Symmetry Group of Ţiţeica Surfaces PDE

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Abstract

Using the theory of the symmetry groups for PDEs of order two ([7], [17], [20]), one finds the symmetry group G associated to Ţiţeica surfaces PDE. One proves that Monge-Ampère-Ţiţeica PDE which is invariant with respect to G', where G' is the maximal solvable subgroup of the symmetry group G, is just the PDE of Ţiţeica surfaces. One studies the inverse problem and one shows that the Ţiţeica surfaces PDE is an Euler-Lagrange equation. One determines the variational symmetry group of the associated functional, and one obtains the conservation laws associated to the Ţiţeica surfaces PDE. One finds some group-invariant solutions of the Ţiţeica surfaces PDE. All these results shows that Ţiţeica surfaces theory is strongly related to variational problems, and hence it is a subject of global differential geometry.

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1 Introduction

The symmetry group (or strong symmetry group [19]) associated to a PDE is a Lie group of local transformations which change the solutions of the equation into its solutions. The theory of symmetry groups has a big importance in Geometry, Mechanics and Physics ([3]-[9], [11], [13], [14], [17]-[21], [25]). We apply this theory in the case of Ţiţeica surfaces PDE in Geometry, determining the symmetry group, some group-invariant solutions, a Ţiţeica Lagrangian, and conservation laws.

The centroaffine invariant

$$I = \frac{K}{d^4},$$

where K is the Gauss curvature of a surface Σ and d is the distance from the origin to the tangent plane at an arbitrary point of Σ, was introduced by Ţiţeica ([22]). A surface Σ for which the ratio $\frac{K}{d^4}$ is constant, is called *Ţiţeica surface*.

For the aplication of the theory of symmetry groups ([17]-[20]), we shall consider the case in which Σ is a simple surface, being given by an explicit Cartesian equation

$$\Sigma : \quad u = f(x, y),$$

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where $f \in C^2(D)$ and $D \subset \mathbf{R^2}$ is a domain. In this case, the Gauss curvature of the surface Σ is

$$K = \frac{u_{xx}u_{yy} - u_{xy}^2}{\left(1 + u_x^2 + u_y^2\right)^2},$$

and the distance from the origin to the tangent plane at an arbitrary point of Σ is

$$d = \frac{|xu_x + yu_y - u|}{\sqrt{1 + u_x^2 + u_y^2}}.$$

Given the nonzero function I (centroaffine invariant), the condition

$$\frac{K}{d^4} = I$$

transcribes like a PDE

$$u_{xx}u_{yy} - u_{xy}^2 = I(xu_x + yu_y - u)^4. \tag{1}$$

Moreover, the conditions $d,\ K \neq 0$ are equivalent to

$$xu_x + yu_y - u \neq 0,\ \ u_{xx}u_{yy} - u_{xy}^2 \neq 0. \tag{2}$$

One proves that the symmetry group G of PDE (1), with $I = constant$, is the unimodular subgroup of the centroaffine group.

PDE (1) is a Monge-Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 = H(x, y, u, u_x, u_y). \tag{3}$$

Therefore PDE (3) will be called *Monge-Ampère -Ţiţeica equation.*

2 Symmetry group of a PDE of order two

Let D be an open set in $\mathbf{R}^2$ and $u \in C^2(D)$. The function $u^{(2)} : D \to U^{(2)} = U \times U_1 \times U_2$,

$$u^{(2)} = (u; u_x, u_y; u_{xx}, u_{xy}, u_{yy})$$

is called *the prolongation of order two of the function u.*

The total space $D \times U^{(2)}$ whose coordinates represent the independent variables, the dependent variable and the derivatives of dependent variable till the order two, is called *jet space of order two of the base space $D \times U$.*

We consider the PDE of order two

$$F(x, y, u^{(2)}) = 0, \tag{4}$$

where $F : D \times U^{(2)} \to \mathbf{R}$ is a differentiable function.

Definition 1. PDE (4) is called of *maximal rank* if the associated Jacobi matrix

$$J_F(x, y, u^{(2)}) = (F_x, F_y; F_u; F_{u_x}, F_{u_y}; F_{u_{xx}}, F_{u_{xy}}, F_{u_{yy}})$$

has *rank* 1 on the set described by the equation $F(x, y, u^{(2)}) = 0$.

In this case the set

$$S = \{(x, y, u^{(2)}) \in D \times U^{(2)} | F(x, y, u^{(2)}) = 0\}$$

is a hypersurface.

Definition 2. *The symmetry group of PDE (4)* is a group of local transformations G acting on an open set $M \subset D \times U$ with the properties:

(a) if $u = f(x, y)$ is a solution of the equation and if $g \cdot f$ has sense for $g \in G$, then $v = g \cdot f(x, y)$ is also a solution.

(b) any solution of the equation can be obtained by a DE associated to PDE (hence any solution is G-invariant $g \cdot f = f$, $\forall g \in G$).

Definition 3. Let

$$X = \zeta(x, y, u)\frac{\partial}{\partial x} + \eta(x, y, u)\frac{\partial}{\partial y} + \phi(x, y, u)\frac{\partial}{\partial u}$$

be a C^∞ vector field on an open set $M \subset D \times U$. *The prolongations of order one respectively two* of the vector field X are the vector fields

$$\mathrm{pr}^{(1)}X = X + \Phi^x \frac{\partial}{\partial u_x} + \Phi^y \frac{\partial}{\partial u_y},$$

