Optimal Approximations on Riemannian Manifolds

M.Toda and C.Udrişte

Abstract

The paper studies some new variational problems, extending certain applications of optimal approximations by piecewise smooth functions in computer vision.

For this aim, we introduce families of functionals as generalizations of those considered by D.Mumford and I.Shah [1], via the idea of p-energy [4]-[7]. Among these functionals, one is of the Finslerian type.

Mathematics Subject Classification: 49F20, 92A27, 53B40
Key words: Riemannian manifold, variational problems, (α, β)-Finsler metric

Sonar, radar or laser "range" data, involve a function $\varphi(x, y)$ of two variables that represents the distance from a fixed point P in direction (x, y) to the nearest solid object. Another example using a function $\varphi(x, y)$ of two variables is the one of a lens at P focussing the light on a planar domain R of Cartesian coordinates x, y, where $\varphi(x, y)$ is the intensity of the light signal striking R at the point (x, y). The function $\varphi(x, y)$ defined on the planar domain R is called an *image*. In [1], authors study the problem of appropriately decomposing the planar domain R. In this scope, the authors introduced certain variational problems. Obviously, in this context, the image $\varphi(x, y)$ presents discontinuities caused by the location of the bodies.

Taking into account that in practice R is not always a planar domain (it might be a lens, a sphere, a parabolic surface and so on) and on the other side that the image may be reconverted by a regular transformation, we consider R as a two-dimensional Riemannian space endowed with the Riemannian metric $g(x, y)$. The set R is a bounded and the function $\varphi(x, y)$ is modeled by a set of smooth functions Ψ_i respectively defined on disjoint connected open subsets R_i of the domain R, each one with a piecewise smooth boundary.

Let Γ be the union of the parts of the boundaries of R_i inside R. The decomposition $R = R_1 \cup R_2 \cup \ldots \cup R_n \cup \Gamma$ must be such that φ varies smoothly and slowly within each R_i and varies discontinuously across most of the set Γ. We look for an optimal approximation of $\varphi(x, y)$ by piecewise smooth functions $\Psi(x, y)$ such that $\Psi_i = \Psi|_{R_i}$ be differentiable. For this aim, we generalize the functional E from [1],

Balkan Journal of Geometry and Its Applications, Vol.4, No.1, 1999, pp. 135-144

$$E(\Psi,\Gamma)=\mu\iint_R(\Psi-\varphi)^2dxdy+\iint_{R-\Gamma}||grad\,\Psi||^2dxdy+\nu|\Gamma|,$$

of a planar domain, by introducing the functional

$$E_p(\Psi,\Gamma)=\mu\int\int_R|\Psi-\varphi|^{2p}d\sigma+\int\int_{R-\Gamma}||grad\,\Psi||^{2p}d\sigma+\nu l_p(\Gamma), \tag{1}$$

where Ψ is assumed to be differentiable on $\cup R_i$ (but can be discontinuous on Γ), p a real positive number, $d\sigma=\sqrt{det\,g}dx^1\wedge dx^2$ is the element of area, μ,ν are positive real numbers,

$$l_p(\gamma)=\int_0^1||\dot\gamma(t)||^pdt=\int_0^1\left[\sqrt{g_{ij}(x^1,x^2)\dot x^i\dot x^j}\right]^pdt$$

is the p-energy of any parametrized trajectory $\gamma(t)$, and $l_p(\Gamma)$ is the total p-energy of the arcs making up Γ. The case $p=1$ was studied in [1].

The best approximation Ψ of φ is the one which minimizes E_p.

Remarks. a) the first term of E_p requires that Ψ approximates φ.

b) the second term of E_p requires that Ψ (and hence φ) does not vary strongly in the sets R_i.

c) the third term of E_p requires that the curve Γ that accomplish these conditions be as short as possible.

First, we shall analyse the first variation for E_p with respect to Γ, considering just the case of a simple point $P\in\Gamma$, near which Γ moves. Such a point P lies on exactly one curve $\gamma_i\subset\Gamma$ of class C^1 which in a small neighbourhood U of P can be considered as a plane curve of type $y=h(x)$, or $x=h(y)$. Moreover, we consider the set U to be planar. We can deform γ_i to the curve

$$\gamma_i(t):y=h(x)+t\delta h(x),$$

where δh is zero outside a small neighbourhood of P.

