On Closed Concircular Almost Contact Riemannian Manifolds

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Abstract

A class of almost contact Riemannian manifolds (ACRM) (which we call closed concircular ACRM) is defined and studied, showing that such a manifold is a local product and obtaining some important harmonic properties.

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Introduction

Almost contact structures are, in some sense, the odd-dimensional analogous to almost complex structures. In fact, if M is an almost contact manifold, it is easily seen that $M \times \mathbf{R}$ become an almost complex manifold and this property has been used by Oubiña [7] to obtain a classification of almost contact metric manifolds. Moreover, almost contact and almost complex structures are the simplest examples of a f-structure (see e.g. [14]).

Additional structures on an almost contact manifold have been introduced in the last decades in several papers (see e.g [6]). In our work we study the geometry of an almost contact Riemannian manifold when its Reeb vector field ξ satisfy two properties: it is concircular and $U^\flat$ is closed, where $U = \nabla_\xi \xi$.

Now we intoduce some notation and explain the main results of the paper. Let (M, ϕ, η, ξ, g) be almost contact Riemannian manifold, where the structure tensors $(\phi, \Omega, \eta, \xi)$ mean the tensor field, the 2-form, the 1-form and its dual $\eta^\sharp = \xi$, respectively. One has

$$\phi^2 = -Id + \eta \otimes \xi, \eta \wedge \Omega^m \neq 0;$$
$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y);$$

for any vector fields X, Y on M, where $\Omega(X,Y) = g(\phi X, Y)$.

We assume in this paper that ξ is U-concircular, i.e.,

$$\nabla \xi = \eta \otimes U$$

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where $U \in Ker\eta$ will be called the generative of ξ and that $U^\flat$ is a closed 1-form. In this case we find

$$d\Omega = 0, d\eta = U^\flat \wedge \eta \neq 0, \eta \wedge d\eta = 0,$$

which implies that M is not a cosymplectic nor a contact manifold.

In section 2 we prove important properties of η, Ω and U. In the last part of this section we assume that the manifold satisfies a technical condition and we prove, among other properties, the following structure equations

(i) $\mathcal{L}_U\eta = \|U\|^2\eta$; $\mathcal{L}_U\Omega = 2\lambda\Omega$; $\lambda = const.$
i.e., U defines an infinitesimal conformal transformation of the almost contact Riemannian structure defined on M.

(ii) Any such M , may be viewed as the locally product of the integral submanifolds of $D_U = Span\{U, \xi\}$ and $D_U^\perp$.

(iii) The transversal curvature forms Θ_0^A ($A \in \{1, \ldots, 2m\}$) are monomial and they define a closed vector valued 2-form

$$\Pi = \sum \Theta_0^A \otimes e_A = \lambda\eta \wedge I + (U^\flat \wedge \eta) \otimes U.$$

In section 3 one afirms that $U^\flat$ and η are eigenfunctions of Δ and if

$$\eta_q = \eta \wedge \Omega^q,$$

then the Lie derivatives with respect to ξ are expressed by

$$\mathcal{L}_\xi\eta_q = -U^\flat \wedge \Omega^q.$$

In the last section a vector field X such that

$$\nabla X = f\phi dp - g(X, U)\eta \otimes \xi$$

is defined as a *ϕ-soldering vector field* (abr. ϕs). If M carries such a vector field, then Ω is invariant by X, and it moves to a *potential form* and M is foliated by 3-dimensional submanifolds M_X tangent to $Span\{\phi X, \xi, U\}$.

It is also proved that ϕX defines an infinitesimal homothety of Ω and both X and ϕX define infinitesimal conformal transformations of ξ.

1 Preliminaries and Notations

First, we shall explain the notation used in the paper.

Let (M, g) be a n-dimensional connected Riemannian manifold and let ∇ be the covariant differential operator defined by the metric tensor g (we assume that M is oriented and ∇ is the Levi-Civita connection).

We denote by $\mathcal{F}(M)$ the ring of real functions, $\mathcal{T}_q^p(M)$ the $\mathcal{F}(M)$-module of (p, q) tensor fields, TM (resp T^*M) the tangent (resp. cotangent) bundle of M and $\bigwedge^r(M)$ the $\mathcal{F}(M)$-module of r-forms, with $\bigwedge M = \oplus_{r=0}^n \bigwedge^r(M)$, $\mathcal{X}M$ the set of sections of the tangent bundle TM and

$$\flat : TM \to T^*M, \sharp : T^*M \to TM$$

the *musical isomorphisms* defined by g.

If $X_i \in \mathcal{X}M$ are vector fields, $Span\{X_1, \ldots, X_k\}$ is the distribution spanned by them.

We denote by $A^q(M,TM) = \Gamma Hom(\bigwedge^q TM, TM)$ the set of vector valued q-forms, and following [8] we write for the covariant derivative with respect to ∇, $d^\nabla : A^q(M,TM) \to A^{q+1}(M,TM)$. The vector valued 1-form $dp \in A^1(M,TM)$ is the identity vector valued 1-form and is called the soldering form of M [2]. As is well known, $A^1(M,TM)$ is isomorphic to $\mathcal{T}_1^1$, and the soldering form dp corresponds with the identity $(1,1)$ tensor field, which will be denoted as I. If $\phi \in \mathcal{T}_1^1$ then $\phi dp \in \mathcal{T}_1^1$ and one has the left inner product $< X, \phi dp >= \phi X =< \phi X, dp >$. As ∇ is a symmetric connection one obtains $d^\nabla(dp) = 0$.

