The Characteristic Elements of Special Nilpotent Lie Algebras of Dimension Ten

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Abstract

Let g be a Nilpotent Lie algebra over a field K of characteristic 0 whose dimension is ten and the dimension of its maximal abelian ideal g_0 is nine. The aim of the present paper is to give the characteristic elements of these Nilpotent Lie algebras.

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1 Introduction

The determination of characteristic elements of a given Nilpotent Lie algebra is an open problem. In this paper we estimate the characteristic elements of the nilpotent Lie algebras of dimension ten whose maximal abelian ideal is of dimension nine.

The whole paper contains four paragraphs. The first paragraph is the introduction and the second is the preliminaries. In the third paragraph are described the nilpotent Lie algebras of dimension ten whose the dimension of maximal abelian ideal is nine. The characteristic elements of these Nilpotent Lie algebras are estimated in the forth paragraph.

2 Preliminaries

We denote a base of a Nilpotent Lie algebra g of dimension n by $\{X_1, ...X_n\}$.For this Lie algebra g we have the relation

$$(1) \qquad [X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k \quad i, j = 1, ...n,$$

where c_{ij}^k are the structure constants of the Lie algebra g.

The invariants of g are the functions $F(x)$ which satisfy the relations

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$$[X_i, F(x)] = 0, \quad i = 1, 2, ...n. \tag{2}$$

It has been discussed [4] to derive, which we adopt of replacing the X' s by c-number differential operators

$$X_i \to x_i = \sum x_k c_{ij}^k \partial x_i, \tag{3}$$

which have the same commutation rule and act on a space of continuously differentiable functions of n real variables. The commutation relations (2) are replaced by the set of partial differential equations

$$x_i F(x) = 0, \tag{4}$$

which can be usually solved by the standard methods.Therefore, the solutions of (2) are obtained from those of (4) by the replacements $x_i \to X_i$ provided that the factors X_i in $F(x)$ can be ordered so that (4) implies (2). This method gives the invariants of the algebra g. We have used here the base $\{X_1, ...X_n\}$ instead of $\{e_1, ...e_n\}$. This method has been used to construct the partial differential equations which form a system.

Definition. Let $Der(g)$ be the Lie algebra of derivations of g. The torus on g is a commutative subalgebra of $Der(g)$ consisting of semi-simple endomorphismus. A torus T is called *maximal*, if it is not contained strictly in any other torus.

Let T be a maximal torus on g. Then g can be decomposed as follows

$$g = \bigoplus_{\beta \in T^*} g^\beta, \tag{5}$$

where T^* is the dual of T and g^β is defined by

$$g^\beta = \{x \in g/t(x) = \beta(t)x, \quad \forall t \in T \ \}. \tag{6}$$

The decomposition (5) is called *weight system*. The other notions, which are the *center of g, T-msg, root system, Generalised Cartan Matrix and Dynkin diagram*, are contained in the paper [5].

All the above characteristic elements will be given for the nilpotent Lie algebras, over a field of characteristic zero and of dimension ten, whose maximal abelian ideal is of dimension nine.

3 Nilpotent Lie algebras of dimension ten whose maximal abelian ideal is of dimension nine

The Nilpotent Lie algebras of dimension ten whose maximal abelian ideal is of dimension nine are the following:

$g_{10,1}:$ $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_5] = e_6, [e_1, e_7] = e_8, [e_1, e_9] = e_{10}$

$g_{10,2}:$ $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_1, e_6] = e_7, [e_1, e_7] = e_8,$
$[e_1, e_9] = e_{10}$

$$
\begin{aligned}
g_{10,3}: \quad & [e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_1,e_7]=e_8\\
& [e_1,e_9]=e_{10}\\
g_{10,4}: \quad & [e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_5]=e_6,[e_1,e_6]=e_7,[e_1,e_8]=e_9,\\
& [e_1,e_9]=e_{10}\\
g_{10,5}: \quad & [e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_1,e_7]=e_8,\\
& [e_1,e_8]=e_9,[e_1,e_9]=e_{10}\\
g_{10,6}: \quad & [e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_1,e_6]=e_7,\\
& [e_1,e_8]=e_9,[e_1,e_9]=e_{10}\\
g_{10,7}: \quad & [e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_1,e_6]=e_7,\\
& [e_1,e_7]=e_8,[e_1,e_9]=e_{10}\\
g_{10,8}: \quad & [e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_1,e_5]=e_6,[e_1,e_6]=e_7,\\
& [e_1,e_7]=e_8,[e_1,e_8]=e_9,[e_1,e_9]=e_{10}
\end{aligned}
$$

