Conformally Closed Finsler Spaces

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Abstract

Let S be a set of a special kind of Finsler spaces. If $F^n \in S$ remains to belong to S by any conformal change of metric, then S is called *conformally closed*. The present paper is devoted mainly to studying conformally closed sets of Berwald spaces and Douglas spaces.

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1 Introduction

Let us denote by $F^n = (M^n, L(x,y))$ an n-dimensional Finsler space on a smooth n-manifold M^n with the fundamental metric function $L(x,y)$, $x = (x^i), y = (y^i)$. In the present paper we are concerned with the theory of conformal changes of metrics:

$$F^n = (M^n, L(x,y)) \to \bar{F}^n = (M^n, \bar{L}(x,y)), \quad \bar{L} = e^{c(x)}L, \tag{1.1}$$

with the conformal factor $c(x)$.

M. Hashiguchi ([5], 1976) has developed the theory based on the new formulation of Finsler connections initiated by the present author [9]. In the first place he found conformally invariant tensors

$$B_{ij} = (g_{ij} - 2l_i l_j)/F, \quad B^{ij} = F(g^{ij} - 2l^i l^j), \tag{1.2}$$

where $F = L^2/2$ and (B^{ij}) is the inverse of the matrix (B_{ij}).

Secondly he dealt with the quantities $G^i(x,y)$, from which the Berwald connection $B\Gamma = (G_j{}^i{}_k, G^i{}_j)$ is constructed; $G^i{}_j = \dot{\partial}_j G^i$ and $G_j{}^i{}_k = \dot{\partial}_k G^i{}_j$. On the conformal change (1.1) he showed

$$\bar{G}^i = G^i - B^{ir} c_r, \quad c_r = \partial_r c(x). \tag{1.3}$$

Then the changed Berwald connection $(\bar{G}_j{}^i{}_k, \bar{G}^i{}_j)$ of $\bar{F}^n$ is given as

$$\bar{G}^i{}_j = G^i{}_j - B^{ir}{}_j c_r, \quad \bar{G}_j{}^i{}_k = G_j{}^i{}_k - B^{ir}{}_{jk} c_r, \tag{1.4}$$

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where $B^{ir}{}_j = \dot{\partial}_j B^{ir}$ and $B^{ir}{}_{jk} = \dot{\partial}_k B^{ir}{}_j$.

Thirdly he constructed the change of the Cartan connection $C\Gamma = (F_j{}^i{}_k, G^i{}_j, C_j{}^i{}_k)$. Let us denote by $({}_{|i}, |i)$ the h- and v-covariant differentiations in $C\Gamma$. Then he showed

$$\bar{F}_j{}^i{}_k = F_j{}^i{}_k - U^{ir}{}_{jk} c_r.$$

We treat the conformal invariants $U^{ir}{}_{jk}$. Putting $V^{ir}{}_{jk} = U^{ir}{}_{jk} - B^{ir}{}_{jk}$, we have

$$g_{hr} g_{is} V^{rs}{}_{jk} = L T_{hijk}, \tag{1.5}$$

where the well-known T-tensor appears ([9]; [1], (3.5.3.1)), defined by

$$T_{hijk} = L C_{hij}|_k + (l_h C_{ijk} + ④\ ).$$

Here and throughout the following we shall use, as in [14], the abbreviations to avoid long expressions of the similar terms:

$$l_h C_{ijk} + ④ = l_h C_{ijk} + l_i C_{hjk} + l_j C_{hik} + l_k C_{hij},$$

$$h_{hi} F_{jk} + ⑥ = h_{hi} F_{jk} + h_{hj} F_{ik} + h_{hk} F_{ij} + h_{ij} F_{hk} + h_{ik} F_{hj} + h_{jk} F_{hi}.$$

Then he obtained the relation between the $(v)hv$-torsion tensors $P^i{}_{jk} = C_j{}^i{}_{k|0}$ as

$$\bar{P}^i{}_{jk} = P^i{}_{jk} + V^{ir}{}_{jk} c_r. \tag{1.6}$$

A Finsler space is called a *Landsberg space* ([1], [9]) if $P^i{}_{jk}$ vanishes identically. As a consequence of (1.6) with (1.5) we obtain Hashiguchi's theorem: A Landsberg space remains to be a Landsberg space by any conformal change of metric, if and only if its T-tensor vanishes identically. We should like to define the notion of "conformally closed" by expressing this theorem as

Theorem H. *A Landsberg space is conformally closed, if and only if its T-tensor vanishes identically.*

Since (1.5) shows $\bar{T}_{hijk} = e^{4c} T_{hijk}$, the condition $T = 0$ is conformally invariant. Hence, if we define the two sets:

- $L(n)$ $\cdots$ Landsberg spaces of dim . n,
- $L_c(n)$ $\cdots$ conformally closed Landsberg spaces of dim . n,

then any $F^n \in L_c(n)$ remains to belong to $L_c(n)$ by any conformal change of metric, while for any $F^n \in L(n) \backslash L_c(n)$ we have a function $c(x)$ such that $\bar{F}^n = (M^n, e^c L) \notin L(n)$. Thus $L_c(n)$ may be said to be *closed in* $L(n)$ with respect to conformal changes of metrics.

