which can be regarded (via canonical izomorphisms) as belonging to $S(\xi'^*) \otimes_{\mathcal{F}(M)} S(\xi)$ and $S(\xi''^*) \otimes_{\mathcal{F}(M)} S(\xi)$ respectively. The vectorial form is given by the formulas (1) above.

A very efficient and explicit approach of mixt tensors and mixt covariant derivative is given in [2] or [3], where a local calculus is used.

Consider now a local base $\{s_A\}_{A\in\overline{1,m}}$ of the local sections on ξ such that $\{s_a\}_{a\in\overline{1,m_1}}$ is a local base of the sections on ξ'. Then P' has a local form

$$P'(S^a s_a + S^{\bar{a}} s_{\bar{b}}) = \left(S^a + P^a_{\bar{a}} S^{\bar{a}}\right) s_a \left(= S^A P^a_A s_a\right) \tag{2}$$

and thus P'' has the local form

$$P''(S^a s_a + S^{\bar{a}} s_{\bar{b}}) = S^{\bar{a}}\left(s_{\bar{a}} - P^a_{\bar{a}} s_a\right). \tag{3}$$

It is easy to see that if $\{s_{\bar{a}}\}_{\bar{a}\in\overline{1,m_2}}$ is a local base of the sections on ξ'', then the local functions $P^a_{\bar{a}}$ vanish. We can get to this situation taking instead of $\{s_{\bar{a}}\}_{\bar{a}\in\overline{1,m_2}}$ the sections $\{\mathbf{s}_{\bar{a}}\}_{\bar{a}\in\overline{1,m_2}}$, where

$$\mathbf{s}_{\bar{a}} = s_{\bar{a}} - P^a_{\bar{a}} s_a \ .$$

Let us denote as $\{\Gamma^A_{iC}\}_{A,C\in\overline{1,m},a=\overline{1,n}}$ the components of the connection ∇ taken in local bases $\{s_A\}_{A\in\overline{1,m}}$ of sections on ξ and $\{X_i\}_{i\in\overline{1,n}}$ of fields on M: $\Gamma^A_{iC} s_A = \nabla_{X_i} s_C$. Consider the dual base of forms $\left\{\theta^i\right\}_{i\in\overline{1,n}}$ and the local connection forms $\omega^A_C = \Gamma^A_{iC}\theta^i$.

We can also consider the components $\left\{\bar{\Gamma}^a_{ib}\right\}$ and $\left\{\bar{\Gamma}^{\bar{a}}_{i\bar{b}}\right\}$ of the linear connections ∇^1 and ∇^2, given by the conditions $\bar{\Gamma}^a_{ib} s_a = \nabla^1_{X_i} s_b$ and $\bar{\Gamma}^{\bar{a}}_{i\bar{b}} \mathbf{s}_{\bar{a}} = \nabla^1_{X_i} \mathbf{s}_{\bar{b}}$. The local connection forms are $\bar{\omega}^a_c = \bar{\Gamma}^a_{ic}\theta^i$ şi $\bar{\omega}^{\bar{a}}_{\bar{c}} = \bar{\Gamma}^{\bar{a}}_{i\bar{c}}\theta^i$.

Proposition 1.2 *There are the following relations between the connection forms:*

$$\bar{\omega}^a_b = \omega^a_b + P^a_{\bar{a}}\omega^{\bar{a}}_b \left(= P^a_A\omega^A_b\right), \ \bar{\omega}^{\bar{a}}_{\bar{b}} = \omega^{\bar{a}}_{\bar{b}} - P^a_{\bar{b}}\omega^{\bar{a}}_a \left(= P^A_{\bar{b}}\omega^{\bar{a}}_A\right) \ . \tag{4}$$

Proof. We use the definition of ∇^1 and ∇^2, and the local forms (2) and (3) of P' and P''. From

$$\bar{\Gamma}^a_{ib}s_a = \nabla^1_{X_i}s_b = P'\left(\nabla_{X_i}s_b\right) = P'\left(\Gamma^A_{ib}s_A\right) \tag{5}$$

we have $\bar{\Gamma}^a_{ib} = \Gamma^a_{ib} + P^a_{\bar{a}}\Gamma^{\bar{a}}_{ib}$, which proves the first formula. From

$$\bar{\Gamma}^{\bar{a}}_{i\bar{b}}\mathbf{s}_{\bar{a}} = \nabla^1_{X_i}\mathbf{s}_{\bar{b}} = P''(\nabla_{X_i}(s_{\bar{b}} - P^c_{\bar{b}}s_c)) = P''\left(\Gamma^A_{i\bar{b}}s_A - X_i(P^c_{\bar{b}})s_c - P^c_{\bar{b}}\nabla_{X_i}s_c\right) \tag{6}$$

we have $\bar{\Gamma}^{\bar{a}}_{i\bar{b}} = \Gamma^{\bar{a}}_{i\bar{b}} - P^a_{\bar{b}}\Gamma^{\bar{a}}_{ia}$ which proves the second formula. q.e.d.