4 Characteristic elements

We calculate the invariants of these Lie algebras. Firstly , we consider the Lie algebras $g_{10,7}$ and $g_{10,8}$.

We consider the system of partial differential equations

$$
\begin{aligned}
g_{10,7}: \quad & e_3\frac{\partial F}{\partial e_2}+e_4\frac{\partial F}{\partial e_3}+e_5\frac{\partial F}{\partial e_4}+e_6\frac{\partial F}{\partial e_5}+e_7\frac{\partial F}{\partial e_6}+e_8\frac{\partial F}{\partial e_7}+e_{10}\frac{\partial F}{\partial e_9}=0\\
& -e_3\frac{\partial F}{\partial e_2}=0,-e_4\frac{\partial F}{\partial e_3}=0,-e_5\frac{\partial F}{\partial e_4}=0,-e_6\frac{\partial F}{\partial e_5}=0\\
& -e_7\frac{\partial F}{\partial e_6}=0,-e_8\frac{\partial F}{\partial e_7}=0,-e_{10}\frac{\partial F}{\partial e_9}=0
\end{aligned}
$$

$$
\begin{aligned}
g_{10,8}: \quad & e_3\frac{\partial F}{\partial e_2}+e_4\frac{\partial F}{\partial e_3}+e_5\frac{\partial F}{\partial e_4}+e_6\frac{\partial F}{\partial e_5}+e_7\frac{\partial F}{\partial e_6}+e_8\frac{\partial F}{\partial e_7}+e_9\frac{\partial F}{\partial e_8}+e_{10}\frac{\partial F}{\partial e_9}=0\\
& -e_3\frac{\partial F}{\partial e_2}=0,-e_4\frac{\partial F}{\partial e_3}=0,-e_5\frac{\partial F}{\partial e_4}=0,-e_6\frac{\partial F}{\partial e_5}=0\\
& -e_7\frac{\partial F}{\partial e_6}=0,-e_8\frac{\partial F}{\partial e_7}=0,-e_9\frac{\partial F}{\partial e_8}=0,-e_{10}\frac{\partial F}{\partial e_9}=0.
\end{aligned}
$$

The solutions of the above linear systems of partial differential equations give the invariants, which are :

$$
\begin{aligned}
g_{10,7}: \quad & e_8, e_{10}, e_{10}e_7-e_9e_8, 2e_6e_8-e_7^2, 3e_5e_8^2-3e_8e_6e_7+e_7^3, 2e_2e_8-2e_3e_7+2e_4e_6-e_5^2,\\
& 4e_4e_8^3-2e_6^2e_8^2-4e_5e_7e_8^2+4e_6e_7^2e_8-e_7^4,\\
& 5e_3e_8^4-5e_4e_7e_8^3-5e_5e_6e_8^3+5e_8^2e_5e_7^2+5e_7e_6^2e_8^2-5e_6e_8e_7^3+e_7^3.\\
g_{10,8}: \quad & e_{10}, 2e_{10}e_8-e_9^2, 3e_7e_{10}^2-3e_8e_9e_{10}+e_9^3, 2e_2e_{10}-2e_3e_9+2e_4e_8-2e_5e_7+e_6^2,\\
& 4e_6e_{10}^3-2e_{10}^2e_8^2-4e_9e_7e_{10}^2+4e_8e_9^2e_{10}-e_9^4,\\
& 5e_5e_{10}^4-5e_6e_9e_{10}^3-5e_7e_8e_{10}^3+5e_7e_{10}^2e_9^2+5e_9e_8^2e_{10}^2-5e_{10}e_8e_9^3+e_9^5,
\end{aligned}
$$