Further we should like to use "conformally closed" in a sense as follows: Let us consider an F^n with the 1-form metric $L(a^\alpha)$ where $a^\alpha = a_i^\alpha(x) y^i$ are n independent 1-forms of y^i [10]. Since $L(a^\alpha)$ is assumed to be positively homogeneous in a^α of degree one, the conformal change (1.1) yields $\bar{L} = e^c L(a^\alpha) = L(e^c a^\alpha)$. Consequently $\bar{L} = L(\bar{a}^\alpha)$ is still a 1-form metric with $\bar{a}^\alpha = e^c a^\alpha$. This property may be said as follows:

Proposition 1. The notion of the 1-form metric is conformally closed.

As a consequence we may say that the notion of the Riemannian metric is conformally closed and any Riemannian space is conformally closed. It is also obvious that any conformally flat Finsler space is conformally closed, because if F^n is conformal to a locally Minkowski space, then the conformally changed $\bar{F}^n$ of F^n is also conformal to the locally Minkowski space.

2 Conformally closed Berwald spaces, I

In Proposition 4 of the paper [13] the present author showed interesting Berwald spaces of dimension three:
Example 1. Berwald spaces which are respectively conformal to Minkowski spaces with the cubic metrics L_1 and L_2:

$$(L_1)^3 = \dot{x}^3 + \dot{y}^3 + \dot{z}^3 - 3\dot{x}\dot{y}\dot{z}, \quad (L_2)^3 = \dot{x}\dot{y}\dot{z}.$$

These Minkowski spaces $(\mathbf{R}^3, L_1)$ and $(\mathbf{R}^3, L_2)$ have the remarkable property: Any conformally changed spaces of these spaces are Berwald spaces.
Definition. A Berwald space is called *conformally closed*, if it remains to be a Berwald space by any conformal change of metric. We denote by $B_c(n)$ the set of all conformally closed Berwald spaces of dimension n.

Consequently the Minkowski spaces $(\mathbf{R}^3, L_1)$ and $(\mathbf{R}^3, L_2)$ in Example 1 belong to $B_c(3)$.

Now (1.4) leads to the conformal change of the hv-curvature tensor $G_j{}^i{}_{kl} = \dot{\partial}_l G_j{}^i{}_k$ of $B\Gamma$:

$$\bar{G}_j{}^i{}_{kl} = G_j{}^i{}_{kl} - B^{ir}{}_{jkl} c_r, \quad B^{ir}{}_{jkl} = \dot{\partial}_l B^{ir}{}_{jk}. \tag{2.1}$$

A Finsler space F^n is called a *Berwald space* ([1], [9]), if $B\Gamma$ is linear, that is, $G_j{}^i{}_k$ are functions of position alone. Therefore F^n is a Berwald space, if and only if $G_j{}^i{}_{kl}$ vanishes identically. Hence (2.1) shows
Proposition 2. A Berwald space is conformally closed, if and only if $B^{ir}{}_{jkl}$ vanishes identically.

Since $B^{ir}{}_{jkl} = \dot{\partial}_j \dot{\partial}_k \dot{\partial}_l B^{ir}$ and B^{ir}, defined by (1.2) is written as $B^{ir} = (L^2/2)g^{ij} - y^i y^j$, we obtain
Theorem 1. *A Berwald space is conformally closed, if and only if $L^2 g^{ij}$ are homogeneous polynomials in (y^i) of degree two.*
Remark. Compare the expression of Theorem 1 with that of the definition of Douglas space ([3], p. 388). Both are expressions peculiar to Finsler geometry. Cf. Theorem 7.
Example 2. We deal with a Finsler space F^n with the *m-th root metric L*:

$$L^m = a_{hi\cdots l}(x) y^h y^i \cdots y^l,$$

where the coefficients $a_{hi\cdots l}(x)$ are components of a covariant symmetric m-tensor ([13], [14], [17], [18], [22]). We define covariant $(m-r)$-tensors

$$a_{hi\cdots j}(x, y) = a_{hi\cdots jk\cdots l}(x) y^k \cdots y^l / L^r,$$

and the inverse (a^{ij}) of the matrix (a_{ij}). Then $a^i = a^{ir} a_r$ is equal to $l^i = y^i/L$ and

$$g^{ij} = \{a^{ij} + (m-2)a^i a^j\}/(m-1).$$

Consequently Theorem 1 leads to
Proposition 3. A Berwald space with a m-th root metric is conformally closed, if and only if $L^2 a^{ij}$ are homogeneous polynomials in (y^i) of degree two.
Example 1. **(1)** We pay attention to the space $(\mathbf{R}^3, L_1)$ again. Putting $(y^i) = (\dot{x}, \dot{y}, \dot{z})$, we get

$$(L_1)^2a^{11} = (y^1)^2 - 4y^2y^3, \quad (L_1)^2a^{12} = -y^1y^2 - 2(y^3)^2.$$

Hence Proposition 3 shows that $(\mathbf{R}^3, L_1)$ is conformally closed as a Berwald space.