We write for T (resp. R) the torsion (resp. curvature) tensor of ∇. Then, for any vector field Z on M , the second covariant differential is defined as $\nabla^2 Z = d^\nabla(\nabla Z)$ and satisfies

$$\nabla^2 Z(V,W) = R(V,W)Z; \quad V, W \in \mathcal{X}M \tag{1.1}$$

The operator $d^\omega = d + e(\omega)$ acting on ΛM, where $e(\omega)$ means the exterior product by the closed 1-form ω is called the *cohomology operator* [3]. One has $d^\omega \circ d^\omega = 0$ and if $d^\omega \pi = 0$, π is said to be d^ω-*closed*. If ω is exact, then π is said to be d^ω-*exact*.

Let $\omega \in \bigwedge^1(M)$. If $\omega \neq 0, \omega = \wedge d\omega = 0$, then ω is called a *class* 1 1-form.

If M is endowed with a 2-form Ω, then one can define the morphism

$$\Omega^\flat : T_x M \to T_x^* M$$

as $\Omega^\flat(Z) = {}^\flat Z = -i_Z \Omega; Z \in T_x = M$, for all $x \in M$, where

$$i_Z \Omega(X) = \Omega(Z, X); \forall X \in T_x M.$$

We shall denote by *grad* (resp. *div*) the gradient (resp. divergence) operator.

Let $G \in \bigwedge^{2m+1} M$ be the Riemannian volume element of M. We denote by $*$ the *Hodge star operator* of (M, g) with respect to G. We recall that the codifferential operator δ (see e.g. [8] p.153) is expressed by

$$\delta \pi = (-1)^{n(r+1)+1} * d * \pi, \quad \pi \in \bigwedge {}^r M.$$

Definition 1.1 [12] A function $f : R^n \to R$ is isoparametric if $\|gradf\|^2$ and *div* $(gradf)$ can be expressed as functions of f.

Now, we shall remember some basic results about almost contact manifolds (see e. g. [14], [6]).

Definition 1.2 A differentiable manifold M is said to have an almost contact structure if admits a vector field ξ (called the *Reeb vector field*), a 1-form η and a $(1,1)$ tensor field ϕ satisfiying

$$\begin{aligned} &\eta(\xi) = 1; \\ &\phi^2 = -I + \eta \otimes \xi \end{aligned} \tag{1.2}$$

The above conditions imply that M is odd dimensional, say $2m+1$, and $\phi(\xi) = 0$; $rank\phi = 2m$; $\eta \circ \phi = 0$. The distribution $ker\eta$ is called *horizontal* and $Span\{\xi\}$ is called *vertical*.

Definition 1.3 A manifold endowed with an almost contact structure (ϕ, ξ, η) and with a Riemannian metric g satisfying

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{1.3}$$

for any vector fields $X, Y \in \mathcal{X}M$ is *called almost conctact Riemannian* manifold.

If M is an almost contact Riemannian manifold, then the 2-form Ω defined by

$$\Omega(X,Y) = g(\phi X, Y); \quad X, Y \in \mathcal{X}M; \tag{1.4}$$

satisfies $\eta \wedge \Omega^m \neq 0$, which implies that M is orientable. Moreover, $rank\ \Omega = 2m$ and $i_\xi \Omega = 0$.

Note that, on an almost contact Riemannian manifold, one has $\eta(X) = g(X, \xi)$; $\forall X \in \mathcal{X}M$; i.e; $\xi = \eta^\sharp$. In the above conditions, it is easily seen that

$$g(Z, \phi Z') + g(\phi Z, Z') = 0; \quad \forall Z, Z' =\in \mathcal{X}M. \tag{1.5}$$

Remark 1.4 Let (M, ϕ, η, ξ, g) be an almost contact Riemannian manifold and $x \in M$. Then one can choose a local field of orthonormal frames over a neighborhood of x

$$\mathcal{O} = \{e_\mu, \mu = 0, 1, \ldots, 2m\} = \{\xi, e_1, \ldots, e_m, \phi e_1, \ldots, \phi e_m\}$$

and its associated coframe $\mathcal{O}^* = \{\omega^0, \omega^1, \ldots, \omega^m, \omega^{m+1}, \ldots, \omega^{2m}\}$, where $\omega^0 = \eta$. Then, with respect to this frame

$$\phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -I_m \\ 0 & I_m & 0 \end{pmatrix}$$

i.e., $\phi = -\omega^{a^*} \otimes e_a + \omega^a \otimes e_{a^*}$ where $a \in \{1, \ldots, m\}$ and $a^* = a + m$.

We shall use the capital (resp. greek) letters when the subindex runs in $\{1, \ldots 2m\}$ (resp $\{0, 1, \ldots 2m\}$).

