Symmetry Groups and Conservation Laws of Certain Partial Differential Equations

Nicoleta Bîlă

Dedicated to Prof.Dr. Constantin UDRIŞTE on the occasion of his sixtieth birthday

Abstract

We study partial differential equations by using invariance under transformations groups due to Sophus Lie. This method, so called the classical Lie method of infinitesimal transformations or symmetry groups theory, has been applied last years to important PDEs arising from mathematics and physics. Based on the finding of a symmetry group associated to the studied PDEs system, we can find a lot of properties related on solutions. A modern presentation of this theory using the jet functions theory is given by Olver in his book [37].

The aim of this paper is to point out the ideas of the Ph.D.Thesis [2], which contains recent results obtained by application of the Lie's method for certain PDEs systems which arises from differential geometry, especially from Ţiţeica (Tzitzeica) surfaces theory, and from physics. The paper is split into four parts:the symmetry group history, symmetry groups in differential geometry, the thesis ideas and original results of thesis.

Mathematics Subject Classification 2000: 58J70, 35A30
Key words: symmetry group, criterion of infinitesimal invariance, group-invariant solutions, conservation laws

1 The Symmetry group history

The *groups theory* has an important role in a lot of domains of the science: mathematics, physics, classical mechanics, electromagnetism, relativity theory, quantum mechanics. This theory was introduced for the necessity to find a mathematical apparatus to study the properties of mathematical or physical objects. After the infinitesimal calculus invention, the concept of *group* is considered the most important discovery in mathematics.

The idea due to Sophus Lie, namely, to study the differential equations systems and the partial differential equations systems by using the transformations groups implied a new theory: the symmetry groups theory, known also as the classical Lie method of infinitesimal transformations.

Balkan Journal of Geometry and Its Applications, Vol.5, No.1, 2000, pp. 39-56

The *symmetry group* of a PDEs system is the largest (connected) local Lie group of transformations acting on the space of the independent and dependent variables of the system, with the property that it conserves the set of solutions. In the Lie's theory this group consists in geometric transformations which act on the set of solutions by transforming their graphs. The infinitesimal generator is a vector field associated with a local one-parameter group of transformations, and from this it follows that the symmetry group is described by the composition of the basic one-parameter groups.

The method of finding the symmetry group associated to a PDEs system is based on the infinitesimal criterion of invariance, and in this case the prolongation of the infinitesimal generator to the derivatives space of the dependent variables of the system is very important. Lie made the remark that, for a given PDEs system, the complicated nonlinear conditions of invariance can be replaced by equivalent linear conditions: an overdetermined linear PDEs system, so-called *the defining equations* for the symmetry group.

The knowledge of the symmetry group implies a lot of properties both of the system and of their solutions: the determining of the group-invariant solutions, the construction of new solutions of the system from known ones, the classification of the group-invariant solutions - two solutions are equivalent if one can be transformed into the other by some element group - using the associated adjoint representation. Moreover, we can find a PDEs class which is invariant under a given group of transformations. In his papers, Lie found the symmetry group associated with several ordinary differential equations and respectively first and second order PDEs. He was the first which made the remark that one can determine invariant solutions under a one-parameter group of transformations. Lie introduced the concept of variational symmetry group and he got the infinitesimal variational criterion too. Cartan [16] studied the equivalence between a given set of PDEs and a set of differential forms. The Noether's theorem implied the development of this research field since 1918. Thus a lot of mathematicians and physicians became interested to study this new theory: Jacobi studied the connection of variational symmetries with the conservation laws, continued later by Shutz and Engel; Birkhoff - applied the groups theory in the study of certain PDEs which arise from hydrodinamics; Ovsiannikov and the Russian school [41] - from 1950 to 1960 - proposed an important systematic programme to apply this method in the case of the most important PDEs which arise from physics, and in that time they introduced the algorithm of classification of solutions with the adjoint representation. Bluman and Cole [13] - found again the symmetry group of the heat equation given by Lie, determined group-invariant solutions for this PDE and group-invariant PDEs invariant under this symmetry group also. Miller [30]- applied the symmetry group theory in the finding of the solutions of the second order linear PDEs by using the separable variables method. Weisner - studied the connection of certain symmetry groups with some functions. Moreover Harrison şi Estabrook developed a similar theory but using differential forms [21], and this point of view was continued by Edelen. A modern exposition of the Lie method was given by Olver in his book [37], using the jet functions theory. In [39] Olver and Rosenau defined again the symmetry group like a strong symmetry group and they introduced the concept of weak symmetry group of a PDEs system. On the other hand, the algorithm to find the symmetry group of a PDEs system is a computational method. This allowed to appear a lot of symbolic programs to eliminate the difficult calculus in the case of particular forms of PDEs [22], [37].