(5)
$$\mathrm{pr}^{(2)}X = \mathrm{pr}^{(1)}X + \Phi^{xx}\frac{\partial}{\partial u_{xx}} + \Phi^{xy}\frac{\partial}{\partial u_{xy}} + \Phi^{yy}\frac{\partial}{\partial u_{yy}},$$

where

$$\Phi^x = \phi_x + (\phi_u - \zeta_x)u_x - \eta_x u_y - \zeta_u u_x^2 - \eta_u u_x u_y,$$

$$\Phi^y = \phi_y - \zeta_y u_x + (\phi_u - \eta_y)u_y - \zeta_u u_x u_y - \eta_u u_y^2$$

and respectively

$$\begin{aligned} \Phi^{xx} &= \phi_{xx} + (2\phi_{xu} - \zeta_{xx})u_x - \eta_{xx}u_y + (\phi_{uu} - 2\zeta_{xu})u_x^2 - \\ &- 2\eta_{xu}u_x u_y - \zeta_{uu}u_x^3 - \eta_{uu}u_x^2 u_y + (\phi_u - 2\zeta_x)u_{xx} - 2\eta_x u_{xy} - \\ &- 3\zeta_u u_x u_{xx} - \eta_u u_y u_{xx} - 2\eta_u u_x u_{xy}, \end{aligned}$$

$$\begin{aligned} \Phi^{xy} &= \phi_{xy} + (\phi_{uy} - \zeta_{xy})u_x + (\phi_{ux} - \eta_{xy})u_y - \zeta_{uy}u_x^2 + (\phi_{uu} - \zeta_{ux} - \\ &- \eta_{uy})u_x u_y - \eta_{ux}u_y^2 - \zeta_y u_{xx} + (\phi_u - \zeta_x - \eta_y)u_{xy} - \eta_x u_{yy} - \\ &- \zeta_u u_y u_{xx} - 2\eta_u u_y u_{xy} - 2\zeta_u u_x u_{xy} - \eta_u u_x u_{yy} - \zeta_{uu}u_x^2 u_y - \eta_{uu}u_x u_y^2, \end{aligned}$$

$$\begin{aligned} \Phi^{yy} &= \phi_{yy} + (2\phi_{uy} - \eta_{yy})u_y - \zeta_{yy}u_x + (\phi_{uu} - 2\eta_{uy})u_y^2 - 2\zeta_{uy}u_x u_y - \\ &- \eta_{uu}u_y^3 - \zeta_{uu}u_x u_y^2 + (\phi_u - 2\eta_y)u_{yy} - 2\zeta_y u_{xy} - 3\eta_u u_y u_{yy} - \\ &- \zeta_u u_x u_{yy} - 2\zeta_u u_y u_{xy}. \end{aligned}$$

For the determination of the symmetry group of PDE (4) is used the following *criterion of infinitesimal invariance* [17].

Theorem 1. *Let*

$$F(x, y, u^{(2)}) = 0,$$

be a PDE of maximal rank defined on an open set $M \subset D \times U$. If G is a local group of transformations on M and

$$pr^{(2)}X[F(x, y, u^{(2)})] = 0 \quad \textit{whenever} \quad F(x, y, u^{(2)}) = 0, \tag{6}$$

for every infinitesimal generator X of G, then G is a symmetry group of the considered equation.

Proposition 1. *If PDE (4) defined on $M \subset D \times U$ is of maximal rank, then the set of infinitesimal symmetries of the equation forms a Lie algebra on M. Moreover, if this algebra is finite-dimensional, then the symmetry group of PDE is a Lie group of local transformations on M.*

Algorithm for determination of the symmetry group G of PDE (4):

-one considers the field X on M and its prolongations of the first and second order, and one writes the infinitesimal invariance condition (6);

-one eliminates any dependence between partial derivatives of the function u using the given PDE;

-one writes the condition (6) like a polynomial in the partial derivatives of u, and we identify this polynomial with zero;

-it follows a PDEs system in the unknown functions ζ, η, ϕ, and the solution of this system defines the symmetry group of PDE (4).

3 Symmetry group of Ţiţeica surfaces PDE

We consider the Ţiţeica surfaces PDE,

$$u_{xx}u_{yy} - u_{xy}^2 = \alpha(xu_x + yu_y - u)^4, \quad \alpha \in \mathbf{R}^* \tag{1'}$$

with the conditions (2), which assure the maximal rank.

Let

$$X = \zeta(x, y, u)\frac{\partial}{\partial x} + \eta(x, y, u)\frac{\partial}{\partial y} + \phi(x, y, u)\frac{\partial}{\partial u}$$

be a C^∞ vector field on the open set $M \subset D \times U$. In the case of PDE $(1')$, the condition (6) becomes

$$-4\alpha\zeta u_x(xu_x + yu_y - u)^3 - 4\alpha\eta u_y(xu_x + yu_y - u)^3 + 4\alpha\phi(xu_x + yu_y - u)^3 -$$
$$-4\alpha x\Phi^x(xu_x + yu_y - u)^3 - 4\alpha y\Phi^y(xu_x + yu_y - u)^3 + \Phi^{xx}u_{yy} -$$
$$-2\Phi^{xy}u_{xy} + \Phi^{yy}u_{xx} = 0.$$

Replacing the functions Φ^x, Φ^y, Φ^{xx}, Φ^{xy}, Φ^{yy} given by the relations (5) and eliminating any dependence between partial derivatives of the function u (determined by the PDE $(1')$), we obtain

$$-uu_{xx}\phi_{yy} + u_xu_{xx}(x\phi_{yy} + u\zeta_{yy}) + u_yu_{xx}(y\phi_{yy} - 2u\phi_{uy} + 2u\eta_{yy}) -$$
$$-xu_x^2u_{xx}\zeta_{yy} + u_xu_yu_{xx}(u\zeta_{uy} - y\zeta_{yy} + 2x\phi_{uy} - x\eta_{yy}) +$$