The local forms of B^1 and B^2 are

$$B^1(X_i, A^a s_a) = P''\left(\nabla_{X_i}(A^a s_a)\right) = A^a\Gamma^{\bar{a}}_{ia}\mathbf{s}_{\bar{a}} \left(= A^a B^{\bar{a}}_{ia}s_{\bar{a}}\right),$$

and

$$B^2(X_i, A^{\bar{a}}\mathbf{s}_{\bar{a}}) = P'\left(\nabla_{X_i}(A^{\bar{a}}\mathbf{s}_{\bar{a}})\right) =$$

$$A^{\bar{a}}\left(\Gamma^a_{i\bar{a}} + P^a_{\bar{b}}\Gamma^{\bar{b}}_{i\bar{a}} - X_i(P^a_{\bar{a}}) - P^c_{\bar{a}}\Gamma^a_{ic} - P^c_{\bar{a}}P^c_{\bar{b}}\Gamma^{\bar{b}}_{ic}\right)s_a \left(= A^{\bar{a}}B^a_{i\bar{a}}s_a\right).$$

Consider now the local forms

$$B^{\bar{a}}_a = B^{\bar{a}}_{ia}\theta^i \ , \quad B^a_{\bar{a}} = B^a_{i\bar{a}}\theta^i.$$

The above relations give

Proposition 1.3 *The following relations hold true:*

$$B^{\bar{a}}_a = \omega^{\bar{a}}_a \, , \ B^a_{\bar{a}} = \omega^a_{\bar{a}} + P^a_{\bar{b}}\omega^{\bar{b}}_{\bar{a}} - d(P^a_{\bar{a}}) - P^c_{\bar{a}}\omega^a_c - P^c_{\bar{a}}P^a_{\bar{b}}\omega^{\bar{b}}_c \ . \tag{7}$$

Let us prove the first main formula.

Proposition 1.4 *(*Gauss-Weingarten formulas*) The covariant derivatives of the inclusions I' and I'' are given by the formulas*

$$\left(DI'\right)(s_a) = B^{\bar{\alpha}}_{\alpha}\mathbf{s}_{\bar{a}} \ , \ \left(DI''\right)(\mathbf{s}_{\bar{a}}) = B^a_{\bar{a}}s_a.$$

Proof. Using the definitions of B^1 and B^2 we have

$$\left(D_{X_i}I'\right)(s_a) = \nabla_{X_i}I'(s_a) - I'\left(\nabla^1_{X_i}s_a\right) = B^{\bar{a}}_{ia}\mathbf{s}_{\bar{a}}$$

$$\left(D_{X_i}I''\right)(\mathbf{s}_{\bar{a}}) = \nabla_{X_i}I''(\mathbf{s}_{\bar{a}}) - I''\left(\nabla^2_{X_i}\mathbf{s}_{\bar{a}}\right) = B^a_{i\bar{a}}s_a$$

which proves the both relations. q.e.d.

The Cartan structure equations of the linear connections ∇, ∇^1 and ∇^2 are

$$d\omega^A_C + \omega^A_B\wedge\omega^B_C = \Omega^A_C \ , \ d\omega^a_c + \omega^a_b\wedge\omega^b_c = \Omega^a_c \ , \ d\omega^{\bar{a}}_{\bar{c}} + \omega^{\bar{a}}_{\bar{b}}\wedge\omega^{\bar{b}}_{\bar{c}} = \Omega^{\bar{a}}_{\bar{c}}. \tag{8}$$