If t is small, the new curve $\gamma_i(t)$ meets no curve γ_j', $j\neq i$ except at its endpoints, and

$$\Gamma(t)=\gamma_i(t)\cup\bigcup_{j\neq i}\gamma_j$$

is a deformation of Γ. We can no longer speak of leaving Ψ *fixed* while Γ moves, since Ψ is of class C^1 on $R-\Gamma$ but Ψ is usually discontinuous across Γ. Instead of this, we set

$$\Psi^+=\Psi|_{U^+},\quad U^+=\{(x,y)|y>h(x)\}\cap U$$
$$\Psi^-=\Psi|_{U^-},\quad U^-=\{(x,y)|y<h(x)\}\cap U.$$

We choose some C^1 extensions of Ψ^+ from U^+ to U, and of Ψ^- from U^- to U. This is possible, since $\Psi\in C^1$ on both sides of Γ at all the simple points of Γ.

Now let

$$\Psi^t(p)=\begin{cases}\Psi(P) & \text{if}\quad P\notin U\\ \text{extension of }\Psi^+ & \text{if}\quad P\in U,\ P\text{ above }\gamma_i(t)\\ \text{extension of }\Psi^- & \text{if}\quad P\in U,\ P\text{ below }\gamma_i(t).\end{cases}$$

Consequently,

$$E_p(\Psi^t,\gamma(t)) - E_p(\Psi,\gamma) = \mu \iint_U \left\{ \left[(\Psi^t-\varphi)^2\right]^p - \left[(\Psi-\varphi)^2\right] \right\} dxdy +$$

$$+ \iint_{U-\Gamma(t)} ||grad\ \Psi^t||^{2p} dxdy - \iint_{U-\Gamma} ||grad\ \Psi||^{2p} dxdy +$$

$$+\nu\left[l_p(\gamma_i(t)) - l_p(\gamma_i)\right] = \mu \int_I \left(\int_{h(x)}^{h(x)+t\delta h(x)} \left\{ \left[(\Psi^- - \varphi)^2\right]^p - \left[(\Psi^+ - \varphi)^2\right]^p \right\} dy \right) dx +$$

$$+ \int_I \left(\int_{h(x)}^{h(x)+t\delta h(x)} \left[||grad\ \Psi^-||^{2p} - ||grad\ \Psi^+||^{2p} \right] dy \right) dx +$$

$$+\nu \int_I \left[(1+((h+t\delta h)')^2)^{\frac{p}{2}} - (1+h'^2)^{\frac{p}{2}} \right] dx,$$

where the symbol $'$ indicates differentiation with respect to x. The calculation above implies

$$\frac{\delta E_p}{\delta \gamma} = \lim_{t\to 0} \frac{E_p(\Psi^t,\gamma(t)) - E_p(\Psi,\gamma)}{t} =$$

$$= \mu \int_I \left\{ \left[(\Psi^- - \varphi)^2\right]^p - \left[(\Psi^+ - \varphi)^2\right]^p \right\} \Big|_{y=h(x)} \delta h dx +$$

$$+ \int_I \left[||grad\ \Psi^-||^{2p} - ||grad\ \Psi^+||^{2p} \right] \Big|_{y=h(x)} \delta h dx + \nu \lim_{t\to 0} \frac{\displaystyle\int_I \sum_{n=1}^{\infty} \frac{d^n F}{dt^n}(0) \cdot \frac{t^n}{n!} dx}{t},$$

where

$$F(x,t) = (1+((h+t\delta h)')^2)^{\frac{p}{2}}.$$

Using integration by parts, the third term in the above sum, i.e.,

$$\nu \int_I 2h'(\delta h)' \frac{p}{2} (1+h'^2)^{\frac{p}{2}-1} dx$$

becomes

$$-\nu p \int_I h''(1+h'^2)^{\frac{p}{2}-2} \delta h (1+(p-1)h'^2) dx,$$

and then

$$\begin{aligned} \frac{\delta E_p}{\delta \gamma_i} &= \int_{\gamma_i} \{ [\mu|\Psi^- - \varphi|^{2p} + ||grad\ \Psi^-||^{2p}]|_{\gamma_i} - [\mu|\Psi^+ - \varphi|^{2p} + \\ &+ ||grad\ \Psi^+||^{2p}]|_{\gamma_i} - \nu p h''(1+(p-1)h'^2)(1+h'^2)^{\frac{p}{2}-2} \} \frac{\delta h}{\sqrt{1+h'^2}} ds, \end{aligned}$$

where $ds = \sqrt{1+h'^2}dx$ is the arc element of γ_i. The coefficient $\dfrac{\delta h}{\sqrt{1+h'^2}}$ represents the displacement of the deformed Γ along the *normal* lines to the initial Γ.