$$+u_y^2u_{xx}(2y\phi_{uy}-u\phi_{uu}-y\eta_{yy}+2u\eta_{uy})-xu_x^2u_yu_{xx}\zeta_{uy}+$$
$$+u_xu_y^2u_{xx}(u\zeta_{uu}-y\zeta_{uy}+x\phi_{uu}-2x\eta_{uy})+u_y^3u_{xx}(y\phi_{uu}-2y\eta_{uy}+u\eta_{uu})-$$
$$-xu_x^2u_y^2u_{xx}\zeta_{uu}-u_xu_y^3u_{xx}(x\eta_{uu}+y\zeta_{uu})-yu_y^4u_{xx}\eta_{uu}+2uu_{xy}\phi_{xy}+$$
$$+2u_xu_{xy}(u\phi_{uy}-x\phi_{xy}-u\zeta_{xy})+2u_yu_{xy}(u\phi_{ux}-y\phi_{xy}-u\eta_{xy})-$$
$$-2u_x^2u_{xy}(x\phi_{uy}-x\zeta_{xy}+u\zeta_{uy})+2u_xu_yu_{xy}(u\phi_{uu}-x\phi_{ux}-y\phi_{uy}+y\zeta_{xy}-u\zeta_{ux}+$$
$$+x\eta_{xy}-u\eta_{uy})-2u_y^2u_{xy}(y\phi_{ux}-y\eta_{xy}+u\eta_{ux})+2xu_x^3u_{xy}\zeta_{uy}+2u_x^2u_yu_{xy}(y\zeta_{uy}+$$
$$+x\zeta_{ux}-u\zeta_{uu}+x\eta_{uy}-x\phi_{uu})+2u_xu_y^2u_{xy}(y\zeta_{ux}+y\eta_{uy}+x\eta_{ux}-u\eta_{uu}-y\phi_{uu})+$$
$$+2yu_y^3u_{xy}\eta_{ux}+2xu_x^3u_yu_{xy}\zeta_{uu}+2u_x^2u_y^2u_{xy}(y\zeta_{uu}+x\eta_{uu})+$$
$$+2yu_xu_y^3u_{xy}\eta_{uu}-uu_{yy}\phi_{xx}+u_xu_{yy}(x\phi_{xx}-2u\phi_{xu}+u\zeta_{xx})+$$
$$+u_yu_{yy}(x\phi_{xx}+u\eta_{xx})+u_x^2u_{yy}(2x\phi_{xu}-u\phi_{uu}-x\zeta_{xx}+2u\zeta_{xu})+$$
$$+u_xu_yu_{yy}(2y\phi_{xu}-y\zeta_{xx}-x\eta_{xx}+2u\eta_{xu})-yu_y^2u_{yy}\eta_{xx}+$$
$$+u_x^3u_{yy}(x\phi_{uu}-2x\zeta_{xu}+u\zeta_{uu})+u_x^2u_yu_{yy}(y\phi_{uu}-2y\zeta_{xu}-2x\eta_{xu}+$$
$$+u\eta_{uu})-2yu_xu_y^2u_{yy}\eta_{xu}-xu_x^4u_{yy}\zeta_{uu}-u_x^3u_yu_{yy}(y\zeta_{uu}+x\eta_{uu})-$$
$$-yu_x^2u_y^2u_{yy}\eta_{uu}-2u_{xy}^2(2\phi-2x\phi_x-2y\phi_y-u(\phi_u-\zeta_x-\eta_y))+$$
$$+u_xu_{xy}^2(4\zeta-4y\zeta_y-2x\zeta_x+2x\eta_y+2x\phi_u-4u\zeta_u)+2u_yu_{xy}^2(2\eta-2x\eta_x-$$
$$-y(\eta_y-\zeta_x-\phi_u)-2u\eta_u)+2u_{xx}u_{yy}(2\phi-2x\phi_x-2y\phi_y-u(\phi_u-\zeta_x-\eta_y))-$$
$$-2u_xu_{xx}u_{yy}(2\zeta-2y\zeta_y+x(\phi_u-\zeta_x+\eta_y)-2u\zeta_u)-$$
$$-2u_yu_{xx}u_{yy}(2\eta-2x\eta_x+y(\phi_u+\zeta_x-\eta_y)-2u\eta_u)=0.$$

Looking at this condition as a polynomial in the partial derivatives of the function u, and identifying with the polynom zero, we obtain the PDE system

$$\begin{array}{cccc}
\zeta_{xy}=0, & \zeta_{yy}=0, & \zeta_{uu}=0, & \zeta_{uy}=0,\\
\eta_{xx}=0, & \eta_{yy}=0, & \eta_{uu}=0, & \eta_{ux}=0,\\
\phi_{xx}=0, & \phi_{xy}=0, & \phi_{yy}=0, & \phi_{uy}=0,\\
\phi_{uu}=2\eta_{uy}, & \phi_{ux}=\eta_{xy}, & \eta_{uy}=\zeta_{ux}, & 2\phi_{xu}=\zeta_{xx},
\end{array}$$

$$2\phi-2x\phi_x-2y\phi_y-u(\phi_u-\zeta_x-\eta_y)=0,$$
$$2\zeta-2y\zeta_y-x(\zeta_x-\eta_y-\phi_u)-2u\zeta_u=0,$$
$$2\eta-2x\eta_x+y(\phi_u+\zeta_x-\eta_y)-2u\eta_u=0,$$

whose solution defines the symmetry group of the equation $(1')$. The general solution of this PDE system is

$$(7)\qquad \left\{\begin{array}{rcl}
\zeta(x,y,u) &=& C_1x+C_3y+C_4u,\\
\eta(x,y,u) &=& C_5x+C_2y+C_6u,\\
\phi(x,y,u) &=& C_7x+C_8y-(C_1+C_2)u,
\end{array}\right.$$

where $C_1, ..., C_8 \in \mathbf{R}$, and consequently the infinitesimal generator of the symmetry group G is

$$X = C_1 \left(x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} \right) + C_2 \left(y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} \right) + C_3 y\frac{\partial}{\partial x} + C_4 u\frac{\partial}{\partial x} +$$

$$+ C_5 x\frac{\partial}{\partial y} + C_6 u\frac{\partial}{\partial y} + C_7 x\frac{\partial}{\partial u} + C_8 y\frac{\partial}{\partial u}.$$

Theorem 2. *The Lie algebra* $\mathbf{g}$ *of the symmetry group* G *associated to Ţiţeica surfaces PDE is generated by the vector fields*

$$X_1 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, \;\; X_2 = y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}, \;\; X_3 = y\frac{\partial}{\partial x}, \;\; X_4 = u\frac{\partial}{\partial x} \tag{8}$$

$$X_5 = x\frac{\partial}{\partial y}, \;\; X_6 = u\frac{\partial}{\partial y}, \;\; X_7 = x\frac{\partial}{\partial u}, \;\; X_8 = y\frac{\partial}{\partial u},$$

and G *is the unimodular subgroup of centroaffine group.*

The constants of the structure of the Lie algebra of the group G are finding from the table