Theorem 1.1 *(*Gauss-Codazzi formulas*) The following relations hold true:*

$$\Omega^c_b + P^c_{\bar{b}}\Omega^{\bar{b}}_b = \bar{\Omega}^c_b + B^c_{\bar{b}}\wedge B^{\bar{b}}_b \ , \ \Omega^{\bar{c}}_{\bar{b}} - P^b_{\bar{b}}\Omega^{\bar{c}}_b = \bar{\Omega}^{\bar{c}}_{\bar{b}} + B^{\bar{c}}_b\wedge B^b_{\bar{b}} \tag{9}$$

$$\Omega^c_{\bar{c}} + P^b_{\bar{c}}\Omega^c_b = dB^c_{\bar{c}} - P^b_{\bar{c}}P^c_{\bar{b}}\Omega^{\bar{b}}_b - \bar{\omega}^c_b\wedge B^b_{\bar{c}} - B^c_{\bar{b}}\wedge\bar{\omega}^{\bar{b}}_{\bar{c}} \tag{10}$$

Proof. The check of the formulas is made by strightforward and long computations.

In order to prove the first formula (9) we differentiate (4), which gives $\omega^c_{\bar b}$, and using the formulas (7), which gives $B^a_{\bar b}$, and the second structure equation (8) of the induced connection. We get

$$d\omega^c_b = d\bar{\omega}^c_b - dP^c_{\bar a} \wedge \omega^{\bar a}_b - P^c_{\bar a} d\omega^{\bar a}_b$$

or

$$\Omega^c_b - \omega^c_a \wedge \omega^a_b - \omega^c_{\bar a} \wedge \omega^{\bar a}_b = \bar{\Omega}^c_b - \left(\omega^c_a + P^c_{\bar d}\omega^{\bar d}_a\right) \wedge \left(\omega^a_b + P^a_{\bar e}\omega^{\bar e}_b\right) -$$
$$- \left(B^c_{\bar d} \wedge \omega^{\bar d}_b + \omega^c_{\bar d} \wedge \omega^{\bar d}_b + P^c_{\bar d}\omega^{\bar d}_{\bar q} \wedge \omega^{\bar q}_b - P^d_{\bar b}\left(\omega^c_d \wedge \omega^{\bar b}_b + P^c_{\bar d}\omega^{\bar d}_d \wedge \omega^{\bar b}_b\right)\right) -$$
$$-P^c_{\bar b}\left(\Omega^{\bar b}_c - \omega^{\bar b}_d \wedge \omega^d_b - \omega^{\bar b}_{\bar d} \wedge \omega^{\bar d}_b\right).$$

Reducing we obtain the first formula (9).

The second formula (9) can be obtained in a similar way. We differentiate the formula (4), which gives $\bar{\omega}^{\bar c}_{\bar b}$, we use the formula (7), which give $B^a_{\bar b}$, and the third equation (8) of the induced connection.

In order to prove formula (10) we differentiate the functions (7),

$$dB^c_{\bar c} = d\omega^c_{\bar c} + P^c_{\bar b} \wedge \omega^{\bar b}_{\bar c} + P^c_{\bar b} d\omega^{\bar b}_{\bar c} - P^b_{\bar c} \wedge \left(\omega^c_b + P^c_{\bar b}\omega^{\bar b}_b\right) - P^b_{\bar c}\left(d\omega^c_b + dP^c_{\bar b} \wedge \omega^{\bar b}_b + P^c_{\bar b} d\omega^{\bar b}_b\right).$$

If we use again the formula (7), which gives us the differential $dP^k_{\bar j}$, and the structure equations of the induced connections, by simplifications, we obtain the formula (10). q.e.d.

Remark 1.1 *The Gauss-Codazzi equation are obtained in a global form in [1]. The above form is very closed to [3], where the tangent and normal connections are induced on vector subbundles by metric d-connections.*

2 Metric induced connections

Let G be a metric and ∇ be a linear connection on ξ. The connection ∇ is *metric* if $\nabla G = 0$, or

$$\nabla_X G(s_1, s_2) = G(\nabla_X s_1, s_2) + G(s_1, \nabla_X s_2),$$

$(\forall) X \in \mathcal{X}(M)$, $s_1, s_2 \in S(\xi)$.

Then G induces the non-degenerate metrics g and $\bar{g}$ on the suplementary vector subbundles ξ' and ξ'', which are orthogonal related to G. The connection ∇ induces the linear connections $\nabla^1 = P' \circ \nabla$ and $\nabla^2 = P'' \circ \nabla$ on ξ' and ξ'' respectively, as in the previous section.