At an extremum of E_p we have

$$
\begin{aligned}
(2)\qquad [\mu|\Psi^+ - \varphi|^{2p} &+ ||grad\,\Psi^+||^{2p}] - [\mu|\Psi^- - \varphi|^{2p} + ||grad\,\Psi^-||^{2p}] + \\
&+ \nu p h''[(p-1)h'^2 + 1](1+h'^2)^{\frac{p}{2}-2} = 0.
\end{aligned}
$$

The terms in the first two square brackets can be interpreted as the *p-energy density* corresponding to the functional E_p, above and below the curbe, respectively. The term

$$curv_p(\gamma_i) = ph''[(p-1)h'^2 + 1](1+h'^2)^{\frac{p}{2}-2}$$

shall be called the *p-curvature* of the curve γ_i. We notice that for $p = 1$, this term becomes

$$k = \frac{h''(x)}{[1 + h'^2(x)]^{3/2}},$$

i.e., the usual curvature of the plane curve $y = h(x)$, which exists and is bounded almost everywhere. Denoting

$$e_p(\Psi; x, y) = \mu|\Psi - \varphi|^{2p} + ||grad\,\Psi(x,y)||^{2p},$$

(2) may be rewritten as

$$\left(e_p(\Psi^+) - e_p(\Psi^-) + \nu \cdot curv_p(\gamma_i)\right)\Big|_{\gamma_i} = 0.$$

Second, we study the first variation of E_p with respect to Ψ. In (1), let R be relative compact with piecewise smooth boundary, φ a continuous function on $\bar{R}, \mu, \nu$ positive constants, $\Psi|_{R-\Gamma} \in C^1$ with continuous derivatives up to all boundary point and Γ made up of a finite set of C^1 curves. Let us now fix Γ and φ and let vary Ψ in the conditions $\delta\Psi \in C^1$, $\Psi \in C^2$ on $R - \Gamma$. Then we find

$$
\begin{aligned}
E_p(\Psi + \delta\Psi, \Gamma) &- E_p(\Psi, \Gamma) = \mu \iint_R \left\{[(\Psi + t\delta\Psi - \varphi)^2]^p - [(\Psi - \varphi)^2]^p\right\} d\sigma + \\
&+ \iint_{R-\Gamma} [||grad\,\Psi + t\,grad\,\delta\Psi||^{2p} - ||grad\,\Psi||^{2p}] d\sigma = \\
&= \mu \iint_R \sum_{n=1}^{\infty} \frac{d^{(n)}F}{dt^n}(0)\frac{t^n}{n!} d\sigma + \iint_{R-\Gamma} \sum_{n=1}^{\infty} \frac{d^{(n)}G}{dt^n}(0)\frac{t^n}{n!} d\sigma,
\end{aligned}
$$

where

$$
\begin{aligned}
F(t) &= |\Psi + t\delta\Psi - \varphi|^{2p}, \\
G(t) &= ||grad\,\Psi + t\,grad\,\delta\Psi||^{2p} = (||grad\,\Psi||^2 + 2t < grad\,\Psi, grad\,\delta\Psi > + \\
&+ t^2||grad\,\delta\Psi||^2)^p, \\
\frac{\delta E_p}{\delta\Psi} &= \lim_{t\to 0} \frac{E_p(\Psi + t\delta\Psi, \Gamma) - E_p(\Psi, \Gamma)}{t} = 2p\mu \iint_R \delta\Psi[(\Psi - \varphi)^2]^{p-1}(\Psi - \varphi) d\sigma + \\
&+ 2p \iint_{R-\Gamma} < grad\,\Psi, grad\,\delta\psi > ||grad\,\Psi||^{2(p-1)} d\sigma = \\
&= 2p \left\{ \mu \iint_R \delta\Psi\, sgn(\Psi - \varphi)|\Psi - \varphi|^{2p-1} d\sigma + \right. \\
&+ \left. \iint_{R-\Gamma} < ||grad\,\Psi||^{2(p-1)} grad\,\Psi, grad\,\delta\Psi > d\sigma \right\}.
\end{aligned}
$$