[.,.]	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	0	$-X_3$	$-2X_4$	X_5	$-X_6$	$2X_7$	X_8
X_2	0	0	X_3	$-X_4$	$-X_5$	$-2X_6$	X_7	$2X_8$
X_3	X_3	$-X_3$	0	0	X_2-X_1	$-X_4$	X_8	0
X_4	$2X_4$	X_4	0	0	X_6	0	$-X_1$	$-X_3$
X_5	$-X_5$	X_5	X_1-X_2	$-X_6$	0	0	0	X_7
X_6	X_6	$2X_6$	X_4	0	0	0	$-X_5$	$-X_2$
X_7	$-2X_7$	$-X_7$	$-X_8$	X_1	0	X_5	0	0
X_8	$-X_8$	$-2X_8$	0	X_3	$-X_7$	X_2	0	0

Now we shall study the converse of the Theorem 2: given the Lie group G of transformations, determine the most general Monge-Ampère-Ţiţeica PDE which admits G like group of symmetries. This implies the using of a maximal chain of Lie subalgebras of the algebra $\mathbf{g}$ of the group G, in the case in which $\mathbf{g}$ is solvable.

Since the Lie algebra $\mathbf{g}$ of the symmetry group G is not solvable, one considers the maximal solvable Lie subalgebra $\mathbf{g}'$, described by the vector fields X_1, X_2, X_3, X_7. Denote $G' \subset G$ the corresponding subgroup.

Theorem 3. *The PDE of type Monge-Ampère-Ţiţeica of maximal rank, which admits* G' *like group of symmetry, is a PDE of type Ţiţeica.*

Proof. We consider the maximal chain of Lie subalgebras of the Lie algebra $\mathbf{g}'$,

$$\{X_8\} \subset \{X_3, X_8\} \subset \{X_3, X_7\} \subset \{X_1, X_3, X_7\} \subset \{X_1, X_2, X_3, X_7\}. \tag{9}$$

We impose the condition that PDE (3) to be invariant with respect to every of these subalgebras, denoting

$$F = u_{xx}u_{yy} - u_{xy}^2 - H(x, y, u, u_x, u_y).$$

1) We start with $\{X_8\}$: $X_8 = y\frac{\partial}{\partial u}$ and $pr^{(2)}X_8 = y\frac{\partial}{\partial u} + \frac{\partial}{\partial u_y}$.
The condition (6) implies $pr^{(2)}X_8(F) = 0$. It follows

$$F = u_{xx}u_{yy} - u_{xy}^2 - H_1(x, y, u_x, yu_y - u).$$

2) If we use $\{X_3, X_8\}$: $X_3 = y\frac{\partial}{\partial x}$ and

$$pr^{(2)}X_3 = y\frac{\partial}{\partial x} - u_x\frac{\partial}{\partial u_y} - u_{xx}\frac{\partial}{\partial u_{xy}} - 2u_{xy}\frac{\partial}{\partial u_{yy}},$$

then we obtain

$$F = u_{xx}u_{yy} - u_{xy}^2 - H_2(y, u, u_x, xu_x + yu_y - u).$$

3) For $\{X_3, X_7\}$: $X_7 = x\frac{\partial}{\partial u}$ and $pr^{(2)}X_7 = x\frac{\partial}{\partial u} + \frac{\partial}{\partial u_x}$,
we find

$$F = u_{xx}u_{yy} - u_{xy}^2 - H_3(y, xu_x + yu_y - u).$$

4) For $\{X_1, X_3, X_7\}$: $X_1 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}$, with

$$pr^{(2)}X_1 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} - 2u_x\frac{\partial}{\partial u_x} - u_y\frac{\partial}{\partial u_y} - 3u_{xx}\frac{\partial}{\partial u_{xx}} - 2u_{xy}\frac{\partial}{\partial u_{xy}} - u_{yy}\frac{\partial}{\partial u_{yy}},$$

we get

$$F = u_{xx}u_{yy} - u_{xy}^2 - (xu_x + yu_y - u)^4 H_4(y).$$

5) Finally, $\{X_1, X_2, X_3, X_7\}$: $X_2 = y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}$ and

$$pr^{(2)}X_2 = y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - u_x\frac{\partial}{\partial u_x} - 2u_y\frac{\partial}{\partial u_y} - u_{xx}\frac{\partial}{\partial u_{xx}} - 2u_{xy}\frac{\partial}{\partial u_{xy}} - 3u_{yy}\frac{\partial}{\partial u_{yy}},$$

imply

$$F = u_{xx}u_{yy} - u_{xy}^2 - \alpha(xu_x + yu_y - u)^4, \quad \alpha \in \mathbf{R},$$

and consequently the Monge-Ampère-Ţiţeica PDE is reduced to Ţiţeica PDE

$$u_{xx}u_{yy} - u_{xy}^2 = \alpha(xu_x + yu_y - u)^4, \quad \alpha \in \mathbf{R}.$$

If $\alpha \neq 0$, then the condition of maximal rank is satisfied.

4 Inverse problem associated to a PDE

The simple form of the inverse problem in the calculus of variations is to determine if an operator with partial derivatives is identically to an Euler-Langrange operator with partial derivatives ([1], [2], [12], [17], [20]). We quote
Theorem 4. *Let T be the operator associated to PDE (4). T is identically to an Euler-Lagrange operator if and only if the integrability Helmholtz conditions*

$$(10)\qquad \begin{cases} \frac{\partial T}{\partial u_x} &= D_x\left(\frac{\partial T}{\partial u_{xx}}\right) + D_y\left(\frac{1}{2}\frac{\partial T}{\partial u_{xy}}\right) \\ \frac{\partial T}{\partial u_y} &= D_x\left(\frac{1}{2}\frac{\partial T}{\partial u_{xy}}\right) + D_y\left(\frac{\partial T}{\partial u_{yy}}\right), \end{cases}$$

are satisfied, where

$$
(11) \qquad \begin{array}{rcl} D_x & = & \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y}, \\ D_y & = & \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{xy} \frac{\partial}{\partial u_x} + u_{yy} \frac{\partial}{\partial u_y}. \end{array}
$$

In this case, there exists a Lagrangian L such that the Euler-Lagrange PDE $E(L)(u) = 0$ is equivalent to the PDE associated to the operator T, in the sense that every solution of the equation $T(u) = 0$ is a solution of the Euler-Lagrange equation $E(L)(u) = 0$ and conversely.