(2) We consider the space $(\mathbf{R}^n, L)$, a generalization of $(\mathbf{R}^3, L_2)$ of Example 1, where L, given by

$$L^n = n!y^1y^2\cdots y^n,$$

is called the *Berwald-Moór metric* ([9], Proposition 24.2; [22]). H. Shimada shows

$$L^2a^{11} = -n(n-2)(y^1)^2, \quad L^2a^{12} = ny^1y^2.$$

Hence $(\mathbf{R}^n, L)$ is conformally closed as a Berwald space.

Example 3. The *ecological metric* L ([1], (5.4.1.2); [22]) is given by

$$L^m = (y^1)^m + \cdots + (y^n)^m.$$

H. Shimada shows $L^2a^{11} = L^m/(y^1)^{m-2}$ and $a^{12} = 0$. Hence the Minkowski space $(\mathbf{R}^n, L)$ is *not* conformally closed as a Berwald space.

Example 4. We treat a simple quartic metric L:

$$L^4 = 6(y^1)^2y^2y^3.$$

It is easy to show typical $L^2a_{11} = 2y^2y^3$, $L^2a_{12} = 2y^1y^3$, $a_{22} = 0$, $L^2a_{23} = (y^1)^2$, and we have

$$L^2a^{11} = -(y^1)^2, \quad L^2a^{12} = 2y^1y^2, \quad L^2a^{22} = -4(y^2)^2, \quad L^2a^{23} = 2y^2y^3.$$

Consequently the Minkowski space $(\mathbf{R}^3, L)$ is conformally closed as a Berwald space.

Example 5. Let us consider a Minkowski space $(\mathbf{R}^3, L)$ with another quartic metric L:

$$L^4 = 6y^1y^2y^3Y, \quad Y = y^1 + y^2 + y^3.$$

We have typical $L^2a_{11} = 2y^2y^3$, $L^2a_{12} = y^3(2Y - y^3)$, and hence

$$Da^{11} = L^2(y^1)^2\{4y^2y^3 - (2Y - y^1)^2\},$$

$$Da^{12} = L^2y^1y^2\{(2Y - y^1)(2Y - y^2) - 2y^3(2Y - y^3)\},$$

where $D = L^4(Y^2 - y^1y^2 - y^2y^3 - y^3y^1)$. Accordingly Proposition 3 shows that $(\mathbf{R}^3, L)$ is *not* conformally closed as a Berwald space.

Example 6. Finally we consider a Minkowski space $(\mathbf{R}^3, L)$ with the quartic metric L:

$$L^4 = 6(y^1)^2\{(y^2)^2 + (y^3)^2\}.$$

We have typical $L^2a_{11} = (y^2)^2 + (y^3)^2$, $L^2a_{12} = 2y^1y^2$, $L^2a_{22} = (y^1)^2$, $a_{23} = 0$, and hence

$$L^2a^{11} = -2(y^1)^2, \quad L^2a^{12} = 4y^1y^2,$$

$$L^2a^{22} = 2\{3(y^3)^2 - (y^2)^2\}, \quad L^2a^{23} = -8y^2y^3.$$

Therefore $(\mathbf{R}^3, L)$ is conformally closed as a Berwald space.

3 Conformally closed Berwald spaces, II

We shall write the condition $\dot\partial_h\dot\partial_i\dot\partial_j(Fg^{ab})=0,\ F=L^2/2$, stated in Theorem 1 in terms of well-known tensors of F^n.

Put $F_{hi\cdots k}=\dot\partial_h\dot\partial_i\cdots\dot\partial_k F$. We have $F_{hi}=g_{hi}$ and hence $g^{ad}F_{dc}=\delta^a_c$, from which we obtain

$$(\dot\partial_h g^{ad})F_{dc}+g^{ad}F_{dch}=0,$$

$$(\dot\partial_h\dot\partial_i g^{ad})F_{dc}+(\dot\partial_h g^{ad})F_{dci}+(\dot\partial_i g^{ad})F_{dch}+g^{ad}F_{dchi}=0,$$

$$(\dot\partial_h\dot\partial_i\dot\partial_j g^{ad})F_{dc}+\{(\dot\partial_h\dot\partial_i g^{ad})F_{dcj}+③\}+\{(\dot\partial_h g^{ad})F_{dcij}+③\}+g^{ad}F_{dchij}=0.$$