There are a lot of papers and books which study this subject. So we have just presented only some of the important stages of the development of this theory: [13], [21], [22], [25], [30], [37]-[40], [41], [45], [60]. In the references [14], [17], [23], [26], [27], [31], [35]-[36], [46], [47], [63] appear similar problems connected with our research work interest.

2 Symmetry groups in differential geometry

We make the remark that PDEs and the PDEs systems which arise from differential geometry were not systematically studied by using the symmetry group theory. That is our motivation to apply the modern symmetry groups theory in the case of certain PDEs connected with differential geometry. This survey is based on our recent papers in which we find the differentiable distributions for which the integral varieties are graphs of the invariant solutions of certain equations and on variational conservation laws too.

We studied Monge-Ampère PDEs, Monge-Ampère-Ţiţeica PDEs respectively, the basic results beeing published in the paper [57]. The study of Ţiţeica surfaces PDEs implied the finding of some infinitesimal symmetries, Lagrangians and conservation laws associated with these equations [11], [59]. All these results show that the theory of Ţiţeica surfaces becomes a theory of variational calculus.

In the chapter 4 we include the papers [10] and [9], which consider the Camassa-Holm PDE from fluid mechanics and Blair PDEs system from solar physics and contact geometry. The papers [6] and [3] contain applications of the symmetry groups theory in the study of the PDE of surfaces with constant Gaussian curvature and the minimal surfaces PDE. In [4] and [5] are shown special classes of ODEs invariant under groups of transformations.

The problem to associate connections, and metrics generated by found symmetry groups is part of our recent work.

Opposite to our published papers, the thesis [2] contains also the original results found by using the Lie Program for analysis of partial differential equations on IBM type PCs [22]. We want to make the remark that in the case of the Monge-Ampère-Ţiţeica PDEs (2.1.1) and (2.1.2) respectively, the symbolic programs cannot be applied in the n-dimensional case. Their using for the PDEs systems (3.1.6), (3.1.8) and (3.1.10), which define the Ţiţeica surfaces, is very difficult also.

3 The Thesis ideas [2]

The theory presented by Olver in his book [37], which is very known and appreciated in the mathematical literature, represents the basis of our ideas.

The first chapter of the thesis contains a short presentation of the basic results of the symmetry groups theory for the PDEs systems and variational symmetry groups theory. Also, we give the explicit formulas for the finding of the symmetry group of certain PDEs studied in the second and in the fourth chapter.

The second chapter studies the symmeries of the Monge-Ampère PDE

$$det(u_{ij}) = f(x, u),$$

and especially in the cases $f = 1$ and $n = 2$ respectively. The second order PDE

$$u_{xx}u_{yy} - u_{xy}^2 = H(x, y, u, u_x, u_y)$$

is called the Monge-Ampère-Ţiţeica PDE. One proves that

$$u_{xx}u_{yy} - u_{xy}^2 = 1,$$

is the only second order Monge-Ampère-Ţiţeica PDE invariant under the symmetry group associated to it. One finds the group-invariant solutions and one proves that Pogorelov's solution [42] and Jörgens's solution [24] are group-invariant solutions. Moreover, one generalises the Calabi's result about the invariance of the Monge-Ampère equation, in the case $f = 1$, under the special linear group, so that this group is a subgroup of the symmetry group associated to the equation.

Studying the tetrahedral surfaces, in 1907, Ţiţeica introduced a new class of surfaces [48], with the property that

$$\frac{K}{d^4} = constant,$$

where K is the Gauss curvature and d the distance from origin to tangent plane of the surface. Ţiţeica called them S-surfaces, and showed that these are invariant under the centroaffine transformations. Now the surfaces S are called Ţiţeica-surfaces. Ţiţeica was the first who discovered centroaffine properties, so he is considered one of the founders of the affine geometry which was developed later as independent domain and for which the S-surfaces were affine spheres. The results due to Mayer, Myller, Vrânceanu, Gheorghiu, etc developed the centroaffine geometry [19], [20], [28], [29], [61], [62].

The study of the Ţiţeica surfaces PDE

$$u_{xx}u_{yy} - u_{xy}^2 = \alpha(xu_x + yu_y - u)^4, \quad \alpha \in R^*,$$

is contained in the second chapter and it is based on the paper [59]. One shows that the symmetry group associated with this PDE is the unimodular subgroup of the centroaffine group.