For the associated Lagrangian of order two

$$
L = L(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}),
$$

the Euler-Lagrange operator of order two is

$$
(12) \qquad \begin{array}{rcl} E(L)(u) & = & \frac{\partial L}{\partial u} - D_x \left(\frac{\partial L}{\partial u_x} \right) - D_y \left(\frac{\partial L}{\partial u_y} \right) + \\ & + & D_{xx} \left(\frac{\partial L}{\partial u_{xx}} \right) + D_{xy} \left(\frac{\partial L}{\partial u_{xy}} \right) + D_{yy} \left(\frac{\partial L}{\partial u_{yy}} \right). \end{array}
$$

Definition 4. *An operator T is equivalent to an Euler-Lagrange operator $E(L)$,* if there exists a nonzero function $f = f(x, y, u, u_x, u_y)$ such that $f \cdot T = E(L)$. The function f is called *variational integrant factor.*

5 Lagrangians associated to Ţiţeica surfaces PDE

We consider the PDE of type Ţiţeica $(1')$ under conditions (2). The operator

$$
T(u) = u_{xx} u_{yy} - u_{xy}^2 - \alpha (x u_x + y u_y - u)^4, \ \ \alpha \in \mathbf{R}^*,
$$

which defines the equation $(1')$, is not identically to an Euler-Lagrange operator, since the integrability conditions (10) are not satisfied.

Theorem 5. *The operator T is equivalent to an Euler-Lagrange operator.*

Proof. Suppose there exists a variational integrant factor,

$$
f = f(x, y, u, u_x, u_y),
$$

such that $f \cdot T = E(L)$. In this case, the integrability conditions (10), for $f \cdot T$, become

$$
\left\{ \begin{array}{rcl} u_{yy} \left(\frac{\partial f}{\partial x} + u_x \frac{\partial f}{\partial u} \right) - u_{xy} \left(\frac{\partial f}{\partial y} + u_y \frac{\partial f}{\partial u} \right) + & & \\ + \alpha \frac{\partial f}{\partial u_x} (x u_x + y u_y - u)^4 + 4 \alpha x f (x u_x + y u_y - u)^3 & = & 0 \\ u_{xx} \left(\frac{\partial f}{\partial y} + u_y \frac{\partial f}{\partial u} \right) - u_{xy} \left(\frac{\partial f}{\partial x} + u_x \frac{\partial f}{\partial u} \right) + & & \\ + \alpha \frac{\partial f}{\partial u_y} (x u_x + y u_y - u)^4 + 4 \alpha y f (x u_x + y u_y - u)^3 & = & 0. \end{array} \right.
$$

Equating to zero the coefficients of partial derivatives of second order of the function u, we obtain the following PDE system

$$\begin{cases} \frac{\partial f}{\partial x} + u_x \frac{\partial f}{\partial u} &= 0 \\ \frac{\partial f}{\partial y} + u_y \frac{\partial f}{\partial u} &= 0 \\ (xu_x + yu_y - u)\frac{\partial f}{\partial u_x} + 4xf &= 0 \\ (xu_x + yu_y - u)\frac{\partial f}{\partial u_y} + 4yf &= 0. \end{cases}$$

The solution of this system is

$$f(x, y, u, u_x, u_y) = \frac{C}{(xu_x + yu_y - u)^4}, \quad C \in \mathbf{R}^*.$$

Hence, PDE of Ţiţeica surfaces, written in the initial form

$$\frac{K}{d^4} = \alpha,$$

is an Euler-Lagrange equation.

Theorem 6. *A Lagrangian of order two associated to Ţiţeica surfaces PDE is*

$$L(x, y, u^{(2)}) = \frac{u(u_{xy}^2 - u_{xx}u_{yy})}{(xu_x + yu_y - u)^4} - \alpha u. \tag{13}$$

Proof. Using formula (12), after tedious computations it follows

$$E(L)(u) = \frac{u_{xx}u_{yy} - u_{xy}^2}{(xu_x + yu_y - u)^4} - \alpha.$$

6 Variational symmetry group. Conservation laws

We will make a short presentation of the variational symmetry group ([17], [20]) for the functionals of the form

$$\mathcal{L}[u] = \int\int_{D_0} L(x, y, u^{(2)}) dx dy,$$

where D_0 is a domain in $\mathbf{R}^2$.

Let $D \subset D_0$ be a subdomain, U an open set in $\mathbf{R}$ and $M \subset D \times U$ an open set. Let $u \in C^2(D)$, $u = f(x, y)$ such that

$$\Gamma_u = \{(x, y, f(x, y)) | (x, y) \in D\} \subset M.$$

Definition 5. A local group G of transformations on M is called *variational symmetry group for the functional*

$$\mathcal{L}[u] = \int\int_{D_0} L(x, y, u^{(2)}) dx dy, \tag{14}$$

if for $g_\varepsilon \in G$, $g_\varepsilon(x, y, u) = (\bar{x}, \bar{y}, \bar{u})$, the function $\bar{u} = \bar{f}(\bar{x}, \bar{y}) = (g \cdot f)(\bar{x}, \bar{y})$ is defined on $\bar{D} \subset D_0$ and

$$\int\int_{\bar{D}} L(\bar{x},\bar{y},pr^{(2)}\bar{f}(\bar{x},\bar{y}))d\bar{x}d\bar{y} = \int\int_{D} L(x,y,pr^{(2)}f(x,y))dxdy.$$

The infinitesimal criterion for the variational problem is given by
Theorem 7. *A connected group G of transformations acting on $M \subset D_0 \times U$ is a group of variational symmetries for the functional (14) if and only if*

$$pr^{(2)}X(L) + L\ Div\xi = 0, \tag{15}$$

for $\forall(x,y,u^{(2)}) \in M^{(2)} \subset D \times U^{(2)}$ *and for any infinitesimal generator*

$$X = \zeta(x,y,u)\frac{\partial}{\partial x} + \eta(x,y,u)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u}$$

of G, *where* $\xi = (\zeta,\eta)$ *and* $Div\xi = D_x\zeta + D_y\eta$.
Theorem 8. *If G is a variational symmetry group of the functional (14), then G is a symmetry group of Euler-Lagrange equation $E(L)(u) = 0$.*

