On General Lipschitzian Kernels

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Abstract

Using an entended version of the Ionescu Tulcea-Marinescu ergodic theorem we give a refined spectral analysis of general lipschitzian kernels.

1 Introduction

The aim of this paper is to obtain some spectral properties of general lipschitzian kernels. To state these properties we use a basic tool which consists in the extended version of the Ionescu Tulcea-Marinescu ergodic theorem. We also bring into evidence that the contraction coefficients play a key role in the study of lipschitzian kernels. The impact of these coefficients being so noted, the spectral radius of the transition operator in the Banach space of continuous functions is established. In Section 3 we obtain a refined spectral analysis of general lipschitzian kernels. The main result of this paper enables us to conclude that the transition operator is quasi-compact on the Banach space of Lipschitz functions and moreover to obtain some informations about the spectral radius, respectively the essential spectral radius of this operator.

2 Preliminaries

Let (W, d) be a compact metric space. For any function $f \in C(W) =$ the collection of all bounded continuous complex-valued functions on W, set

$$| f |_\infty = \sup\{| f(w) | \mid w \in W\},$$

$$s(f) = \sup \left\{ \frac{|f(w) - f(w')|}{d(w, w')} \mid w, w' \in W, \quad w \neq w' \right\},$$

$$\|f\| = | f |_\infty + s(f).$$

Let $L(W) = \{f \mid f \in C(W), \quad \|f\| < \infty\}$ be the collection of all bounded Lipschitz complex-valued functions on W. As is well known, $C(W)$ and $L(W)$ are Banach spaces under $| \ |$ and $\| \ \|$, respectively.

Let $\{(W,\mathcal{W}),\ (\mathcal{X},\mathcal{X}),\ \sqcap,\mu\}$ be a quadruple, where

    1) $(W,\mathcal{W})$ and $(X,\mathcal{X})$ are arbitrary measurable spaces;

    2) $u:W\times X\to W$ is a $(\mathcal{W}\otimes\mathcal{X},\mathcal{W})$ - measurable map;

    3) μ is a transition measure from $(W,\mathcal{W})$ to $(X,\mathcal{X})$ i.e.

        $\forall w\in W\ \mu(w,\ )$ is a finite positive measure on $(X,\mathcal{X})$

        $\forall A\in\mathcal{X}\ \mu(\ ,\mathcal{A})$ is a measurable function on $(W,\mathcal{W})$.

We also assume that

$$\sup\{|\ \mu(\ ,A)\mid_\infty|\ A\in\mathcal{X}\}<\infty,$$

$$\sup\{s(\mu(\ ,A))\mid A\in\mathcal{X}\}<\infty.$$

We denote by $x^{(n)}$ the element $(x_1,...,x_n)\in X^n$. For any $n\in\mathbf{N}^\star$, let us define recursively the maps $u^{(n)}:W\times X^n\to W$ by the equation

$$u^{(n+1)}(w,x^{(n+1)})=\left\{\begin{array}{ll} u(w,x_1),&\text{if}\quad n=0;\\[2mm] u(u^{(n)}(w,x^{(n)}),x_{n+1}),&\text{if}\quad n\in\mathbf{N}^\star.\end{array}\right.$$

We shall simply write $wx^{(n)}$ for $u^{(n)}(w,x^{(n)})$, whenever no confusion is possible. Using this convention, the above equation becomes

$$wx^{(n+1)}=\left\{\begin{array}{ll} wx_1,&\text{if}\quad n=0;\\[2mm] (wx^{(n)})x_{n+1},&\text{if}\quad n\in\mathbf{N}^\star.\end{array}\right.$$

For every $w\in W$, $n\in\mathbf{N}^\star$ and $A\in\mathcal{X}^{\backslash}$ let us define

$$\mu_n(w,A)=\left\{\begin{array}{ll} \mu(w,A),&\text{if}\quad n=1;\\[2mm] \prod_{i=0}^{n-1}\int_X\mu(wx^{(i)},dx_{i+1})\mathcal{X}_\mathcal{A}(\S^{(\backslash)}),&\text{if}\quad \backslash>\infty.\end{array}\right.$$

It is obvious that, for $n\in\mathbf{N}^\star$ fixed, μ_n is a transition measure from $(W,\mathcal{W})$ to $(X^n,\mathcal{X}^{\backslash})$.