We obtain

$$\begin{aligned}\frac{1}{2p}\frac{\delta E_p}{\delta\Psi} &= \iint_{R-\Gamma}\delta\Psi[\mu sgn(\Psi-\varphi)|\Psi-\varphi|^{2p-1}-\\ &- div(||grad\,\Psi||^{2(p-1)}grad\,\Psi)]d\sigma+\\ &+\int_{\partial(R-\Gamma)} <\vec{n},\delta\Psi||grad\psi||^{2(p-1)}grad\Psi> ds,\end{aligned}$$

using the Gauss-Ostrogradski formula.

Further, denoting $B=\partial(R-\Gamma)$, we obtain by direct calculation

$$\begin{aligned}\frac{1}{2p}\frac{\delta E_p}{\delta\Psi} &= \iint_{R-\Gamma}\delta\Psi[\mu sgn(\Psi-\varphi)|\Psi-\varphi|^{2p-1}-\\ &- div(||grad\,\Psi||^{2(p-1)}grad\,\Psi)]d\sigma+\\ &+\int_{B}\delta\Psi||grad\psi||^{2(p-1)}\frac{\partial\Psi}{\partial n}ds.\end{aligned}$$

In the following, we shall apply the fundamental lemma of Variational Calculus twice. Taking $\partial\Psi$ to be a test function, which is non-zero near one point of $R-\Gamma$ and zero elswere, we deduce that Ψ satisfies:

$$div\,(||grad\,\Psi||^{2(p-1)}grad\,\Psi)=\mu sgn(\Psi-\varphi)|\Psi-\varphi|^{2p-1}\quad \text{on}\quad R-\Gamma; \tag{3}$$

$$||grad\,\Psi||^{2(p-1)}\frac{\partial\Psi}{\partial n}=0, \tag{4}$$

on ∂R and on the two sides $\gamma_i^{\pm}$ of each γ_i.

Obviously, $grad\,\Psi$ may be zero only at the singular points of B. Excepting them, we get $\dfrac{\partial\Psi}{\partial n}=0$ on ∂R and on the two sides $\gamma_i^{\pm}$ of each γ_i .

Remark. For $p=1$, the PDE (3) becomes the Poisson equation $\Delta\Psi=\mu(\Psi-\varphi)$ on $R-\Gamma$, and (4) becomes the boundary condition

$$\frac{\partial\Psi}{\partial n}=0\quad \text{on}\quad \partial R\bigcup\left(\bigcup_i\gamma_i^t\right). \tag{4'}$$

So, for $p=1$, we obtain the Neumann problem (3), (4′).

By analogy with [1], it can be shown that the natural limit functional of E_p as $\mu\to 0$ is

$$E_p^0(\Gamma)=\sum_i\iint_{R_i}\left[(\varphi-mean_{R_i}\varphi)^2\right]^p d\sigma+\nu_0 l_p(\Gamma), \tag{5}$$

because, when Γ is fixed and $\mu\to 0$, the function Ψ which minimizes E_p tends to the piecewise constant limit $\Psi|_{R_i}=mean_{R_i}\varphi$.

Using geometric measure theory, it can be shown that the problem of minimizing E_p^0 is well-posed: for any continuous φ, there exists a set Γ made up of a finite number of singular points joined by a finite set of E_p^0-minimizing C^2-arcs.

It is shown that E_p^0 may be viewed as a modification of the usual 2-dimensional Plateau problem functional, of fixed length Γ, by an external force term that keeps the minimal surfaces R_i (in the Plateau problem) from collapsing. It is known that the two-dimensional Plateau problem has only rather uninteresting extrema with Γ as union of line-segments. If we should generalize $E_p^0(\Gamma)$ for dimensions greater than two, the minimizing problem becomes difficult.

We recall the *oriented Plateau problem*:

Given an (n-1)-dimensional oriented submanifold A of an m-dimensional Riemannian manifold, find an n-dimensional orientable submanifold C of least area with $\partial C = A$.

In addition to this problem, our problem of minimizing E_p^0 has "pressure" terms, which renders it more difficult.