The converse of Theorem 8 is generally false.
Definition 6. Let PDE (4) and let $P = (P^1, P^2)$ with $Div\ P = D_xP^1 + D_yP^2$, *the total divergence.* The consequence $Div\ P = 0$ of PDE (4) is called *conservation law.* The function P^1 is called *flow* , and P^2 is called *conserved density associated to the conservation law.*

By the preceding Definition, there exists a function Q such that

$$Div\ P = Q \cdot F. \tag{16}$$

The relation (16) is called *the characteristic form of the conservation law*, and Q is called *the characteristic of the conservation law.*
Definition 7. Let

$$X = \zeta(x,y,u)\frac{\partial}{\partial x} + \eta(x,y,u)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u}$$

be a vector field on M. The vector field

$$X_Q = Q\frac{\partial}{\partial u},\ \ Q = \phi - \zeta u_x - \eta u_y,$$

is called *vector field of evolution associated to* X, and Q is called *the characteristic associated to* X.
Theorem 9 (Noether Theorem). *Let G be a local Lie group of transformations, which is a symmetry group of the variational problem (14) and let*

$$X = \zeta(x,y,u)\frac{\partial}{\partial x} + \eta(x,y,u)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u}$$

the infinitesimal generator of G. The characteristic Q of the field X is also a characteristic of the conservation law for the associated Euler-Lagrange equation $E(L)(u) = 0$.

There follows the existence of $P = (P^1, P^2)$, such that

$$Div\ P = Q \cdot E(L) = 0$$

to be a conservation law (in the caracteristic form) for the Euler-Lagrange equation $E(L) = 0$.

One proves ([17], 356) that for the Lagrangian $L = L(x, y, u^{(2)})$ we have

$$P = -(A + L\xi) = -(A^1 + L\zeta, A^2 + L\eta) = (P^1, P^2), \quad A = (A^1, A^2), \tag{17}$$

where

$$A^1 = Q \cdot E^{(x)}(L) + D_x \left(Q \cdot E^{(xx)}(L)\right) + \frac{1}{2} D_y \left(Q \cdot E^{(xy)}(L)\right), \tag{18}$$

$$A^2 = Q \cdot E^{(y)}(L) + \frac{1}{2} D_x \left(Q \cdot E^{(xy)}(L)\right) + D_y \left(Q \cdot E^{(yy)}(L)\right),$$

and

$$E^{(x)}(L) = \frac{\partial L}{\partial u_x} - 2D_x \left(\frac{\partial L}{\partial u_{xx}}\right) - D_y \left(\frac{\partial L}{\partial u_{xy}}\right), \tag{19}$$

$$E^{(y)}(L) = \frac{\partial L}{\partial u_y} - D_x \left(\frac{\partial L}{\partial u_{xx}}\right) - 2D_y \left(\frac{\partial L}{\partial u_{xy}}\right),$$

$$E^{(xx)}(L) = \frac{\partial L}{\partial u_{xx}}, \quad E^{(xy)}(L) = \frac{\partial L}{\partial u_{xy}}, \quad E^{(yy)}(L) = \frac{\partial L}{\partial u_{yy}},$$

are *Euler operators of superior order.*

7 Group of variational symmetries of the functional attached to Ţiţeica PDE. Conservation laws

We consider the functional

$$\mathcal{L}[u] = \int\int_D u \left(\frac{u_{xx}u_{yy} - u_{xy}^2}{(xu_x + yu_y - u)^4} - \alpha\right) dxdy, \quad \alpha \in \mathbf{R}^*, \tag{20}$$

where D is a domain in $\mathbf{R}^2$, $u \in C^2(D)$ and the condition (2) is satisfied for any $(x, y) \in D$.

Theorem 10. *The Lie algebra of the variational symmetry group of the functional (20) is described by the vector fields*

$$Y_1 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, \quad Y_2 = y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}, \tag{21}$$

$$Y_3 = y\frac{\partial}{\partial x}, \quad Y_4 = x\frac{\partial}{\partial y}.$$

Proof. According Theorem 8, the vector fields which determine the Lie algebra of the variational symmetry group are founded between the vector fields of the Lie algebra of the symmetry group of the associated Euler-Lagrange equation. The condition (15) must be verified only for the vector fields in the Lie algebra (8) of the symmetry group of PDE (1′). One considers

$$X = \sum_{i=1}^{8} C_i X_i,$$

where $C_i \in \mathbf{R}$ and X_i are the infinitesimal generators of the symmetry group G associated to Ţiţeica surfaces PDE. One determines the real constants C_i such that the relation (15) is satisfied. Using the relation (5), the second prolongation of the vector field

$$X = (C_1x + C_3y + C_4u)\frac{\partial}{\partial x} + +(C_5x + C_2y + C_6u)\frac{\partial}{\partial y} + (C_7x + C_8y - (C_1 + C_2))\frac{\partial}{\partial u},$$

is given by the functions

$$\Phi^x = C_7 - (2C_1 + C_2)u_x - C_5u_y - C_4u_x^2 - C_6u_xu_y,$$

$$\Phi^y = C_8 - C_3u_x - (C_1 + 2C_2)u_y - C_4u_xu_y - C_6u_y^2,$$

$$\Phi^{xx} = -(3C_1 + C_2)u_{xx} - 2C_5u_{xy} - 3C_4u_xu_{xx} - C_6u_yu_{xx} - 2C_6u_xu_{xy},$$

$$\Phi^{xy} = -C_3u_{xx} - 2(C_1 + C_2)u_{xy} - C_5u_{yy} - C_4u_yu_{xx} - 2C_6u_yu_{xy} - -2C_4u_xu_{xy} - C_6u_xu_{yy},$$

$$\Phi^{yy} = -(C_1 + 3C_2)u_{yy} - 2C_3u_{xy} - 3C_6u_yu_{yy} - C_4u_xu_{yy} - 2C_4u_yu_{xy}.$$