Then one has

$$\Omega = \sum_{a=1}^{m} \omega^a \wedge \omega^{a^*}; \quad I = \eta \otimes \xi + \omega^A \otimes e_A.$$

Observe that: $X = \sum_{\mu=0}^{2m} X^\mu e_\mu \Rightarrow X^\flat = \sum_{\mu=0}^{2m} X^\mu \omega^\mu$.

With respect to $\mathcal{O}$ and $\mathcal{O}^*$ E. Cartan's structure equations can be written as

$$\nabla e_\lambda = \theta_\lambda^\mu \otimes e_\mu \tag{1.6}$$

$$d\omega^\lambda = -\theta_\mu^\lambda \wedge \omega^\mu \tag{1.7}$$

(1.8) $$d\theta^\lambda_\mu = -\theta^\gamma_\mu \wedge \theta^\lambda_\gamma + \Theta^\lambda_\mu$$

In the above equations θ (resp. Θ) are the local connection forms in the bundle $O(M)$ (resp. the curvature 2-forms on M).

From the orthonormality of the frame $\mathcal{O}$ and $\nabla g = 0$ one can deduce that the matrix

$$\theta = \begin{pmatrix} \theta^0_0 & \dots & \theta^0_{2m+1} \\ \vdots & \vdots & \vdots \\ \theta^{2m+1}_0 & \dots & \theta^{2m+1}_{2m+1} \end{pmatrix}$$

is hemisymmetric, i.e., $\theta^\lambda_\mu = -\theta^\mu_\lambda$. In particular $\theta^\mu_\mu = 0$.

2 Closed concircular almost contact Riemannian manifolds

We introduce the following notion:

Definition 2.1 Let (M, ϕ, η, ξ, g) be an almost contact Riemannian manifold. A vector field U is called the *generative* of ξ if U verifies the following two conditions

i) ξ is U-concircular, i.e;

(2.9) $$\nabla\xi = \eta \otimes U,$$

ii) $U^\flat$ is closed, i.e;

(2.10) $$dU^\flat = 0,$$

Then the manifold $(M, \phi, \eta, \xi, g, U)$ is called a *closed concircular almost contact Riemannian* manifold (abr. CCACR manifold).

Now we shall prove some equations and identities, which will be used in this work.

Lemma 2.2 *Let $(M, \phi, \eta, \xi, g, U)$ be a CCACR manifold. Then:*

(1) $U = \nabla_\xi \xi \in ker \eta$ and $\nabla_U \xi = 0$.

(2) $\theta^A_0 = U^A \eta; \theta^0_0 = 0$.

(3) the next Kähler relations are satisfied:

$$\theta^a_b = \theta^{a^*}_{b^*};\ \theta^{a^*}_b = \theta^{b^*}_a;\ a \in \{1, \dots m\};\ a^* = a + m.$$

(4) The covariant differential of any vector field Z on M is expressed by

$$\nabla Z = (dZ^A + Z^B \theta^A_B) \otimes e_A + (dZ^0 - g(Z, U) = \eta) \otimes \xi + Z^0 \eta \otimes U.$$

(5) For any vector field Z on M, we have

$$(\nabla\phi)Z = -\eta(Z)\eta \otimes \phi U + g(\phi U, Z)\eta \otimes \xi;$$

where $((\nabla\phi)Z)(X) = \phi\nabla_X Z - \nabla_X \phi Z; \forall X =\in \mathcal{X}M$.

Proof. (1) By (2.9) and (1.2)

$$(\nabla\xi)(\xi) = (\eta \otimes U)(\xi) \Rightarrow \nabla_\xi \xi = \eta(\xi)U = U.$$

Let X be a vector field on M. We have $0 = (\nabla_X g)(\xi,\xi) \Rightarrow g(\nabla_X, \xi) = 0$. Then $\nabla_X \xi \in Span\{\xi\}^\perp = ker\eta; \quad \forall X \in \mathcal{X}M$. In particular $U = \nabla_\xi \xi \in ker\eta$. Then by (2.9) one obtains $\nabla_U \xi = \eta(U)U = 0$.

(2) Set $U = U^A e_A$. Let X be a vector field on M. By 1) $\nabla_X \xi \in ker\eta$. One has $\nabla_X \xi = \theta_0^0(X)\xi + \theta_0^A(X)e_A$. Then $\theta_0^0(X) = 0; \forall X \in \mathcal{X}M$.

If $X \in ker\eta$, by (2.9) $\nabla_X \xi = 0$. Then $\theta_0^A(X) = 0; \forall X \in ker\eta$.

If $X = \xi$, $U = \theta_0^A(\xi)e_A = U^A e_A$.

From the above equations one can states $\theta_0^A = U^A \eta$.

(3) See e. g. [5], [13].

(4) Let $Z, X \in \mathcal{X}M$ and set $Z = Z^0\xi + Z^A e_A$. Then, one has $\nabla_X Z = \nabla_X Z^0 \xi + \nabla Z^A e_A$.

Using (2.9) one derives that $\nabla_X Z^0 \xi = (dZ^0 \otimes \xi)(X) + (Z^0 \eta \otimes U)(X)$.