$$36e_4e_{10}^5 - 36e_5e_9e_{10}^4 + 18e_6e_9^2e_{10}^3 - 6e_7e_{10}^2e_9^3 + 6e_{10}^3e_8^3 - 9e_8^2e_9^2e_{10}^2 + 6e_{10}e_8e_9^4 - e_9^6,$$
$$210e_3e_{10}^6 - 210e_4e_9e_{10}^5 + 210e_5e_8e_{10}^5 - 210e_6e_7e_{10}^5 + 210e_9e_{10}^4e_7^2 - 210e_7e_8^2e_{10}^4 +$$
$$210e_9e_8^3e_{10}^3 - 210e_8^2e_9^3e_{10}^2 + 84e_8e_{10}e_9^5 - 12e_9^7.$$

From the above we have the

Theorem. *The invariants of the Lie algebras $g_{10,i}$,i=1,2,...8 are the following*

$g_{10,1}$: $e_4, e_6, e_8, e_{10}, 2e_2e_4 - e_3^2, e_3e_6 - e_4e_5, e_5e_8 - e_7e_6, e_7e_{10} - e_8e_9.$

$g_{10,2}$: $e_5, e_8, e_{10}, 2e_3e_5 - e_4^2, 2e_6e_8 - e_7^2, e_4e_{10} - e_5e_9, e_7e_{10} - e_8e_9.$

$g_{10,3}$: $e_6, e_8, e_{10}, 2e_6e_4 - e_5^2, e_5e_{10} - e_6e_9, e_5e_8 - e_7e_6, e_7e_{10} - e_8e_9,$
$3e_3e_6^2 - 3e_4e_6e_5 + e_5^3.$

$g_{10,4}$: $e_4, e_7, e_{10}, 2e_2e_4 - e_3^2, 2e_8e_{10} - e_9^2, 2e_5e_7 - e_6^2, e_3e_7 - e_4e_6, e_3e_{10} - e_4e_9.$

$g_{10,5}$: $e_6, e_{10}, 2e_6e_4 - e_5^2, e_5e_{10} - e_6e_9, 2e_{10}e_8 - e_9^2, 3e_7e_{10}^2 - 3e_8e_9e_{10} + e_9^3,$
$3e_3e_6^2 - 3e_4e_6e_5 + e_5^3, 2e_2e_6 - 2e_3e_5 + e_4^2.$

$g_{10,6}$: $e_7, e_{10}, 2e_5e_7 - e_6^2, e_6e_{10} - e_7e_9, 2e_{10}e_8 - e_9^2, 3e_4e_7^2 - 3e_5e_6e_7 + e_6^3,$
$4e_3e_7^3 - 2e_5^2e_7^2 - 4e_4e_6e_7^2 + 4e_5e_6^2e_7 - e_6^4,$
$5e_2e_7^4 - 5e_4e_5e_7^3 - 5e_3e_6e_7^3 + 5e_5^2e_6e_7^2 + 5e_4e_6^2e_7^2 - 5e_5e_6^3e_7 + e_6^5.$

$g_{10,7}$: $e_8, e_{10}, 2e_{10}e_7 - e_9e_8, 2e_6e_8 - e_7^2, 3e_5e_8^2 - 3e_8e_6e_7 + e_7^3, 2e_2e_8 - 2e_3e_7 + 2e_4e_6 - e_5^2,$
$4e_4e_8^3 - 2e_6^2e_8^2 - 4e_5e_7e_8^2 + 4e_6e_7^2e_8 - e_7^4,$
$5e_3e_8^4 - 5e_4e_7e_8^3 - 5e_5e_6e_8^3 + 5e_8^2e_5e_7^2 + 5e_7e_6^2e_8^2 - 5e_6e_8e_7^3 + e_7^5.$