Since $F_{dcj}=\dot\partial_j g_{dc}=2C_{dcj}$, the above three equations give respectively

$$\dot\partial_h g^{ab}=-2C^{ab}{}_h, \tag{3.1}$$

$$\dot\partial_h\dot\partial_i g^{ab}=4(C^{ar}{}_hC_r{}^b{}_i+C^{ar}{}_iC_r{}^b{}_h)-F^{ab}{}_{hi}, \tag{3.2}$$

$$\dot\partial_h\dot\partial_i\dot\partial_j g^{as}=2(F^{ar}{}_{hi}C_r{}^b{}_j+F^{br}{}_{hi}C_r{}^a{}_j+③)$$
$$-8\{(C_r{}^a{}_hC_s{}^b{}_i+C_r{}^b{}_hC_s{}^a{}_i)C^{rs}{}_j+③\}-F^{ab}{}_{hij}, \tag{3.3}$$

where $F^{ab}{}_{hi}=g^{ar}g^{bs}F_{rshi}$ and $F^{ab}{}_{hij}=g^{ar}g^{bs}F_{rshij}$.

Let us transvect these equations with $y_a=g_{ar}y^r$. (3.1) and (3.2) give respectively $y_a\dot\partial_h g^{ab}=0$ and $y_a\dot\partial_h\dot\partial_i g^{ab}=F^b{}_{hi}=2C_h{}^b{}_i$. Further (3.3) yields

$$y_a\dot\partial_h\dot\partial_i\dot\partial_j g^{ab}=2F^b{}_{hij}-4(C_h{}^r{}_iC_r{}^b{}_j+③).$$

Now we have $B^{ab}{}_{hij}=\dot\partial_h\dot\partial_i\dot\partial_j(Fg^{ab})$ of the form

$$B^{ab}{}_{hij}=F_{hij}g^{ab}+(F_{hi}\dot\partial_j g^{ab}+F_h\dot\partial_i\dot\partial_j g^{ab}+③)+F\dot\partial_h\dot\partial_i\dot\partial_j g^{ab}. \tag{3.4}$$

Hence we have

$$y_aB^{ab}{}_{hij}=2C_{hij}y^b+2L(l_hC_i{}^b{}_j+③)+F\{2F^b{}_{hij}-4(C_h{}^r{}_iC_r{}^b{}_j+③)\}.$$

In the author's paper [14] the following equation has been given:

$$FF_{hijk}=LT_{hijk}-(y_hC_{ijk}+④)+L^2C^2{}_{hijk},$$

$$C^2{}_{hijk}=C_{hir}C_j{}^r{}_k+③,$$

where the tensor C^2, defined first in [14], is symmetric. Therefore we obtain $y_aB^{ab}{}_{hij}=2LT_{hijr}g^{rb}$, and consequently $B^{ab}{}_{hij}=0$ implies T-tensor $=0$.

This fact, T-tensor $=0$, is certainly in expectation. In fact, first $B^{ab}{}_{hij}=0$ is the necessary and sufficient condition for F^n to belong to $B_c(n)$, and $B_c(n)\subset L_c(n)$ is clear. Consequently Theorem H shows T-tensor $=0$ of F^n. We have another reason: (2.1) gives

$$\bar y_i\bar G_j{}^i{}_{kl}=e^{2c}(y_iG_j{}^i{}_{kl}-y_iB^{ir}{}_{jkl}c_r).$$

As it is well-known ([2], §2), F^n is a Landsberg space, if and only if $y_i G_j{}^i{}_{kl} = 0$. Therefore the above implies that F^n is a conformally closed as a Landsberg space, if and only if $y_i B^{ir}{}_{jkl} = 0$, and hence Theorem H shows that $y_i B^{ir}{}_{jkl} = 0$ is equivalent to T-tensor = 0.

Now we shall continue to calculate $B^{ab}{}_{hij} = 0$ from(3.4) on the assumption "T-tensor = 0". By substituting from(3.1), (3.2) and (3.3), after the long computation, we conclude

$$(3.5)\quad \frac{1}{2}B^{ab}{}_{hij} = C_{hij}h^{ab} + (L^2 C_h{}^r{}_i C^{2ab}{}_{rj} - h_{ij}C^{ab}{}_h + ③\ ) - L^2 C^{abr}C^2{}_{rhij} - T^{ab}{}_{hij},$$

where $T^{ab}{}_{hij} = g^{ar}g^{bs}T_{rshij}$ is defined in [14] as

$$(3.6)\qquad 2T_{hijkl} = LT_{hijk}|_l - (h_{lh}C_{ijk} + h_{li}C_{hjk} + h_{lj}C_{hik} + h_{lk}C_{hij})$$
$$+L^2(C_l{}^r{}_h C^2{}_{rijk} + C_l{}^r{}_i C^2{}_{rhjk} + C_l{}^r{}_j C^2{}_{rhik} + C_l{}^r{}_k C^2{}_{rhij}).$$

It should be remarked that T_{hijk}, $C^2{}_{hijk}$ and further T_{hijkl} are completely symmetric tensors.