From (1.6) and the identities $\theta_A^0 = -\theta_0^A$ and $\theta_0^A = U^A\eta$ one can states

$$\nabla_X Z^A e_A = ((dZ^A + Z^B \theta_B^A) \otimes e_A)(X) - (g(Z,U)\eta) \otimes \xi)(X).$$

Then, one can obtains

$$\nabla Z = (dZ^A + Z^B\theta_B^A) \otimes e_A + (dZ^0 - g(Z,U) = (\eta) \otimes \xi + Z^0 \eta \otimes U.$$

(5) Using 3) and 4) and by a straightforward calculation, one can state

$$(\nabla\phi)Z = -\eta(Z)\eta \otimes \phi U + g(\phi U, Z)\eta \otimes \xi,$$

thus finishing the proof. □

We shall give a sufficient condition which implies that $U^\flat$ is an exact form.

Remark 2.3 Let Y be any $ker\eta$ vector field on M, and assume that Y defines a skew-symmetric Killing vector field (abr. SSK) on M having the structure vector field U as generative (see [6]). Then one may write

$$\nabla Y = U^\flat \otimes Y - Y^\flat \otimes U. \tag{2.11}$$

Since U is also a $ker\eta$ vector field, it follows by Lemma 2.2 (4) that necesarily $g(U,Y) = 0$, and with the help of the structure equations (1.6) and (1.7) one derives by a standard calculation

$$dY^\flat = 2U^\flat \wedge Y^\flat. \tag{2.12}$$

This proves that $Y^\flat$ is an exterior recurrent form which has $2U^\flat$ as recurrence form (see [1]). We notice that (2.12) is the standard equation of SSK vector fields (see [9]).

In addition setting $2l = ||Y||^2$ and taking account that $g(U,Y) = 0$ one gets from (2.11)

$$dl = 2lU^\flat \tag{2.13}$$

which shows that the existence of Y implies that $U^\flat$ be an exact form.

Finally, we shall prove that the existence of such a vector field Y is assured by an exterior differential system in involution. If we denote by Σ the exterior differential system which defines Y, it follows (see [2]) by E. Cartan's that the characteristic numbers are $r = 3, s_0 = 1, s_1 = 2$. Since $r = s_0 + s_1$ it follows that Σ is an involution and the existence of Y depends of 2 arbitrary functions of 1 argument.

Now we shall obtain the main properties of η, Ω and U in a CCACR manifold.

Proposition 2.4 (Properties of η.) *Let $(M, \phi, \eta, \xi, g, U)$ be a CCACR manifold. Then*

(1) $d\eta = U^\flat \wedge \eta$, *and then η is a exterior recurrent.*

(2) $\delta\eta = 0$.

(3) $d^{-U^\flat}\eta = 0$.

(4) $d\eta \neq 0, \quad \eta \wedge d\eta = 0$, *i.e; η is a class 1 1-form.*

Proof. (1) Taking account $\theta_0^A = U^A\eta$ (see Lemma 2.2 (2)) and applying the equation (1.7) to $\omega^0 = \eta$ one finds

$$d\eta = -\theta_B^0 \wedge \omega^B = \theta_0^B \wedge \omega^B = U^B\eta \wedge \omega^B = -U^B\omega^B \wedge \eta = -U^\flat \wedge \eta;$$

(which says that η is *exterior recurrent* 1-form and has the closed 1-form $U^\flat$ as recurrence form [1]).

(2) By definition of δ one has $\delta\eta = - * d * \eta$ and on behalf of Lemma 2.2 (2) one derives $\delta\eta = 0$, which afirms that η is a coclosed form.

(3) On the other hand by the cohomological operator d^ω one may also write the equation $d\eta = U^\flat \wedge \eta$ as $d^{-U^\flat}\eta = 0$, and say that η is $d^{-U^\flat}$-*closed.*

(4) In consequence of 1) and 2) η is a coclosed exterior recurrent form. In addition by Lemma 2.2 (2) one may write $d\eta \neq 0; \eta \wedge d\eta = 0$, these equations proves that η is of class 1 1-form. □

Remark 2.5 We consider the flow $\mathcal{F}$ defined by the unitary vector field ξ. By Prop. 2.4, (1) we have that the distribution $ker\eta = Span\{\xi\}^\perp$ is involutive. Remember that $d\eta \neq 0$, then by virtue of [10] Theorem 10.6 we concludes that $U_x = (\nabla_\xi \xi)_x \neq 0; \quad \forall x \in M$, and that the vector field ξ is not geodesible, i.e; the orbits of ξ are not geodesics on M.

Corollary 2.6 *Let $(M, \phi, \eta, \xi, g, U)$ be a CCACR manifold, then M is foliated by 1-codimensional almost Hermitian submanifolds.*

Proof. Let $x \in M$, we consider N the maximal integral submanifold verifiying that $x \in N; \quad T_yN = ker\eta_y \quad \forall y, \in N$.