$g_{10,8}$: $e_{10}, 2e_{10}e_8 - e_9^2, 3e_7e_{10}^2 - 3e_8e_9e_{10} + e_9^3, 2e_2e_{10} - 2e_3e_9 + 2e_4e_8 - 2e_5e_7 + e_6^2,$
$4e_6e_{10}^3 - 2e_{10}^2e_8^2 - 4e_9e_7e_{10}^2 + 4e_8e_9^2e_{10} - e_9^4,$
$5e_5e_{10}^4 - 5e_6e_9e_{10}^3 - 5e_7e_8e_{10}^3 + 5e_7e_{10}^2e_9^2 + 5e_9e_8^2e_{10}^2 - 5e_{10}e_8e_9^3 + e_9^5,$
$36e_4e_{10}^5 - 36e_5e_9e_{10}^4 + 18e_6e_9^2e_{10}^3 - 6e_7e_{10}^2e_9^3 + 6e_{10}^3e_8^3 - 9e_8^2e_9^2e_{10}^2 + 6e_{10}e_8e_9^4 - e_9^6,$
$210e_3e_{10}^6 - 210e_4e_9e_{10}^5 + 210e_5e_8e_{10}^5 - 210e_6e_7e_{10}^5 + 210e_9e_{10}^4e_7^2 - 210e_7e_8^2e_{10}^4 +$
$210e_9e_8^3e_{10}^3 - 210e_8^2e_9^3e_{10}^2 + 84e_8e_{10}e_9^5 - 12e_9^7.$

Now we shall determine all the other characteristic elements associated to the above eight Nilpotent Lie algebras $g_{10,1}, ...g_{10,8}$.

We shall estimate these elements only for the Lie algebra $g_{10,1}$. With the same method we can estimate all these for the other Lie algebras.

Let T be the maximal torus on $g_{10,1}$ such that $\{e_1, ...e_m\}$ are root vectors. If $t \in T$, then we have

$$t(e_i) = \beta_i(t)e_i. \tag{7}$$

From the fact that $t \in T \subseteq Derg$ and if we apply t to the Lie bracket $[e_1, e_2] = e_3$ of the Lie algebra $g_{10,1}$ we obtain

$$t(e_3) = [t(e_1), e_2] + [e_1, t(e_2)],$$

which by means of (7) implies

$$\beta_3(t)e_3 = [\beta_1(t)e_1, e_2] + [e_1, \beta_2(t)e_2] = (\beta_1(t) + \beta_2(t))[e_1, e_2] = (\beta_1(t) + \beta_2(t))e_3$$

which finally gives

$$\beta_3 = \beta_1 + \beta_2. \tag{8}$$

From the other Lie brackets of the Lie algebra $g_{10,1}$ we have the relations

$$\beta_1 + \beta_3 = \beta_4,\ \beta_1 + \beta_5 = \beta_6,\ \beta_1 + \beta_7 = \beta_8,\ \beta_1 + \beta_9 = \beta_{10}. \tag{9}$$

The solutions of (8) and (9) give

$$\beta_3 = \beta_1 + \beta_2\ ,\ \beta_4 = 2\beta_1 + \beta_2,\ \beta_6 = \beta_1 + \beta_5,\ \beta_8 = \beta_1 + \beta_7,\ \beta_{10} = \ \beta_1 + \beta_9.$$

Then the Nilpotent Lie algebra $g_{10,1}$ can be written

$$\begin{aligned} g_{10,1} &= g = g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_1+\beta_2} \oplus g^{2\beta_1+\beta_2} \oplus g^{\beta_5} \oplus \\ &\oplus\ g^{\beta_1+\beta_5} \oplus g^{\beta_7} \oplus g^{\beta_1+\beta_7} \oplus g^{\beta_9} \oplus g^{\beta_1+\beta_9}, \end{aligned}$$

which also can be written

$$g_{10,1} = \mathbf{C}e_1 \oplus \mathbf{C}e_2 \oplus \mathbf{C}e_3 \oplus \mathbf{C}e_4 \oplus \mathbf{C}e_5 \oplus \mathbf{C}e_6 \oplus \mathbf{C}e_7 \oplus \mathbf{C}e_8 \oplus \mathbf{C}e_9 \oplus \mathbf{C}e_{10},$$

from which we obtain

$$[g, g] = \mathbf{C}e_3 \oplus \mathbf{C}e_4 \oplus \mathbf{C}e_6 \oplus \mathbf{C}e_8 \oplus \mathbf{C}e_{10}.$$