Substituting L and X with $\xi = (C_1x + C_3y + C_4u, C_5x + C_2y + C_6u)$, and $Div\xi = C_1 + C_2 + C_4u_x + C_6u_y$ in the relation (15), after computation, it follows

$$C_7x + C_8y + C_4uu_x + C_6uu_y = 0,$$

and thus $C_4 = C_6 = C_7 = C_8 = 0$. It results that the infinitesimal generator of the variational symmetry group for the functional (20) is

$$X = C_1X_1 + C_2X_2 + C_3X_3 + C_5X_5.$$

Denote $Y_1 = X_1,\ Y_2 = X_2,\ Y_3 = X_3$ and $Y_4 = X_5$.
Proposition 2. *For the vector field*

$$-Y_3 = -y\frac{\partial}{\partial x},$$

the flow and respectively the conserved density of the conservation law are

$$P^1 = -\alpha yu + \frac{u_x}{(xu_x + yu_y - u)^4}(u_{xy}(yu_y - u) - yu_xu_{yy}), \tag{22}$$

$$P^2 = -\frac{u_x}{(xu_x + yu_y - u)^4}(u_{xx}(yu_y - u) - yu_xu_{xy}).$$

Proof. The caracteristic associated to the vector field $-Y_3$ is

$$Q = yu_y.$$

Replacing in the relations (18), we obtain

$$A^1 = \frac{uy(u_{xy}^2 - u_{xx}u_{yy})}{(xu_x + yu_y - u)^4} + \frac{u_{xy}(uu_x - yu_y)}{(xu_x + yu_y - u)^4} + \frac{yu_x^2 u_{yy}}{(xu_x + yu_y - u)^4},$$

$$A^2 = \frac{u_x}{(xu_x + yu_y - u)^4}(u_{xx}(yu_y - u) - yu_x u_{xy}).$$

Introducing $\xi = (-y, 0)$ in the relations (17) it follows that the functions P^1, P^2 have the form (22).

Analogously one determines the conservation laws corresponding to the characteristics of the vector fields (21).

8 Strong/weak symmetry group

The symmetry group introduced in the Definition 2 is called *strong symmetry group.*
Definition 8. *The weak symmetry group of PDE* (4) is a group of transformations acting on $M \subset D \times U$ and which satisfies only the condition (b) in the Definition 2 of the strong symmetry group.

Consequently a weak symmetry group did not transforms solutions of PDE into its solutions.
Proposition 3. *Let G be a connected Lie group of transformations on M, with infinitesimal generators $X_1, ..., X_s$. Let $Q^1, ..., Q^s$ be the characteristics associated to these vector fields. Then any G-invariant function $u = f(x, y)$ must satisfy the system of equations*

$$Q^k(x, y, u^{(1)}) = 0, \quad k = 1, ..., s. \tag{23}$$

Any G-invariant solution $u = f(x, y)$ of PDE (4) is also a solution of the system (23), and hence of the system

$$\begin{cases} F(x, y, u^{(2)}) & = & 0 \\ Q^k(x, y, u^{(1)}) & = & 0, \quad k = 1, ..., s. \end{cases} \tag{24}$$

The converse is true only for the case in which G is a strong symmetry group.

One proves ([19])
Theorem 11. *Let G be a group of transformations acting on $M \subset D \times U$ and (4) a PDE of order two defined on D. Then G is always a symmetry group of the system (24) and hence always a weak symmetry group.*

Every s-parameter subgroup H of the strong symmetry group G (8) determines a family of group-invariant solutions. The problem of classification of the group-invariant solutions is reduced to the problem of classification of Lie subalgebras of the Lie algebra $\mathbf{g}$ of the group G ([14], 186). For the 1-dimensional subalgebras one considers a general element X and this can be simplified as much as possible, using the adjoint transformations.

We shall determine some solutions of PDE (1′) which are invariant with respect to the strong symmetry group G.
Remarks.

1) The finding of the adjoint representation $Ad\ G$ of the Lie group G, can be realised using the Lie series

$$Ad(exp(\varepsilon X)Y) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(adX)^n(Y) = Y - \varepsilon[X,Y] + \frac{\varepsilon^2}{2}[X,[X,Y]] - ... \tag{25}$$

2) If $u = f(x,y)$ is a solution of PDE $(1')$, then the following functions

$$u^{(1)} = e^{-\varepsilon} f(xe^{-\varepsilon}, y),\ \ u^{(2)} = e^{-\varepsilon} f(x, ye^{-\varepsilon}),\ \ u^{(3)} = f(x - \varepsilon y, y),$$

$$u^{(4)} = f(x - \varepsilon u^{(4)}, y),\ \ u^{(5)} = f(x, y - \varepsilon x),\ \ u^{(6)} = f(x, y - \varepsilon u^{(6)}),$$

$$u^{(7)} = f(x,y) + \varepsilon x,\ \ u^{(8)} = f(x,y) + \varepsilon y,\ \ \varepsilon \in \mathbf{R},$$

are also solutions of the equation since every 1-parameter subgroup G_i generated by $X_i,\ i = 1, ..., 8$, is a symmetry group.

3) The adjoint representation $Ad\ G$ of the Lie group G which invariates the Ţiţeica equation, is determined using the Lie series (25). This way we obtain

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	$e^{\varepsilon}X_3$	$e^{2\varepsilon}X_4$
X_2	X_1	X_2	$e^{-\varepsilon}X_3$	$e^{\varepsilon}X_4$
X_3	$X_1-\varepsilon X_3$	$X_2+\varepsilon X_3$	X_3	X_4
X_4	$X_1-2\varepsilon X_4$	$X_2-\varepsilon X_4$	X_3	X_4
X_5	$X_1+\varepsilon X_5$	$X_2-2\varepsilon X_5$	$X_3-\varepsilon(X_1-X_2)-\varepsilon^2 X_5$	$X_4+\varepsilon X_6$
X_6	$X_1-\varepsilon X_6$	$X_2-2\varepsilon X_6$	$X_3-\varepsilon X_4$	X_4
X_7	$X_1+2\varepsilon X_7$	$X_2+\varepsilon X_7$	$X_3+\varepsilon X_8$	$X_4-\varepsilon X_1-\varepsilon^2 X_7$
X_8	$X_1+\varepsilon X_8$	$X_2+2\varepsilon X_8$	X_3	$X_4-\varepsilon X_3$