Therefore we can conclude

Theorem 2. *A Berwald space is conformally closed, if and only if the T-tensor vanishes and*

$$(3.7)\qquad C_{hij}h^{ab} + (L^2 C_h{}^r{}_i C^{2ab}{}_{rj} - h_{hi}C^{ab}{}_j + ③\ ) - L^2 C^{abr}C^2{}_{rhij} - T^{ab}{}_{hij} = 0.$$

4 Conformally closed Berwald spaces of dimension two

The present section is devoted to conformally closed Berwald spaces of dimension two. Let us apply Berwald's theory of two-dimensional Finsler spaces in terms of the Berwald frame field (l, m) ([1], [2]). Then the angular metric tensor h_{ij} is written as $h_{ij} = \epsilon m_i m_j$ with the signature $\epsilon = \pm 1$ and the C-tensor is $LC_{hij} = Im_{hij}$ with the main scalar $I(x, y)$, where m_{hij} is the abbreviation of $m_h m_i m_j$. Cartan's h-covariant derivatives $C_{hij|k}$ is

$$LC_{hij|k} = m_{hij}(I_{,1}l_k + I_{,2}m_k),$$

where $I_{,1}l_k + I_{,2}m_k = I_{|k} = \partial_k I - (\dot{\partial}_r I)G^r{}_k$.

F^2 is a Berwald space if and only if $I_{,1} = I_{,2} = 0$. Then one of the Ricci formulae $I_{,1,2} - I_{,2,1} = -RI_{;2}$, where $L\dot{\partial}_i I = I_{;2}m_i$, yields $I_{;2} = 0$ or the Gauss curvature $R = 0$. It is obvious that $I_{,1} = I_{,2} = I_{;2} = 0$ show I = const., and a Berwald space with $R = 0$ is locally Minkowski. Consequently we have Berwald's theorem ([1], Theorem 3.5.3.1; [9], Theorem 28.2): We define five sets of Finsler spaces of dimension two such that

- $B(2) \cdots$ Berwald spaces,
- $B_1(2) \cdots$ spaces with I = const. and $R \neq 0$,
- $B_2(2) \cdots$ spaces with I = const. and $R = 0$,
- $B_3(2) \cdots$ spaces with $I_{;2} \neq 0$ and $R = 0$,
- $M(2) \cdots$ locally Minkowski spaces.

Then we have

$$B(2) = B_1(2) \cap B_2(2) \cap B_3(2) \qquad \text{(direct sum)}, \tag{4.1}$$

$$M(2) = B_2(2) \cap B_3(2). \tag{4.2}$$

The T-tensor is written as $LT_{hijk} = I_{;2} m_{hijk}$ ([1], [9]). Hence, if we consider the set $B_c(2)$ of conformally closed Berwald spaces of dimension two, any $F^2 \in B_c(2)$ has $I_{;2} = 0$ and belongs to $B_1(2) \cap B_2(2)$.

We have to pay attention to the second condition (3.7). As has been shown in [14], if $I_{;2} = 0$ holds, then we have

$$L^2 C^2_{hijk} = 3\epsilon I^2 m_{hijk}, \quad LT_{hijkl} = (6I^3 - 2\epsilon I) m_{hijkl}.$$

As a consequence it is observed that (3.7) holds automatically and we obtain

Theorem 3. *A Berwald space of dimension two is conformally closed, if and only if it has the constant main scalar, that is,*

$$B_c(2) = B_1(2) \cap B_2(2). \tag{4.3}$$

Remark 1. The main scalar I is conformally invariant. This will suggest Theorem 3.

Remark 2. The condition (3.7) does not give any restriction on the assumption "T-tensor = 0" for the two-dimensional case, as it has been shown above. This remarkable fact is also verified from the "Reduction theorem of certain Landsberg spaces to Berwald spaces" ([2], [12]): If F^2 is a Landsberg space with vanishing T-tensor, then F^2 is a Berwald space. Thus we get $B_c(2) = L_c(2)$.