3   Spectral Properties

Let $B(W,\mathcal{W})$ be the Banach space of bounded $\mathcal{W}$-measurable complex-valued functions f on W, under the supremum norm.

Let us define the transition operator $U:B(W,\mathcal{W})\to\mathcal{B}(\mathcal{W},\mathcal{W})$,

$$Uf(w)=\int_X f(wx)\mu(w,dx).$$

The iterates of the operator U are given by

$$U^n f(w)=\int_{X^n}f(wx^{(n)})\mu_n(w,dx^{(n)}),\quad n\in N^\star.$$

Let us define the contraction coefficient, $c(U^n)$, of the operator U^n by

$$\sup\left\{\int_{X^n}\frac{d(wx^{(n)},w'x^{(n)})}{d(w,w')}\mu_n(w,dx^{(n)})\mid\ w,w'\in W,\ w\neq w'\right\}.$$

Taking into account that $c(U^{m+n}) \leq c(U^m)c(U^n)$, let us put

$$\bar{c}(U) = \lim_{n \to \infty} c(U^n)^{1/n}.$$

U acts on $C(W)$ and $L(W)$ with spectral radius $\tilde{r}(U)$ and $r(U)$ respectively. Clearly, we obtain

$$\tilde{r}(U) = \lim_{n \to \infty} \mid U^n \mid_\infty^{1/n} = \lim_{n \to \infty} \mid U^n 1 \mid_\infty^{1/n} .$$

But the aim of this paper is to give a more refined spectral analysis of U. So, we also consider the essential spectral radius $r_{ess}(U)$ of U. In the sequel we shall obtain deeper properties of U using an extended version of the Ionescu Tulcea-Marinescu ergodic theorem.

Let us recall this theorem.

Theorem [3], [5]. *If a bounded linear operator U on a Banach space Y with spectral radius $r(U)$ and essential spectral radius $r_{ess}(U)$ has the Doeblin-Fortet property $DF(r)$, where $r \geq 0$, i.e.,*

1. $U : (Y, \| \ \|) \to (Y, | \ |)$ is a compact operator

2. $\forall n \in N^\star$, $\exists R_n$, $r_n \in R_+$ such that

$$\lim_{n \to \infty} \inf (r_n)^{1/n} = r < r(U)$$

$$\|U^n y\| \leq r_n \|y\| + R_n \mid y \mid, \quad \forall y \in Y,$$

then $r_{ess}(U) \leq r < r(U)$. Moreover $U/r(U)$ is a quasi-compact operator on $(Y, \| \ \|)$.

The above theorem allows us to study the action of the operator U on the class of Lipschitz functions and to obtain a much more precise result.

Theorem. *If $c(U) < \infty$, then the operator U acts on $C(W)$ and $L(W)$ and has on $C(W)$ the spectral radius*

$$\tilde{r}(U) = \lim_{n \to \infty} \mid U^n 1 \mid_\infty^{1/n} .$$

If $\bar{c}(U) < \tilde{r}(U)$, then $U/r(U)$ is a quasi-compact operator on $L(W)$, $r_{ess}(U) \leq \bar{c}(U)$, $r(U) = \tilde{r}(U)$, $r(U)$ is an eigenvalue of U of maximal order between the eigenvalues of modulus $r(U)$ and the corresponding eigenfunction is positive.

Moreover, if U is an irreducible operator, i.e. for any $w \in W$ and for any positive and non-zero function $f \in C(W)$, there exists an integer $n \geq 1$ such that $U^n f(w) > 0$, then $r(U)$ is the only eigenvalue of modulus $r(U)$ and the corresponding eigenspace is generated by a strict positive function.

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