We defined $J \in \mathcal{T}_1^1(M)$ by $J = \phi|_N$. Then, by (1.2) $J^2(X) = -X; \quad \forall X \in \mathcal{X}N$, and then $(N, J, g|_N)$ is an almost Hermitian manifold. □

Proposition 2.7 *Let $(M, \phi, \eta, \xi, g, U)$ be a CCACR manifold and let Ω the 2-form defined by (1.4), then $d\Omega = 0$.*

Proof. Due to M has an almost contact Riemannian structure the horizontal connection forms θ^A_B satisfy the Kähler relations of Lemma 2.2 (3). By Remark 1.4 one has $\Omega = \sum w^a \wedge w^{a^*}$, then by a standard calculation one finds using $\theta^A_0 = U^A \eta$ and Prop. 2.4 (4), $d\Omega = 0$. $\square$

Remark 2.8 Since $U^\flat = \sum_{A=1}^{2m} U^A \omega^A$ is a closed form, taking into account (1.7) one derives

$$\sum_{A,B=1}^{2m} (dU^A + U^B \theta^A_B) \wedge \omega^A = 0.$$

If there exists $\lambda \in \mathcal{F}(M)$ such that

$$dU^A + U^B \theta^A_B = \lambda \omega^A; \forall A \in \{1, \dots 2m\};$$

the above equation is satisfied.

This remark allows us to introduce the following

Definition 2.9 Let $(M, \phi, \eta, \xi, g, U)$ be a CCACR manifold. We say that M is a λ-CCACR manifold if there exists $\lambda \in \mathcal{F}(M)$ such that

$$dU^A + U^B \theta^A_B = \lambda \omega^A; \quad \forall A \in \{1, \dots 2m\}. \tag{2.14}$$

In the rest of the paper, we shall work with λ-CCACR manifolds, although some results will be also true for general CCACR manifolds.

Proposition 2.10 (Properties of U.) *Let $(M, \phi, \eta, \xi, g, U)$ be a λ-CCACR manifold. Then*

(1) $\mathcal{L}_U \eta = \|U\|^2 \eta$ *and* $\mathcal{L}_U \Omega = 2\lambda\Omega$; $\quad \lambda = const.$, *and then* U *is an infinetesimal conformal transformation of the almost contact structure.*

(2) $(\nabla\phi)U = 0$.

(3) $\nabla_U U = \lambda U$; *i.e., U is an affine geodesic.*

(4) $[U, \xi] = \|U\|^2 \xi$, *which shows that ξ admits an infinitesimal transformation of generators U.*

(5) $g(\nabla_Z U, Z') = g(\nabla_{Z'} U, Z); \forall Z, Z' \in \mathcal{X}M$. *i.e., U is a* $ker\eta$*-gradient vector field.*

Proof. (1) Recall that ${}^\flat U = \sum (U^{a^*} \omega^a - U^a \omega^{a^*})$. Next making use of (2.14) one gets

$$d({}^\flat U) = -2\lambda\Omega \tag{2.15}$$

and consequently since Ω is closed one has $\lambda = const.$ and so one infers $\mathcal{L}_U \Omega = 2\lambda\Omega$. Moreover since $\eta(U) = 0$, one derives from Prop. 2.4, (1) $\mathcal{L}_U \eta = \|U\|^2 \eta$ and by the two last obtained equations we conclude that the generative U of ξ defines and infinitesimal conformal transformation of the almost contact structure of M.

(2) $(\nabla\phi)U = 0$ can be obtained quickly from Lemma 2.2 (5), taking $Z = U$.

(3) Using Lemma 2.2 (4) and taking $Z = U$ vector field U is expressed by

$$\nabla U = (dU^A + U^B\theta_B^A) \otimes e_A + (-g(U,U)\eta) \otimes \xi.$$

Then by (2.14)

$$\nabla U = \lambda\omega^A \otimes e_A - \|U\|^2\eta \otimes \xi = \lambda dp - (\lambda + \|U\|^2)\eta \otimes \xi. \tag{2.16}$$

Since $\eta(U) = 0$, the above equation implies $\nabla_U U = \lambda U$. Moreover $\nabla_\xi U = -\|U\|^2\xi$.

(4) Because ∇ is symmetric $T(\xi, U) = 0$. Then $[\xi, U] = -\|U\|^2\xi$.

(5) It also should be noticed that by (2.16) one has : $g(\nabla_Z U, Z') = g(\nabla_{Z'} U, Z)$; for any vector fields Z, Z' on M, which afirms that U is a gradient vector field (in the sense of Okumura; also [4] and by $d^{-U^\flat}\eta = 0$ one may say that η is a $d^{-U^\flat}$-*exact* 1-form). □

Proposition 2.11 *Let $(M, \phi, \eta, \xi, g, U)$ be a λ-CCACR manifold.*

(1) *If $D_U = Span\{U, \xi\}$ then M may be viewed as the locally product of integral submanifolds of D_U and $D_U^\perp$.*

(2) *Let Y be any $ker\eta$ SSK vector field on M. If $D_Y = Span\{U, Y, \xi\}$ then D_Y defines a foliation on M. Moreover, U defines an infinitesimal conformal transformation of Y and $[\xi, Y] = 0$.*

Proof. (1) By Prop. 2.10 (4), D_U is an involutive distribution. Let U', U'' be vector fields of D_U. By Lemma 2.2 and Prop. 2.10 one has:

$$\nabla_\xi\xi = U;\ \nabla_\xi U = -\|U\|^2\xi;\ \nabla_U\xi = 0;\ \nabla_U U = \lambda U;$$

and then it is easily seen that $\nabla_{U'}U'' \subset D_U$, and consequently the leaves of D_U are totally geodesic.