5 Conformally closed spaces with cubic metric

We are concerned with the conformal closedness of Finsler spaces with cubic metric L: $L^3 = a_{hij}(x) y^h y^i y^j$, where a_{hij} are components of a covariant symmetric 3-tensor. As in Example 2, we put

$$a_{hi} = a_{hij}(x) y^j / L, \quad a_h = a_{hij}(x) y^i y^j / L^2.$$

Then we get [17]

$$l_i = a_i, \quad h_{ij} = 2(a_{ij} - a_i a_j), \quad LC_{hij} = a_{hij} - (a_{hi} a_i + ③\ ) + 2a_h a_i a_j.$$

Throughout the theory of m-th root metrics ([17], [22]) the regularity of the metric, $\det(a_{ij}) \neq 0$, is assumed. By the inverse (a^{ij}) of the matrix (a_{ij}) we define $a^i = a^{ir} a_r$ and $a^h{}_{ij} = a^{hr} a_{rij}$. Then $a^i = l^i$ and

$$2LC_i{}^h{}_j = a^h{}_{ij} - \delta^h_i a_i - \delta^h_j a_i + a^h (2a_i a_j - a_{ij}).$$

The C^2-tensor, defined in (3.5) is written in the form

$$\begin{aligned} 2L^2 C^2{}_{hijk} &= (a_{hir} a^r{}_{jk} - a_{hi} a_{jk} + ③\ ) \\ &- 3(a_h a_{ijk} + ④\ ) + 4(a_h a_i a_{jk} + ⑥\ ) - 12 a_h a_i a_j a_k. \end{aligned} \tag{5.1}$$

As has been shown in [17], the characteristic property of the cubic metric is $\dot\partial_h\dot\partial_i\dot\partial_j\dot\partial_k L^3 = 0$, from which we obtain the theorem: A Finsler space is equipped with cubic metric, if and only if its T-tensor has the form

$$2LT_{hijk} = -2L^2{C^2}_{hijk} - (h_{hi}h_{jk} + ③).$$

Therefore Theorem H leads to the theorem: A Landsberg space with cubic metric is conformally closed, if and only if

$$2L^2{C^2}_{hijk} + (h_{hi}h_{jk} + ③) = 0. \tag{5.2}$$

In the case of the cubic metric we have

$$h_{hi}h_{jk} + ③ = 4(a_{hi}a_{jk} + ③) - 4(a_h a_i a_{jk} + ⑥) + 12a_h a_i a_j a_k.$$

This together with (5.1) leads (5.2) to the concrete form

$$(a_{hir}{a^r}_{jk} + 3a_{hi}a_{jk} + ③) - 3(a_h a_{ijk} + ④) = 0. \tag{5.2$'$}$$

However the Reduction theorem has been shown in [13]: If F^n with cubic metric is a Landsberg space, then it is a Berwald space. Therefore we have finally

Theorem 4. *A Berwald space with cubic metric is conformally closed, if and only if its T-tensor $= 0$, that is,* (5.2) *or* (5.2′) *holds identically.*

Remark. If the T-tensor $= 0$, then from (5.2) we get the following form of T_{hijkl}, defined by (3.6):

$$2T_{hijkl} = -(h_{hi}C_{jkl} + ⑩).$$

Then it is easy to show that the condition (3.7) holds automatically.

Example 1. We consider the Minkowski spaces, treated in Example 1 again.

(1) $(\mathbf{R}^3, L_1)$, $(L_1)^3 = (y^1)^3 + (y^2)^3 + (y^3)^3 - 3y^1y^2y^3$.

We have

$$(L_1)^2{a^1}_{11} = (y^1)^2 - 4y^2y^3, \quad (L_1)^2{a^1}_{12} = \{y^1y^3 + 2(y^2)^2\}/2,$$
$$(L_1)^2{a^1}_{22} = -\{y^1y^2 + 2(y^3)^2\}, \quad (L_1)^2{a^1}_{23} = -\{(y^1)^2 - 4y^2y^3\}/2.$$

Consequently (5.2′) gives $T_{1111} = T_{1112} = T_{1122} = T_{1123} = 0$, that is, $T_{hijk} = 0$, $h,i,j,k = 1,2,3$.

(2) $(\mathbf{R}^3, L_2)$, $(L_2)^3 = y^1y^2y^3$.

We have

$$ {a^1}_{11} = a^1_{22} = 0, \quad (L_2)^2{a^1}_{12} = y^1y^3/2, \quad (L_2)^2{a^1}_{23} = -(y^1)^2/2.$$

Similarly to (1), we get $T_{hijk} = 0$.

The space $(\mathbf{R}^3, L_2)$ has $C_i = C_i{}^r{}_r = 0$ ([9], Proposition 24.2). Similarly $(\mathbf{R}^3, L_1)$ has $C_i = 0$.

The necessary and sufficient condition for a Finsler space with cubic metric to be a Berwald space has been discussed in §2 of [13], but it is a difficult problem to write the condition in terms of the coefficients $a_{ijk}(x)$. Here we write the equation

$$La_{hr}G_i{}^r{}_{jk} + (a_{hir}G_j{}^r{}_k + ③) = \{ijk, h\}$$

where $G_i{}^r{}_{jk}$ is the hv-curvature tensor of the Berwald connection and $\{ijk, h\}$ are generalized Christoffel symbols, defined first in [18].