Proposition 2.1 *If the linear connection ∇ is metric related to G, then the linear connections ∇^1 and ∇^2 are also metric related to g and g', respectively.*

Proof. Let $\{s_A\}_{A \in \overline{1,m}}$ be a local orthonormal base (related to G) of the local sections on ξ such that $\{s_a\}_{a \in \overline{1,m_1}}$ and $\{s_{\bar a}\}_{\bar a \in \overline{m_1+1,m}}$ are local orthonormal bases (related to g and $\bar{g}$ respectively) of the local sections on ξ' and ξ'' respectively. Consider a local base $\{X_i\}_{i\in\overline{1,n}}$ of vector fields on M and denote as $\{\Gamma^A_{iC}\}_{A,C\in\overline{1,m},a=\overline{1,n}}$ the local components of ∇. Thus we have $\Gamma^A_{iC} = G(s_A, \nabla_{X_i} s_C)$ and the condition on ∇ to be metric is is equivalent to the conditions $\Gamma^A_{iC} = -\Gamma^C_{iA}$. The linear connections ∇^1 and ∇^2 has the components $\{\Gamma^a_{ic}\}$ and $\{\Gamma^{\bar{a}}_{i\bar{c}}\}$ respectively. Since $\Gamma^a_{ic} = -\Gamma^c_{ia}$ and $\Gamma^{\bar{a}}_{i\bar{c}} = -\Gamma^{\bar{c}}_{i\bar{a}}$ it follows that the linear connections ∇^1 and ∇^2 are metric with respect to g and $\bar{g}$ respectively. q.e.d.

In the following we suppose that ∇ is metric with respect to G ande we use the local bases, not necessarly orthonormal ones, considered before the Proposition 2.1. Denote the components of the metrics G, g and $\bar{g}$:

$$G_{AB} = G(s_A, s_B)\ ,\ g_{ab} = g(s_a, s_b) = G(s_a, s_b),\ \bar{g}_{\bar{a}\bar{b}} = \bar{g}(\mathbf{s}_{\bar{a}}, \mathbf{s}_{\bar{b}}) = G(\mathbf{s}_{\bar{a}}, \mathbf{s}_{\bar{b}}).$$

The projection P' onto the fibres of the vector subbundle ξ' is given by the matrix with elements $P^a_A = g^{ab}G_{bA}$, thus $P^a_{\bar{a}} = g^{ab}G_{b\bar{a}}$.

Let us define now the curvature covariant tensor, which is a mixt tensor $\tilde{R} : S(\xi) \times S(\xi) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{F}(M)$,

$$\tilde{R}(S,T,X,Y) = G\left(R(Y,X)S,T\right),$$

where R is the curvature tensor of ∇. The components $R^{\ A}_{B ij}$ and R_{ABij} of the tensors R and $\tilde{R}$, are

$$R\left(X_i, X_j\right) s_B = R^A_{Bij} s_A\ ,\ \tilde{R}\left(s_A, s_B, X_i, X_j\right) = R_{ABij}.$$

Proposition 2.2 *1. The curvature covariant tensor $\tilde{R}$ is antisymmetric related to the pairs (S,T) and (X,Y).*

2. The following relations hold true: $R^A_{Bij} = R_{CBij}G^{CA} = -R^B_{Aij}$.

Proof. 1) The antisymmetry concerning (X,Y) is obvious. In order to prove antisymmetry concerning (S,T) the property $\nabla G = 0$ is used in the formula which defines $\tilde{R}$.

2) The first equality follows from the definition of $\tilde{R}$. The second equality follows from the first one and the the antisymmetry of tensor $\tilde{R}$ in arguments (S,T), proved in 1). q.e.d.

We consider now the mixt covariant tensors $\tilde{B}^1$ şi $\tilde{B}^2$, which correspond to B^1 and B^2,

$$\tilde{B}^1(X,s,t) = \bar{g}\left(B^1\left(X,s\right),t\right)\ ,\ \tilde{B}^2(X,t,s) = g\left(B^2\left(X,t\right),s\right),$$

$(\forall)\ s \in S(\xi')$ and $t \in S(\xi'')$.