Hence

$$T - msg = \{e_1, e_2, e_5, e_7, e_9\} = \{x_1, x_2, x_3, x_4, x_5\}.$$

Therefore, the Generalized Cartan Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}, \tag{10}$$

of $g_{10,1}$ can be computed as follows

$$-a_{ij} = \min\{n \in N/(adx_i)^{n+1}x_j = 0, x_i, x_j \in T - msg\}.$$

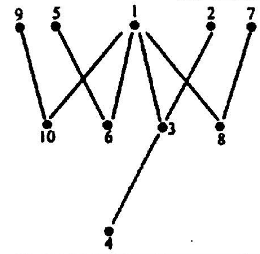
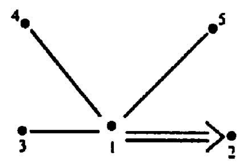
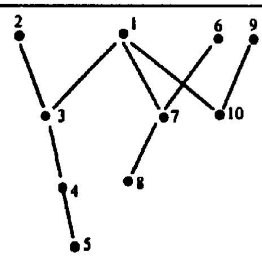
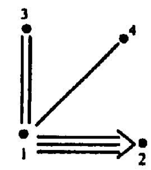
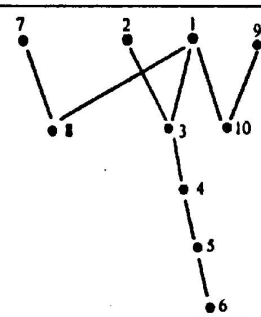
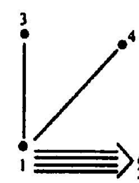
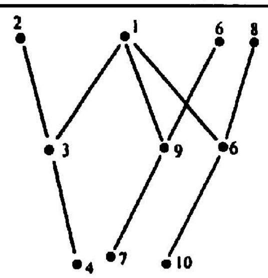
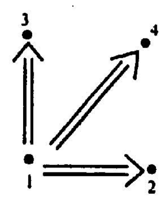
If we take under the consideration the T-msg which is $\{x_1, x_2, x_3, x_4, x_5\}$ and the Lie brackets of the Lie algebra $g_{10,1}$, then the Generalized Cartan Matrix A takes the form

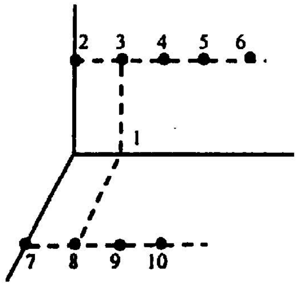
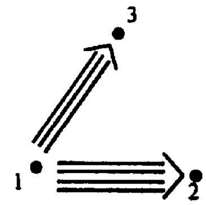
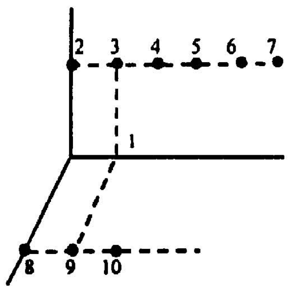
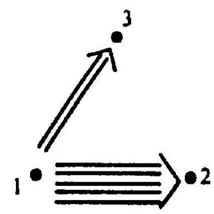
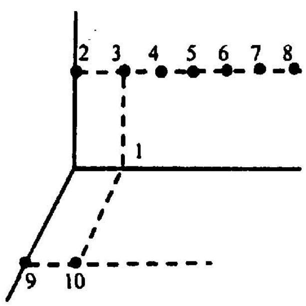
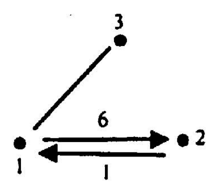
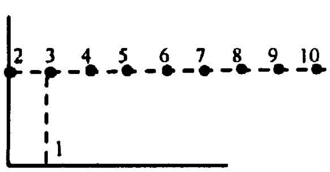
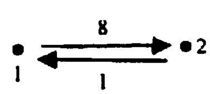
$$A = \begin{bmatrix} 2 & -2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