6 Conformal closedness of Berwald spaces with (α, β)-metric

We consider a generalized Randers space F^n [20], which is a Finsler space with (α, β)-metric $L(\alpha, \beta)$ ([1], [9]). If we deal with a conformal change of metric (1.1), then we have

$$\bar{L} = e^c L(\alpha, \beta) = L(e^c \alpha, e^c \beta),$$

because $L(\alpha, \beta)$ is assumed to be positively homogeneous in (α, β) of degree one. Consequently we get $\bar{L} = L(\bar{\alpha}, \bar{\beta}), \quad (\bar{\alpha}, \bar{\beta}) = (e^c \alpha, e^c \beta)$ [6], and hence the changed space $\bar{F}^n$ is still a generalized Randers space. Therefore, similarly to the case of the 1-form metric, we have

Proposition 4. The notion of the generalized Randers space is conformally closed.

We now consider a Randers space F^n with $L = \alpha + \beta$. Theorem 4 of the paper [8] states that if F^n has the T-tensor $= 0$, then β should vanish, that is, F^n is reduced to a Riemannian space with the Riemannian metric α. Therefore Theorem 2 leads to

Theorem 5. *If a Randers space is conformally closed Berwald space, then it is a Riemannian space.*

Here we shall show a direct proof of this theorem. A Randers space F^n with $L = \alpha + \beta, \ \ \alpha^2 = a_{ij}(x) y^i y^j, \ \ \beta = b_i(x) y^i$, is a Berwald space ([7], [8]), if and only if $b_{i;j} = 0$ in the Levi-Civita connection $\{\gamma_j{}^i{}_k(x)\}$ of the associated Riemannian space with α. The conformally changed space $\bar{F}^n$ of F^n has $\bar{a}_{ij} = e^c a_{ij}$ and $\bar{b}_i = e^c b_i$. From $\bar{\gamma}_j{}^i{}_k = \gamma_j{}^i{}_k + \delta^i{}_j c_k + \delta^i{}_k c_j - c^i a_{jk}, \ \ c_k = \partial_k c, \ \ c^i = a^{ir} c_r$ we have

$$\bar{b}_{i;j} = e^c (b_{i;j} - c_i b_j + b_r c^r a_{ij}). \tag{6.1}$$

If both F^n and $\bar{F}^n$ are Berwald spaces, then (6.1) gives $c_r(b^r a_{ij} - \delta^r{}_i b_j) = 0$. If this is satisfied for any $c(x)$, then we have $b^r a_{ij} = \delta^r{}_i b_j$, which implies $b_i = 0$. Thus we proved Theorem 5.

Secondly we consider a Kropina space F^n with $L = \alpha^2/\beta$. Theorem 2 of C. Shibata's paper [21] states that the T-tensor of F^n never vanishes. Therefore we have

Proposition 5. A Kropina space is not a conformally closed Berwald space.

We shall treat a conformal change of a Kropina space of Berwald type in detail. A Kropina space F^n is a Berwald space ([7], [10], [21]) if and only if we have function $f_i(x)$ satisfying

$$b_{i;j} = f_r b^r a_{ij} + b_i f_j - b_j f_i, \quad b^r = a^{ri} b_i. \tag{6.2}$$

The conformally changed Kropina space $\bar{F}^n$ is also assumed to be a Berwald space:

$$\bar{b}_{i;j} = \bar{f}_r \bar{b}^r \bar{a}_{ij} + \bar{b}_i \bar{f}_j - \bar{b}_j \bar{f}_i = e^c (\bar{f}_r b^r a_{ij} + b_i \bar{f}_j - b_j \bar{f}_i).$$

We have (6.1) and (6.2), and hence the above gives

$$(\bar{f}_r - f_r - c_r) b^r a_{ij} = (\bar{f}_i - f_i - c_i) b_j - (\bar{f}_j - f_j) b_i.$$

This yields $(2\bar{f}_i - 2f_i - c_i) b_j = (2\bar{f}_j - 2f_j - c_j) b_i$, so that we have a function $\kappa(x)$ satisfying $2\bar{f}_i - 2f_i - c_i = 2\kappa b_i$. Hence $(c_r - 2\kappa b_r) b^r a_{ij} = b_i c_j + b_j c_i$, and $n = 2$ is necessary, provided $(c_r - 2\kappa b_r)b^r \neq 0$. In the case of $n > 2$ we have $c_i = 0$. Therefore we have

Theorem 6. *Let a Kropina space F^n, $n > 2$, be a Berwald space. The conformally changed Kropina space F^n is still a Berwald space, if and only if the change is homothetic.*

7 Conformal change of Douglas spaces

A Finsler space is by definition a *Douglas space* ([3], [4]), if $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomials in (y^i) of degree three.