We consider now the local base of sections on ξ given by $\{s_a\} \cup \{\mathbf{s}_{\bar{a}}\}$, where $\{s_a\}$ is a base of the sections on ξ', and $\{\mathbf{s}_{\bar{a}} = s_{\bar{a}} - P^a_{\bar{a}} s_a\}$ a base of the sections on ξ''. Denote as $(g_{ab} = G(s_a, s_b))$ the matrix of the metric g in the base $\{s_a\}$ and as$\{g_{\bar{a}\bar{b}}\}$ the matrix of the metric $\bar{g}$ in the base $\{\mathbf{s}_{\bar{a}} = s_{\bar{a}} - P^a_{\bar{a}} s_a\}$. We denote also $(g^{ab}) = (g_{ab})^{-1}$ and $(g^{\bar{a}\bar{b}}) = (g_{\bar{a}\bar{b}})^{-1}$.

The local coefficients of the mixt tensors B^1, B^2, $\tilde{B}^1$ and $\tilde{B}^2$ are

$$B^{\bar{a}}_{ib}\mathbf{s}_{\bar{a}} = B^1\left(X_i, s_b\right)\ ,\ B^a_{i\bar{b}} s_a = B^2\left(X_i, \mathbf{s}_{\bar{b}}\right)$$

$$B_{ib\bar{a}} = \tilde{B}^1(X_i, s_b, \mathbf{s}_{\bar{a}})\ ,\ B_{i\bar{b}a} = \tilde{B}^2(X_i, \mathbf{s}_{\bar{b}}, s_a)$$

Proposition 2.3 *The folowing relations hold true*

$$\tilde{B}^1(X,s,t) = -\tilde{B}^2(X,t,s),$$

(11) $$B^{\bar{a}}_{ib} = h^{\bar{a}\bar{b}} B_{ib\bar{b}}\ ,\ B^{a}_{i\bar{b}} = g^{ab} B_{i\bar{b}b}\ ,\ B_{ia\bar{b}} = -B_{i\bar{b}a}\ ,\ B^{\bar{a}}_{ib} = -g_{ba} B^{a}_{i\bar{b}} h^{\bar{b}\bar{a}}.$$

Proof. 1) For every $s \in S(\xi')$ and $t \in S(\xi'')$ we have

$$\tilde{B}^1(X,s,t) = \bar{g}\left(B^1\left(X,s\right),t\right) = G\left(\left(\nabla_X s - \nabla^1_X s\right),t\right) = G\left(\nabla_X s, t\right) =$$

$$-G\left(s, \nabla_X t\right) = G\left(s, \nabla_X t - \nabla^2_X t\right) = g\left(s, B^2\left(X,t\right)\right) = -\tilde{B}^2(X,t,s).$$

2) The first two equalities follow from the definitions of the tensors $\tilde{B}^1$ and $\tilde{B}^2$. The last two equalities follows from 1). q.e.d.

We notice that if $\{s_{\bar{a}}\}$ is a base of the sections on ξ'', then $P^a_{\bar{a}} = 0$ and the formulas (4) become

(12) $$\bar{\omega}^a_b = \omega^a_b\ ,\ \bar{\omega}^{\bar{a}}_{\bar{b}} = \omega^{\bar{a}}_{\bar{b}}\ .$$

In this case, using Proposition 2.2, the formulas (7) become

$$B^{\bar{a}}_a = \omega^{\bar{a}}_a = -B^a_{\bar{a}} = -\omega^a_{\bar{a}}\ .$$

Theorem 1.1 can pe restated as follows (compare with [3, Theorem 5.1, pg. 73]):

Theorem 2.1 *(*Gauss-Codazzi formulas*) The following relations hold true:*

(13) $$\Omega_{bc} + P^c_{\bar{b}} g^{\bar{b}\bar{c}} \Omega_{b\bar{c}} = \bar{\Omega}_{bc} + B_{\bar{b}c} \wedge B^{\bar{b}}_b\ ,\ \Omega_{\bar{b}\bar{c}} - P^b_{\bar{b}} \Omega_{b\bar{c}} = \bar{\Omega}_{\bar{b}\bar{c}} + B_{b\bar{c}} \wedge B^b_{\bar{b}}$$

(14) $$\Omega_{\bar{c}c} + P^b_{\bar{c}} \Omega_{bc} = g_{cd}\left(dB^d_{\bar{c}} - P^b_{\bar{c}} P^d_{\bar{b}} g^{\bar{b}\bar{d}} \Omega_{b\bar{d}} - \bar{\omega}^d_b \wedge B^b_{\bar{c}} - B^d_{\bar{b}} \wedge \bar{\omega}^{\bar{b}}_{\bar{c}}\right).$$

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