Let D be the domain of the body in $\mathbf{R}^3$ endowed with the Riemannian metric g. Let $d\omega = \sqrt{det\, g}dx \wedge dy \wedge dz$. We divide D by a finite set of smooth surfaces meeting only at their boundaries and thus we obtain open "cells" D_i, $i = \overline{1,n}$ in which we assume that the density varies smoothly and slowly. Thus,

$$D = D_1 \cup D_2 \cup \ldots \cup D_n \cup \sum .$$

In this case we can introduce

$$E_p^0 = \sum_i \iiint_{D_i} \left[(\varphi - mean_{D_i}\varphi)^2\right]^p d\omega + \nu_0 \mathcal{A}_p(\sum),$$

where $\sum$ is the union of the parts of the boundaries of D_i inside D, whose p-area is

$$\mathcal{A}_p = \iint \left[\sqrt{EG - F^2}\right]^p d\sigma,$$

and E, F, G are the coefficients of the metric induced on $\sum$ by g. By nontrivial calculations, we can obtain also the natural limit functional E_p^∞, as $\mu \to \infty$.

We are interested here especially in the case $p = 1$, for which

$$E^\infty(\Gamma) = \lim_{\mu\to\infty} E_1 = \int_\Gamma \left[\nu_\infty - \left(\frac{\partial\varphi}{\partial n}\right)^2\right] ds. \tag{6}$$

In this case, we notice that the integrand is a generalized Finsler metric. Concerning the minimization of E^∞ *over all* Γ, this is not a well-posed problem in most cases. If $||grad\,\varphi||^2 \le \nu_\infty$ everywhere, then $E^\infty \ge 0$ and the simple choice $\Gamma = \emptyset$ minimizes E^∞. But if $||grad\,\varphi||^2 > \nu_\infty$ on a non-empty open set U, then consider Γ made up of many pieces of level curves of φ within U. On such $\Gamma's, E^\infty$ tends to $-\infty$. Minimizing E^∞ on a suitable restricted class of Γ can be a well-posed problem (see [1]).

Now we focus on the geometrical meaning of the functional (6). The integrand is the generalized Finsler metric

$$\begin{aligned} L(x^1, x^2, dx^1, dz^2) &= \left[\nu_\infty - < \vec{n}, grad\,\varphi >^2\right] ds = \\ &= \left\{\nu_\infty - \left[g_{ij} n^i \left(g^{jk}\frac{\partial\varphi}{\partial x^k}\right)\right]^2\right\} \sqrt{g_{ij}dx^i dx^j}, \end{aligned}$$

where n^i are the direction cosines of the normal to the curve, i.e., $n^i = \dfrac{\nu^i}{\sqrt{g_{kl}\nu^k\nu^l}}$, where ν^i satisfy the condition

$$g_{ij}\nu^i \frac{dx^j}{ds} = 0.$$

Hence,

$$\nu^1 = -(g_{12}dx^1 + g_{22}dx^2), \quad \nu^2 = +(g_{11}dx^1 + g_{12}dx^2),$$

modulo a certain scaling. One can check that

$$g_{kl}\nu^k\nu^l = (det\, g) \cdot g_{kl}dx^k dx^l,$$

and consequently

$$\begin{aligned} L(x^1, x^2, dx^1, dx^2) &= \left[\nu_\infty - \left(n^i \frac{\partial \varphi}{\partial x^i}\right)^2\right] \sqrt{g_{kl}dx^k dx^l} = \\ &= \left[\nu_\infty - \frac{\left(\nu^i \dfrac{\partial \varphi}{\partial x^i}\right)^2}{(det\, g) g_{kl}dx^k dx^l}\right] \sqrt{g_{kl}dx^k dx^l}. \end{aligned}$$

Denoting $y^1 = dx^1,\ y^2 = dx^2$, we notice that $L(x^1, x^2, dx^1, dx^2)$ can be rewritten as

$$L(\alpha, \beta) = \nu_\infty \alpha - \frac{\beta^2}{\alpha}, \tag{7}$$

where

$$\alpha = \alpha(x^1, x^2, y^1, y^2) = \sqrt{g_{kl}y^k y^l},$$

and

$$\beta(x, y) = b_i(x) y^i,$$

with

$$b_1 = \frac{1}{\sqrt{\Delta}} \left(g_{11}\frac{\partial \varphi}{\partial x^2} - g_{12}\frac{\partial \varphi}{\partial x^1}\right),$$

$$b_2 = \frac{1}{\sqrt{\Delta}} \left(g_{12}\frac{\partial \varphi}{\partial x^2} - g_{22}\frac{\partial \varphi}{\partial x^1}\right), \ \Delta = \det g.$$