Ad	X_5	X_6	X_7	X_8
X_1	$e^{-\varepsilon}X_5$	$e^{\varepsilon}X_6$	$e^{-2\varepsilon}X_7$	$e^{-\varepsilon}X_8$
X_2	$e^{\varepsilon}X_5$	$e^{2\varepsilon}X_6$	$e^{-\varepsilon}X_7$	$e^{-2\varepsilon}X_8$
X_3	$X_5-\varepsilon(X_1-X_2)-\varepsilon^2 X_3$	$X_6+\varepsilon X_4$	$X_7-\varepsilon X_3$	X_8
X_4	$X_5-\varepsilon X_6$	X_6	$X_7+\varepsilon X_1-\varepsilon^2 X_4$	$X_8+\varepsilon X_3$
X_5	X_5	X_6	X_7	$X_8-\varepsilon X_7$
X_6	X_5	X_6	$X_7+\varepsilon X_5$	$X_8+\varepsilon X_2-\varepsilon^2 X_6$
X_7	X_5	$X_6-\varepsilon X_5$	X_7	X_8
X_8	$X_5+\varepsilon X_7$	$X_6-\varepsilon X_2-\varepsilon^2 X_8$	X_7	X_8

Finally, we determine some group-invariant solutions of the equation $(1')$, corresponding to 1-dimensional subalgebras generated by $X_1 - X_2,\ \ X_5 - X_3$.

a) For the vector field

$$X_1 - X_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y},$$

one looks for solutions of the form $u = \varphi(xy)$. In this case the PDE $(1')$ becomes

$$2t\varphi'\varphi'' + \varphi'^2 + \alpha(2t\varphi' - \varphi)^4 = 0,$$

where $t = xy$. This DE admits particular solutions of the form $\phi(t) = t^p$, with the condition imposed in (2). For $p = -1$ and $\alpha = \frac{1}{27}$, we obtain

$$u(x,y) = \frac{1}{xy}$$

as a solution of PDE (1′). According to the preceding remark 2, it follows that

$$u(x,y) = \frac{1}{xy} + \varepsilon x, \quad u(x,y) = \frac{1}{xy} + \varepsilon y, \quad u(x,y) = \frac{1}{(x-\varepsilon_1 y)y} + \varepsilon_2 x, \quad \varepsilon, \varepsilon_1, \varepsilon_2 \in \mathbf{R},$$

are also solutions.

Other particular solution of the preceding DE is $\varphi(t) = \sqrt{1+at}$, for $a^2 + 4\alpha{=}0$. For $\alpha < 0$, it follows the solution

$$u(x,y) = \sqrt{1+axy}, \quad a \in \mathbf{R}^*,$$

of PDE (1′). According to remark 2, the functions

$$u(x,y) = \sqrt{1+axy} + \varepsilon x, \quad u(x,y) = \sqrt{1+a(x-\varepsilon_1 y)y} + \varepsilon_2 x, \quad \varepsilon, \varepsilon_1, \varepsilon_2 \in \mathbf{R},$$

are also solutions of PDE (1′).

b) For the vector field

$$X_5 - X_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

one looks for solutions of the form $u = \varphi(r)$, where $r = \sqrt{x^2+y^2}$. Replacing in the PDE (1′) we obtain the DE

$$\frac{1}{r}\varphi'\varphi'' = \alpha(r\varphi' - \varphi)^4.$$

This DE admits particular solutions of the form $\phi(r) = r^p$. For $p = -2$ and $\alpha = -\frac{4}{27}$, it follows

$$u(x,y) = \frac{1}{x^2+y^2}$$

as solution of PDE (1′). According to remark 2, the functions

$$u(x,y) = \frac{1}{x^2+y^2} + \varepsilon x, \quad u(x,y) = \frac{1}{(x-\varepsilon_1 y)^2 + y^2} + \varepsilon_2 x, \quad \varepsilon, \varepsilon_1, \varepsilon_2 \in \mathbf{R},$$

are also solutions of PDE (1′). The DE admits also a particular solution of the form $\varphi(r) = \sqrt{1+ar^2}$, $a \in \mathbf{R}^*$ for $\alpha = a^2$. If $\alpha > 0$, then it follows the implicit solution

$$u^2 + a(x^2+y^2) = 1, \quad u > 0,$$

of PDE (1′), and according to remark 2, the equations

$$u^2 + a((x-\varepsilon y)^2 + y^2) = 1, \quad u > 0,$$

$$(u-\varepsilon x)^2 + a(x^2+y^2) = 1, \quad u - \varepsilon x > 0, \quad \varepsilon \in \mathbf{R},$$

define also solutions of PDE (1′).

Now we refer to weak symmetry groups and the corresponding solutions of PDE $(1')$.

a) Let

$$X = -x^2 y \frac{\partial}{\partial x} + \frac{\partial}{\partial u}.$$

We obtain $C_1 = u - \frac{1}{xy}, \;\; C_2 = y$. Hence

$$u(x,y) = \frac{1}{xy} + h(y).$$

Replacing in the PDE $(1')$, it follows the DE

$$\frac{3}{x^4y^4} + \frac{2h''}{x^3y} = \alpha\left(-\frac{3}{xy} + yh' - h\right)^4.$$

As $h = h(y)$, by identification we deduce $h'' = 0, \;\; yh' - h = 0, \;\; \alpha = \frac{1}{27}$, hence $h(y) = Cy, \; C \in \mathbf{R}$. Consequently

$$u(x,y) = \frac{1}{xy} + Cy, \;\; C \in \mathbf{R}$$

is a solution of PDE $(1')$.

b) Let

$$X = 2ux\frac{\partial}{\partial x} + (u^2 - 1)\frac{\partial}{\partial u}.$$

Since $C_1 = \frac{u^2-1}{x}, \;\; C_2 = y$, it follows

$$u(x,y) = \sqrt{1 + xh(y)},$$

for $u > 0$.

Replacing in PDE $(1')$ we obtain $h(y) = Cy, \; C \in \mathbf{R}^*$. The corresponding solution of PDE $(1')$ is

$$u(x,y) = \sqrt{1 + Cxy}, \;\; u > 0, \;\; C \in \mathbf{R}^*, \;\; 1 + Cxy > 0.$$

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