The vector fields ξ and U are not singular and then the distributions $Span\{\xi\} = ker\eta$ and $Span\{\xi\}^\perp = kerU^\flat$ are well defined. Both distributions are involutive because $d\eta = U^\flat \wedge \eta$ (see Prop. 2.4 (1)), and $dU^\flat = 0$ (see equation (1.6)), and then $D_U^\perp = ker\eta \cap kerU^\flat$ is also involutive.

(2) We consider the 3-form $\varphi = U^\flat \wedge Y^\flat \wedge \eta$; it is easily seen by (2.12) and Prop. 2.4 (1) that $d\varphi = 0$, and consequently the 3-distribution $D_Y = Span\{U, Y, \xi\}$ defines a foliation. Since $i_Y\varphi = \|Y\|^2 U = \wedge\varphi$ one also derives $\mathcal{L}_Y\varphi = 0$, that is φ is invariant by Y.

By (2.11), (2.9) and (2.16) a short calculation gives $[U, Y] = (\|U\|^2 - \lambda)Y$ and $[\xi, Y] = 0$ which says that U defines an infinitesimal conformal transformation of Y, and ξ and Y commute. □

Proposition 2.12 *Let $(M, \phi, \eta, \xi, g, U)$ be a λ-CCACR manifold. The transversal curvature forms $\Theta_0^A (A \in \{1, \ldots, 2m\})$ are monomial an expressed by*

$$\Theta_0^A = (U^A U^\flat + \lambda\omega^A) \wedge \eta$$

and the vector valued 2-form $\Pi = \sum \Theta_0^A \otimes e_A = -\lambda\eta \wedge dp + (U^\flat \wedge \eta) \otimes U$ is d^∇-closed, and U defines an almost conformal transformation of Π, i.e.,

$$\mathcal{L}_U\Pi = \|U\|^2\Pi + 2\lambda(U^\flat \wedge \eta) \otimes U.$$

Proof. With the help of the structure equations (1.8) corresponding to the curvature forms Θ, one derives by (2.14) and (2.16)

$$\Theta_0^A = (U^A U^\flat + \lambda\omega^A) \wedge \eta \tag{2.17}$$

and this shows that Θ_a^A are monomial 2-forms.

Further by (2.17) we deduce the following vector valued 2-form $\Pi = \sum \Theta_0^A \otimes e_A = -\lambda\eta \wedge dp + (U^\flat \wedge \eta) \otimes U$ and we agree to denominate Π the *structure curvature vector valued* form. Now operating on Π by the exterior covariant derivate operator d^∇, and since $d(U^\flat \wedge \eta) = 0$ (see Prop. 2.4, (1)), a short calculation gives $d^\nabla \Pi = 0$. Hence Π enjoys the significative property to be a closed vector valued form. Taking now the Lie derivative of Π with respect to U , one finds by Lemma 2.2 (3) $\mathcal{L}_U \Pi = -\lambda \mathcal{L}_U \eta \wedge dp + \mathcal{L}_U (U^\flat \wedge \eta) \otimes U = \|U\|^2 \Pi + 2\lambda (U^\flat \wedge \eta) \otimes U$.

3 Harmonic Properties

In this section we shall discuss some harmonic properties induced by the almost contact structure of the λ-CCACR manifold M under consideration.

Note that $*U^\flat = \sum (-1)^A \omega^1 \wedge \ldots \wedge \widehat{\omega^A} \wedge \ldots \wedge \omega^{2m} \wedge \eta$ and then one finds that the codifferential $\delta U^\flat$ of $U^\flat$ is expressed by $\delta U^\flat = \|U\|^2 - 2m\lambda$.

Hence since $dU^\flat = 0$, the harmonic operator $\Delta U^\flat$ is expressed by

$$\Delta U^\flat = \lambda \|U\|^2 U^\flat.$$

This is known afirms that $U^\flat$ is an *eigenfunction* of Δ, having $\lambda\|U\|^2$ as eigenvalue.

By similar devices and since it has been shown in section 2 that η is coclosed one finds by Prop. 2.4, (1) that

$$\Delta\eta = 2m\lambda\eta.$$

Hence η is also an *eigenfunction* of Δ. Next according to the concept of conformal adjoint transformation, one has regarding the paring (η, U) $\mathcal{L}_U^* \eta = \delta(U^\flat \wedge \eta) + U^\flat = \wedge \delta\eta$. Then since η is coclosed, one derives by Prop. 2.4, (1)

$$\mathcal{L}_U^* \eta = \; - 2m\lambda\eta,$$

that is U defines a *infinitesimal homothety* of the conformal adjoint $\mathcal{L}^*\eta$.

Let now $\mathbf{L}$ be the Weil's $(1,1)$ operator [11] defined by $\mathbf{L} : \pi \to \pi \wedge \Omega, \pi \in \bigwedge^2 M$. (note that one has $d\mathbf{L}\pi = \mathbf{L} d\pi + \pi \wedge \mathbf{L} = \pi$), and set $\eta_q = \eta \wedge \Omega^q$.