We can see that

$$L(x^1, x^2, ty^1, ty^2) = |t| L(x^1, x^2, y^1, y^2).$$

Thus, $E^\infty(\Gamma) = \int_\Gamma L(\alpha, \beta)$. According to [2], we have the following

Definition. Let $\alpha(x, y) = \sqrt{g_{ij}(x) y^i y^j}$ be a Riemannian metric and $\beta(x, y) = b_i(x) y^i$ a 1-form. Any 1-homogeneous Finsler metric of the form $L(\alpha, \beta)$ is called an (α, β)-*metric*.

Notice. If g is the Euclidean metric in the plane, then

$$n = \frac{(-dx^2, dx^1)}{\sqrt{(dx^1)^2 + (dx^2)^2}}, \quad b_1 = \frac{\partial \varphi}{\partial x^2}, \quad b_2 = -\frac{\partial \varphi}{\partial x^1}$$

$$L = \frac{\nu_\infty[(dx^1)^2 + (dx^2)^2] - \left(\dfrac{\partial\varphi}{\partial x^1}dx^2 - \dfrac{\partial\varphi}{\partial x^2}dx^1\right)^2}{\sqrt{(dx^1)^2 + (dx^2)^2}}.$$

Our metric (7) is among the $(\alpha,\beta) - metrics$. Though it is neither a Randers metric $L_R(\alpha,\beta) = \alpha + \beta$, nor a Kropina metric $L_K(\alpha,\beta) = \dfrac{\alpha^2}{\beta} + \beta$, we have

$$L = \nu_\infty \alpha - L_{-1;2},$$

where

$$L_{p;q} = \alpha^p \beta^q, \quad p + q = 1$$

are extensively studied metrics.

In [2], Matsumoto presented a way of easier calculating the fundamental tensor $\tilde{g}$ of the Finsler space when the metric L is given. We apply this expression to our L. Starting from the definition ($\tilde{g} = \nabla^0\nabla^0(L^2/2)$) of $\tilde{g}$, we use the formula

$$\tilde{g}(u,v) = [L(\nabla^0, \nabla^0 L)](u,v) + (\nabla^0 L)(u)\cdot(\nabla^0 L)(v),$$

$$\tilde{g}_{ij} = L(\dot{\partial}_i, \dot{\partial}_j L) + \dot{\partial}_i L \dot{\partial}_j L,$$

where $\dot{\partial}_i = \dfrac{\partial}{\partial y_i}$ and so we find

$$\tilde{g}_{ij} = p g_{ij} + q_0 b_i b_j + q_{-1}(b_i Y_j + b_j Y_i) + q_{-2} Y_i Y_j,$$

where we used

$$\alpha = g_{ij}(x) y^i y^j, \quad \beta = b_i(x) y^i,\ Y_i = g_{ij} y^j, \quad h_{ij} = L(\dot{\partial}_i \dot{\partial}_j L)$$

$$p = \frac{L}{\alpha} L_\alpha,\ q_0 = L\, L_{\beta\beta},\ q_{-1} = \frac{L}{\alpha} L_{\alpha\beta},\ q_{-2} = \frac{L}{\alpha^2}\left(L_{\alpha\alpha} - \frac{L_\alpha}{\alpha}\right).$$

The subscripts of q_0, q_{-1}, q_{-2} are used to indicate the respective degrees of homogeneity.

By the homogeneity, we obtain two identities

$$p + q_{-1}\beta + q_{-2}\alpha^2 = 0, \quad q_0\beta + q_{-1}\alpha^2 = 0.$$

We also denote

$$Y_i = g_{ij} y^j = L(\dot{\partial}_i L) = p Y_i + L\, L_\beta b_i.$$

From the above relations, the fundamental tensor is

$$\tilde{g}_{ij} = p g_{ij} + p_0 b_i b_j + p_{-1}(b_i Y_j + b_j Y_i) + p_{-2} Y_i Y_j,$$

where the coefficients are: p as above and

$$p_0 = q_0 + (L_\beta)^2 = L\, L_{\beta\beta} + (L_\beta)^2,$$

$$p_{-1} = q_{-1} + \frac{p}{L} L_\beta = \frac{1}{\alpha}(L\, L_{\alpha\beta} + L_\alpha L_\beta),$$

$$p_{-2} = q_{-2} + \left(\frac{p}{L}\right)^2 = \frac{1}{\alpha^2}\left\{L\, L_{\alpha\alpha} + (L_\alpha)^2 - \frac{L}{\alpha} L_\alpha\right\}.$$