From the above we conclude the following

Theorem. *Let g be a Nilpotent Lie algebra of dimension ten over* $\mathbf{C}$ *whose maximal abelian ideal g_o is of dimension nine. The table below gives the characteristic elements of such Lie algebras.*

g	type	dim $z(g)$	T-msg	weight system
$g_{10,1}$	$10,5,1$	4	e_1,e_2,e_5,e_7,e_9	$g=g^{\beta_1}\oplus g^{\beta_2}\oplus g^{\beta_1+\beta_2}\oplus g^{2\beta_1+\beta_2}\oplus g^{\beta_5}\oplus g^{\beta_1+\beta_5}\oplus g^{\beta_7}\oplus g^{\beta_1+\beta_7}\oplus g^{\beta_9}\oplus g^{\beta_1+\beta_9}=\mathbf{C}e_1\oplus\mathbf{C}e_2\oplus\mathbf{C}e_3\oplus\mathbf{C}e_4\oplus\mathbf{C}e_5\oplus\mathbf{C}e_6\oplus\mathbf{C}e_7\oplus\mathbf{C}e_8\oplus\mathbf{C}e_9\oplus\mathbf{C}e_{10}$
$g_{10,2}$	$10,6,3,1$	3	e_1,e_2,e_6,e_9	$g=g^{\beta_1}\oplus g^{\beta_2}\oplus g^{\beta_1+\beta_2}\oplus g^{2\beta_1+\beta_2}\oplus g^{3\beta_1+\beta_2}\oplus g^{\beta_6}\oplus g^{\beta_1+\beta_6}\oplus g^{2\beta_1+\beta_6}\oplus g^{\beta_9}\oplus g^{\beta_1+\beta_9}=\mathbf{C}e_1\oplus\mathbf{C}e_2\oplus\mathbf{C}e_3\oplus\mathbf{C}e_4\oplus\mathbf{C}e_5\oplus\mathbf{C}e_6\oplus\mathbf{C}e_7\oplus\mathbf{C}e_8\oplus\mathbf{C}e_9\oplus\mathbf{C}e_{10}$
$g_{10,3}$	$10,6$ $3,2,$ $1)$	3	e_1,e_2,e_7,e_9	$g=g^{\beta_1}\oplus g^{\beta_2}\oplus g^{\beta_1+\beta_2}\oplus g^{2\beta_1+\beta_2}\oplus g^{3\beta_1+\beta_2}\oplus g^{4\beta_1+\beta_2}\oplus g^{\beta_7}\oplus g^{\beta_1+\beta_7}\oplus g^{\beta_9}\oplus g^{\beta_1+\beta_9}=\mathbf{C}e_1\oplus\mathbf{C}e_2\oplus\mathbf{C}e_3\oplus\mathbf{C}e_4\oplus\mathbf{C}e_5\oplus\mathbf{C}e_6\oplus\mathbf{C}e_7\oplus\mathbf{C}e_8\oplus\mathbf{C}e_9\oplus\mathbf{C}e_{10}$
$g_{10,4}$	$10,6,3$	3	e_1,e_2,e_5,e_8	$g=g^{\beta_1}\oplus g^{\beta_2}\oplus g^{\beta_1+\beta_2}\oplus g^{2\beta_1+\beta_2}\oplus g^{\beta_5}\oplus g^{\beta_1+\beta_5}\oplus g^{2\beta_1+\beta_5}\oplus g^{\beta_8}\oplus g^{\beta_1+\beta_8}\oplus g^{2\beta_1+\beta_8}=\mathbf{C}e_1\oplus\mathbf{C}e_2\oplus\mathbf{C}e_3\oplus\mathbf{C}e_4\oplus\mathbf{C}e_5\oplus\mathbf{C}e_6\oplus\mathbf{C}e_7\oplus\mathbf{C}e_8\oplus\mathbf{C}e_9\oplus\mathbf{C}e_{10}$
$g_{10,5}$	$10,7,5,3,1$	2	e_1,e_2,e_5,e_8	$g=g^{\beta_1}\oplus g^{\beta_2}\oplus g^{\beta_1+\beta_2}\oplus g^{2\beta_1+\beta_2}\oplus g^{3\beta_1+\beta_2}\oplus g^{4\beta_1+\beta_2}\oplus g^{\beta_7}\oplus g^{\beta_1+\beta_7}\oplus g^{2\beta_1+\beta_7}\oplus g^{3\beta_1+\beta_7}=\mathbf{C}e_1\oplus\mathbf{C}e_2\oplus\mathbf{C}e_3\oplus\mathbf{C}e_4\oplus\mathbf{C}e_5\oplus\mathbf{C}e_6\oplus\mathbf{C}e_7\oplus\mathbf{C}e_8\oplus\mathbf{C}e_9\oplus\mathbf{C}e_{10}$