Then by Prop. 2.4 (1) one gets $d\eta_q = U^\flat \wedge \eta_q$ and

$$\mathcal{L}_\xi \eta_q = -U^\flat \wedge \Omega^q = -(U^\flat)_q$$

i.e., the Lie derivatives of the $(2q+1)$-form η_q are equated (up to the sign) by $(2q+1)$-forms $(U^\flat)_q$.

Finally in an other order of ideas, one finds by Prop. 2.7, (4)

$$\|grad\,(\|U\|^2)\|^2 = 4\lambda^2 \|U\|^2.$$

Now by a short calculation one derives $div\,U = 2m\lambda - \|U\|^2$ and

$$div\ grad\ (\|U\|^2) = 8m\lambda^2 - 4\lambda\|U\|^2.$$

Since $\lambda = const.$, we may afirm that $\|U\|^2$ is an *isoparametric* function.

Then we can state:

Theorem 3.1 *Let $M(\phi, \eta, \xi, U, g)$ be a λ-CCACR manifold . Then the tuple of structure tensors (ξ, U, η) satisfies the following properties:*

(1) *The 1-forms $U^\flat$ and η are eigenfunctions of Δ and U defines an infinitesimal homothety of the conformal adjoint $2m$-form $^*\eta$.*

(2) *The Lie derivatives with respect to ξ of the $(2q+1)$-forms $\eta_q = \eta \wedge \Omega^q$ are expressed by*

$$\mathcal{L}_\xi \eta_q = -U^\flat \wedge \Omega^q.$$

(3) *$\|U\|^2$ is an isoparametric function.*

4 ϕ-soldering vector fields

Taking into account Lemma 2.2 (4) one has

$$\forall X \in ker\eta;\ \nabla X = (dX^A + X^B\theta_B^A) \otimes e_A - g(X, U)\eta \otimes \xi.$$

Then we introduce the following

Definition 4.1 Let $(M, \phi, \eta, \xi, g, U)$ be a λ-CCACR manifold. A *ϕ-soldering vector field* is a nonsingular vector field $X \in ker\eta$ satisfying

$$\nabla X = f\phi dp - g(X, U)\eta \otimes \xi, \quad f \in \mathcal{F}(M) \tag{4.18}$$

We study the main properties of such a vector field:

Theorem 4.2 *Let $(M, \phi, \eta, \xi, g, U)$ a $(2m+1)$-dimensional λ-CCACR manifold endowed with a ϕ-soldering vector field X. Then:*

(1) *X defines an infinitesimal automorphism of Ω which moves to a potential form.*

(2) *f is a constant function and ϕX defines an infinitesimal homotethy of Ω.*

(3) *X is a $Ker\eta$ Killing vector field, i.e.,*

$$g(\nabla_Z X, Z') + g(\nabla_{Z'} X, Z) = 0$$

for any $\eta(Z) = 0, \eta(Z') = 0$.

(4) *X and ϕX commute.*

(5) *ϕX is a $ker\eta$ gradient vector field. i.e.,*

$$g(\nabla_Z \phi X, Z') = g(\nabla_{Z'} \phi X, Z)$$

for any $Z, Z' \in \mathcal{X}M$.

(6) X *and* ϕX *define infinitesimal conformal transformations of* ξ.

(7) *The sectional curvature of the plane spanned by* X *and* ϕX *vanishes.*

(8) M *is locally the product of a* 3*-dimensional submanifold* M_X *tangent to* $D_X = Span\{\phi X, \xi, U\}$ *and of a* $(2m-2)$*-dimensional submanifold* $M_X^{\perp}$ *tangent to* $D_X^{\perp}$.

Proof. (1) By Lemma 2.2 (4) and (4.18) we find

$$\text{(4.19)} \qquad \left\{ \begin{array}{l} dX^a + X^A \theta_A^a = -f\omega^{a^*} \\ dX^{a^*} + X^A \theta_A^{a^*} = f\omega^a, a \in \{1, \dots, m\}, a^* = a + m \end{array} \right\}$$

and setting $2\ell = \|X\|^2, \beta = (\phi X)^\flat$ one derives from (4.19) $d\ell = -f\beta$ and $d\beta = 0$.

Note that

$$\beta(Y) = g(\phi X, Y) = \Omega(X, Y); -^\flat X(Y) = (i_X \Omega)Y = \Omega(X, Y);$$

for any vector field Y on M. Then, one has $\beta = -^\flat X$.

Recall that $d\Omega = 0$ (see Prop. 2.7), then one may write

$$\mathcal{L}_X \Omega = (i_X \circ d + d \circ i_X)\Omega = d(-^\flat X) = d\beta = 0.$$

(2) One has $X^\flat = \sum_{A=1}^{2m} X^A \omega^A$. With the help of (4.19) one deduces that $dX^\flat = -f\Omega$ wich implies by Prop. 2.7 that $f = constant$.