Definition. A Douglas space is called *conformally closed*, if it remains to be a Douglas space by any conformal change of metric.

For a conformal change (1.1) we have (1.3). Thus, for D^{ij} we get $\bar{D}^{ij} = D^{ij} - D^{ijr}c_r$, where $D^{ijr} = B^{ir}y^j - B^{jr}y^i$. From (1.2) we have

$$D^{ijr} = F(g^{ir}y^j - g^{jr}y^i), \quad F = L^2/2. \tag{7.1}$$

Therefore we have

Theorem 7. *A Douglas space is conformally closed, if and only if D^{ijr} of* (7.1) *are homogeneous polynomials in (y^i) of degree three.*

It is shown [3] that a Finsler space is a Douglas space, if and only if the projective invariants $Q^i = G^i - G^r{}_r y^i/(n+1)$ are homogeneous polynomials in (y^i) of degree two. (1.3) and (1.4) show

$$\bar{Q}^i = Q^i - \{B^{ir} - B^{rs}{}_s y^i/(n+1)\}c_r.$$

Consequently we have

Proposition 6. A Douglas space is conformally closed, if and only if $B^{ij} - B^{jr}{}_r y^i/(n+1)$ are homogeneous polynomials in (y^i) of degree two.

We treat a Kropina space with $L = \alpha^2/\beta$ again. Since a two-dimensional Kropina space is Douglas space without any restriction [3], we have immediately

Theorem 8. *A two-dimensional Kropina space is a conformally closed Douglas space.*

For a Kropina space of arbitrary dimension we have ([9], [21])

$$2(\alpha/\beta)^2 g^{ij} = a^{ij} - b^i b^j/b^2 + (2\beta/b^2\alpha^2)(b^i y^j + b^j y^i) + (2/b^2\alpha^4)(b^2\alpha^2 - 2\beta^2)y^i y^j.$$

Hence we have

$$2D^{ijk} = \{\alpha^2 a^{ik}/2 - (\alpha^2/2b^2)b^i b^k + (\beta/b^2)b^i y^k\}y^j - [i,j],$$

where $[i,j]$ denotes the interchange of i, j. Thus Theorem 7 yields

Theorem 9. *If a Kropina space F^n, $n > 2$, is Douglas space, then it is a conformally closed Douglas space.*

If we consider a conformal change of an (α, β)-metric, then we get $\bar{L} = e^c L(\alpha, \beta) = L(\bar{\alpha}, \bar{\beta})$, where

$$\bar{\alpha} = e^c\alpha \quad (\bar{a}_{ij} = e^c a_{ij}); \quad \bar{\beta} = e^c\beta \quad (\bar{b}_i = e^c b_i).$$

For $s_{ij} = (\partial_j b_i - \partial_i b_j)/2$ we get

$$\bar{s}_{ij} = e^c\{s_{ij} + (b_i c_j - b_j c_i)/2\}. \tag{7.2}$$

We are concerned with the conformal change of a Randers space F^n with $L = \alpha+\beta$. F^n is a Douglas space [3], if and only if $s_{ij} = 0$, that is, there exists locally a function $b(x)$ such that $b_i = \partial_i b$. Then (7.2) is reduced to $\bar{s}_{ij} = e^c(b_i c_j - b_j c_i)/2$. Hence we get $\bar{s}_{ij} = 0$, that is, $\bar{F}^n$ is still a Douglas space, if and only if $b_i c_j - b_j c_i = 0$, that is, c_i is proportional to b_i. Therefore we have

Theorem 10. *Let F^n be a Randers space of Douglas type with $L = \alpha + \beta$, provided that $\beta = (\partial_i b(x))y^i \neq 0$.*

(1) *The conformally changed $\bar{F}^n$ is not of Douglas type in general.*

(2) *$\bar{F}^n$ is also of Douglas type, if and only if the conformal factor $c(x)$ is such that $\partial_i c$ is proportional to ∂b_i.*

From (7.2) it is follows that if we put $s_i = b^r s_{ri}$, then we get

$$\bar{s}_i = s_i + (b^2 c_i - b^r c_r b_i)/2, \quad b^2 = b^r b_r,$$

$$\bar{s}_{ij} - (\bar{b}_i \bar{s}_j - \bar{b}_j \bar{s}_i)/\bar{b}^2 = e^c\{s_{ij} - (b_i s_j - b_j s_i)/b^2\}.$$

Consequently, if F^n has $s_{ij} - (b_i s_j - b_j s_i)/b^2 = 0$, so is the conformally changed $\bar{F}^n$. The condition $s_{ij} - (b_i s_j - b_j s_i)/b^2 = 0$ is necessary and sufficient for a Kropina space F^n, $n > 2$, to be a Douglas space [16]. Therefore we could obtain another proof of Theorem 9 not due to Theorem 7.

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