For

$$L(\alpha,\beta) = \nu_\infty \alpha - \frac{\beta^2}{\alpha},$$

we get

$$\begin{aligned}
p &= \left(\nu_\infty - \frac{\beta^2}{\alpha^2}\right)\left(\nu_\infty + \frac{\beta^2}{\alpha^2}\right) = \nu_\infty^2 - \left(\frac{\beta}{\alpha}\right)^4, \\
p_0 &= \left(\nu_\infty \alpha - \frac{\beta^2}{\alpha}\right)\left(-\frac{2}{\alpha}\right) + \frac{4\beta^2}{\alpha^2} = -2\nu_\infty + \frac{6\beta^2}{\alpha^2}, \\
p_{-1} &= \frac{1}{\alpha}\left[\left(\nu_\infty \alpha - \frac{\beta^2}{\alpha}\right)\frac{2\beta}{\alpha^2} + \left(\nu_\infty + \frac{\beta^2}{\alpha^2}\right)\left(-\frac{2\beta}{\alpha}\right)\right] = -\frac{4\beta^3}{\alpha^4}, \\
p_{-2} &= \frac{1}{\alpha^2}\left[\left(\nu_\infty \alpha - \frac{\beta^2}{\alpha}\right)\left(-\frac{2\beta^2}{\alpha^3}\right) + \left(\nu_\infty + \frac{\beta^2}{\alpha^2}\right)^2 - \nu_\infty^2 + \frac{\beta^4}{\alpha^4}\right] = \frac{4\beta^4}{\alpha^6}.
\end{aligned}$$

Hence,

$$\tilde{g}_{ij} = \left[\nu_\infty^2 - \left(\frac{\beta}{\alpha}\right)^4\right] g_{ij} = \left(\frac{6\beta^2}{\alpha^2} - 2\nu_\infty\right) b_i b_j - \frac{4\beta^3}{\alpha^4}(b_i Y_j + b_j Y_i) + \frac{4\beta^4}{\alpha^6} Y_i Y_j.$$

By direct calculation we can state when $\tilde{g}_{ij}$ is positive definite.

We remind that in a Finsler space with an (α,β)-metric, we have a linear connection $(\Gamma^i_{jk}(x))$ (that is, the Riemannian connection of the Riemannian metric α) and therefore we obtain a Finsler connection $(\Gamma^i_{jk},\ N^i_j = y^k\Gamma^i_{jk},\ C^i_{jk})$, where C^i_{jk} are usual quantities [2]. Making use of this Finsler connection, Hashiguchi and Ichijyo developed in [3] an interesting theory and presented examples of Berwald and Wagner spaces among Finsler spaces with (α,β)-metric. The study of the Finsler space with the metric (4) is the subject of a forecoming paper.

References

[1] D.Mumford, J.Shah, *Optimal Approximations by Piecewise Smooth Functions and Associated Variational Problems Communications on Pure and Applied Mathematics*, 42, (1989), 577-685.

[2] M.Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces*, Karseisha, Otsu, 1986.

[3] M.Hashiguchi, K.Ichijyo, *On Some Special* (α,β)*-Metrics*, Preliminary Report.

[4] C.Udrişte, *Properties of a functional defined on the path space of a Riemann manifold*, Analele Universităţii Bucureşti, (1978), 97-105.

[5] C.Udrişte, *Properties of a family of functionals*, Proceedings of the National Seminar on Finsler Spaces, Braşov (1980), 219-221.

[6] C.Udrişte, *A finite dimensional approximation to* Ω_p^c, Proceedings of the National Seminar on Finsler Spaces, Braşov (1983), 225-226.

[7] C.Udrişte, *Convex functions and optimization methods on Riemannian manifolds*, Mathematics and Its Applications, 297, Kluwer Academic Publishers, 1994.

University of Kansas
Mathematics Department
KUID 613495, M.Toda, G.T.A.
405 Snow Hall
Lawrence KS 66045, USA

University Politehnica of Bucharest
Department of Mathematics I
Splaiul Independentei 313
77206 Bucharest, Romania
email:udriste@mathem.pub.ro