As $X \in ker\eta$; one obtains $X^\flat = -^\flat(\phi X) = i_{\phi X}\Omega$, and then

$$\mathcal{L}_{\phi X}\Omega = (i_{\phi X} \circ d + d \circ i_{\phi X})\Omega = dX^\flat = -f\Omega.$$

(3) By (4.18) one finds $g(\nabla_Z X, Z') + g(\nabla_{Z'} X, Z) = 2\lambda\eta(Z)\eta(Z')$ for any vector fields Z, Z' on M. Therefore if $Z, Z' \in Ker\eta$ one may write $g(\nabla_Z X, Z') + g(\nabla_{Z'} X, Z) = 0$. In this case we agree to say that X is a $ker\eta$ Killing vector field.

(4) Recall that $(\nabla\phi)Z = -\eta(Z)\eta \otimes \phi(U) + g(\phi U, Z)$ for any Z vector field on M (see Lemma 2.2 (5)).

Making use of the equations (4.18) and (1.2) one has

$$\nabla_Y \phi X = \phi(\nabla_Y X) - (\nabla\phi)(X)(Y) = -fY + f\eta(Y)\xi - g(\phi U, X)\eta(Y)\xi,$$

for any vector field Y on M. Then one infers

$$\text{(4.20)} \qquad \nabla\phi X = -fdp + (f - g(\phi U, X))\eta \otimes \xi.$$

Then by (4.18) and the above equation $T(X, \phi X) = 0 \Rightarrow [X, \phi X] = 0$, which shows that X and ϕX commute.

(5) From (4.20) one may write

$$g(\nabla_Z \phi X, Z') = g(\nabla_{Z'}\phi X, Z), \forall Z, Z' \in ker\eta$$

which says that ϕX is a $ker\eta$ gradient vector field.

(6) Taking into account that ξ is U-concircular, X is a ϕ-soldering vector field and the equation (4.20) one finds $[X, \xi] = g(X, U)\xi$; $[\phi X, \xi] = g(\phi U, X)\xi$; and so following the above calculations prove that both X and ϕX define infinitesimal conformal transformations of ξ.

(7) Operating on (4.18) by the operator d^∇ one gets by a standard calculation

$$d^\nabla(\nabla X) = \nabla^2 X = ((\lambda U^\flat - f(\phi U)^\flat \wedge \eta) \otimes \xi.$$

By (1.1) one has

$$K_{\phi X \wedge X} = \frac{g(R(\phi X, X)X, \phi X)}{\|X\|^2\|\phi X\|^2 - g(X, \phi X)^2} = \frac{g(((\lambda U^\flat - f(\phi U)^\flat \wedge \eta) \otimes \xi(X, \phi X), \phi X}{\|X\|^2\|\phi X\|^2 - g(X, \phi X)^2}$$

where $K_{\phi X \wedge X}$ is the sectional curvature of the plane spanned by X and ϕX.

Then, a short calculation gives $K_{\phi X \wedge X} = 0$.

(8) By Lemma 2.2 (3), it is seen by similar consideration as in Prop. 2.11(1), then M is locally the product of a 3-dimensional submanifold M_X tangent to $D_X = Span\{\xi, U, \phi X\}$ and of a $(2m-2)$-dimensional submanifold $M_X^\perp$ tangent to $D_X^\perp$. □

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References

[1] D. K. Datta, *Exterior recurrent forms on a manifold*, Tensor N. S., 36 (1982), 115-120.

[2] J. Dieudonné, *Treatise on Analysis*, vol. 4. Ac. Press, N. York, London, 1974.

[3] F. Guedira and A. Lichnerowicz, *Géométrie des algèbres de Lie locales de Kirilov*, J. Math. Pures Appl., 63 (1984), 407-484.

[4] K. Matsumoto, I. Mihai and R. Rosca, *On gradient almost torse forming vector fields*, Mathem. Journal of Toyama University vol. 19 (1996), 143-159.

[5] K. Matsumoto, I. Mihai and R. Rosca, *A certain locally conformal almost cosymplectic manifold and its submanifolds*, Tensor N. S. 51, (1992) 91-102 (196), 143-159.

[6] I. Mihai, R. Rosca and L. Verstraelen, *Some aspects of the Differential Geometry of vector fields*, Pure and Applied Differential Geometry, Leuven-Brussel vol. 2 (1996).

[7] J. A.Oubiña, *New classes of almost contact metric manifolds*, Publ. Math. Debrecen, 32 (1985), 187-193.

[8] W.A. Poor. *Differential Geometric Structures*, Mc. Graw Hill; New York (1981).

[9] R. Rosca, *On exterior concurrent SSK vector fields*, Rend. Sem. Mat. Messina, 2 (1993), 131-145.

[10] P. Tondeur, *Foliations on Riemannian Manifolds*, Springer-Verlag, N. York, 1988.

[11] A. Weil, *Sur la théorie des formes differentielles attachies a une varieté analitique complexe*, Com. Math. Helv., 20 (1947), 110-116.

[12] A. West, *Isoparametric systems. Geometry and Topology of Submanifolds. 222-230*, World Scientific Riena Edge, USA (1989).

[13] K. Yano and M. Kon, *Anti-invariant submanifolds of Sasakian space forms II*, J. Korean Math. Soc., 13 (1976), 1-14.

[14] K. Yano and M. Kon, *Structures on manifolds*, World Scientific, Singapore (1984).

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