$g_{10,6}$	$10,7,5,3,2,1$	2	e_1,e_2,e_8	$g=g^{\beta_1}\oplus g^{\beta_2}\oplus g^{\beta_1+\beta_2}\oplus g^{2\beta_1+\beta_2}\oplus g^{3\beta_1+\beta_2}\oplus g^{4\beta_1+\beta_2}\oplus g^{5\beta_1+\beta_2}\oplus g^{\beta_8}\oplus g^{\beta_1+\beta_8}\oplus g^{2\beta_1+\beta_8}=\mathbf{C}e_1\oplus\mathbf{C}e_2\oplus\mathbf{C}e_3\oplus\mathbf{C}e_4\oplus\mathbf{C}e_5\oplus\mathbf{C}e_6\oplus\mathbf{C}e_7\oplus\mathbf{C}e_8\oplus\mathbf{C}e_9\oplus\mathbf{C}e_{10}$
$g_{10,7}$	$10,7,5,4,3,2,1$	2	e_1,e_2,e_9	$g=g^{\beta_1}\oplus g^{\beta_2}\oplus g^{\beta_1+\beta_2}\oplus g^{2\beta_1+\beta_2}\oplus g^{3\beta_1+\beta_2}\oplus g^{4\beta_1+\beta_2}\oplus g^{5\beta_1+\beta_2}\oplus g^{6\beta_1+\beta_2}\oplus g^{\beta_9}\oplus g^{\beta_1+\beta_9}=\mathbf{C}e_1\oplus\mathbf{C}e_2\oplus\mathbf{C}e_3\oplus\mathbf{C}e_4\oplus\mathbf{C}e_5\oplus\mathbf{C}e_6\oplus\mathbf{C}e_7\oplus\mathbf{C}e_8\oplus\mathbf{C}e_9\oplus\mathbf{C}e_{10}$
$g_{10,8}$	$10,8,7,6,5,4,3,2,1$	1	e_1,e_2	$g=g^{\beta_1}\oplus g^{\beta_2}\oplus g^{\beta_1+\beta_2}\oplus g^{2\beta_1+\beta_2}\oplus g^{3\beta_1+\beta_2}\oplus g^{4\beta_1+\beta_2}\oplus g^{5\beta_1+\beta_2}\oplus g^{6\beta_1+\beta_2}\oplus g^{7\beta_1+\beta_2}\oplus g^{8\beta_1+\beta_2}=\mathbf{C}e_1\oplus\mathbf{C}e_2\oplus\mathbf{C}e_3\oplus\mathbf{C}e_4\oplus\mathbf{C}e_5\oplus\mathbf{C}e_6\oplus\mathbf{C}e_7\oplus\mathbf{C}e_8\oplus\mathbf{C}e_9\oplus\mathbf{C}e_{10}$

root system	Cartan matrix	Dynkin diagram
	$\begin{bmatrix} 2 & -2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 2 \end{bmatrix}$	
	$\begin{bmatrix} 2 & -3 & -2 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix}$	
	$\begin{bmatrix} 2 & -4 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix}$	
	$\begin{bmatrix} 2 & -2 & -2 & -2 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix}$	

root system	**Cartan matrix**	**Dynkin diagram**
	$\begin{bmatrix} 2 & -4 & -3 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$	
	$\begin{bmatrix} 2 & -5 & -2 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$	
	$\begin{bmatrix} 2 & -6 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$	
	$\begin{bmatrix} 2 & -8 \\ -1 & 2 \end{bmatrix}$	

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