One proves that the only Monge-Ampère-Ţiţeica PDE invariant under the unimodular group is the Ţiţeica surfaces PDE. Also one finds group-invariant solutions and weak symmetry groups [39] associated to this PDE. By studying the inverse problem [37], [40], one shows that the Ţiţeica surfaces PDE is an Euler-Lagrange equation and one determines an associated second order Lagrangian. One considers variational problem and one gets the variational symmetry group and respectively conservations laws. In this way the Ţiţeica theory becomes related to variational calculus on differential manifolds. Also one shows that the Ţiţeica surfaces PDE can be included in the following class of Euler-Lagrange equations

$$\frac{u_{xx}u_{yy} - u_{xy}^2}{(xu_x + yu_y - u)^m} = \alpha,$$

where $m \neq 0, 2, 3$ and $\alpha \neq 0$.

The third chapter of the thesis analyses infinitesimal symmetries associated to the completely integrable system

$$\begin{cases} r_{uu} &= ar_u + br_v \\ r_{uv} &= hr \\ r_{vv} &= a''r_u + b''r_v, \end{cases}$$

which define the Ţiţeica surfaces [49], where the functions a, b, h, a'', b'' satisfy the conditions of integrability

$$ah = h_u, \quad a_v = ba'' + h, \quad b_v + bb'' = 0,$$

$$h_v = b''h, \quad a''_u + aa'' = 0, \quad h = b''_u + a''b.$$

One considers the associated scalar system

$$\begin{cases} \theta_{uu} &= a\theta_u + b\theta_v \\ \theta_{uv} &= h\theta \\ \theta_{vv} &= a''\theta_u + b''\theta_v, \end{cases}$$

and one finds associated infinitesimal symmetries. Ţiţeica proved that three independent solutions x, y, z of this second order completely integrable PDEs system define an S-surface, where u, v are the coordinates of the asymptotic lines of the surface [49]. Thus, one determines the following subgroups of the symmetry groups of these systems: the subgroups which act on the space of the independent variables and on the space of the dependent variables respectively associated with the system. One proves that the symmetry subgroup which acts on the space of the dependent variables, is the unimodular subgroup of the centroaffine group, and one finds again the above result.

Gheorghiu proved the existence of a space A_2 with affine connection associated to the scalar system which defines the Ţiţeica surfaces, and he introduced [19] a new class of affine spaces A_2^0. New examples of these spaces were given by Udrişte [52]. The problem to associate our results with the Ţiţeica connection [61] and the result due to Gheorghiu is still open [64].

One finds the symmetry groups associated with Liouville-Ţiţeica PDE

$$(\ln h)_{uv} = h,$$

and Ţiţeica PDE

$$(\ln h)_{uv} = h - \frac{1}{h^2},$$

respectively, the equations which defines the conditions of completely integrability of the systems which define the Ţiţeica ruled surfaces and Ţiţeica surfaces which are not ruled. One proves that these equations are Euler-Lagrange PDEs and by studying the associated variational problems, one finds variational symmetry groups and conservation laws. Our results are different from those presented in Bobenko [14] and Wolf [63]. By using the CRACK Program, in the paper [63], Wolf finds conservation laws by solving the conservation law condition directly (it is not assumed that any Lie-symmetries are known, nor that the equations are equivalent to the Euler-Lagrange equations of a variational problems).

The fourth chapter contains the study of the symmetry group associated with the third order PDE

$$uu_{xxx} + u_{xxy} + 2u_x u_{xx} - 3uu_x - u_y = 0,$$

so-called Camassa-Holm (CH) equation. This PDE was introduced by Fuchssteiner and Fokas, and Camassa and Holm [15] discovered again that it is the model of the shallow wave equation. Holm, Marsden and Ratiu [23] showed that the n-dimensional CH equation describes geodesic motion on the diffeomorphism group $\mathcal{D}iff(R^n)$, Misiolek [31] has shown that the CH equation represents a geodesic flow on the Bott-Virasoro group $\mathcal{D}iff(S^1)$, and Kourbaeva [26] has shown that this equation is a geodesic spray of the weak Riemannian metric on the diffeomorphism group $\mathcal{D}iff(R)$ or on $\mathcal{D}iff(S^1)$.

For a certain class of third order PDEs, namely

$$u_y - \varepsilon u_{xxy} + 2ku_x = uu_{xxx} + \alpha uu_x + \beta u_x u_{xx}$$

which includes the CH PDE, Clarkson, Mansfield and Priestley applied the nonclassical symmetries method [18].

In 4.1 one gets again the associated Lie group to CH PDE and moreover one finds a class of third order PDEs invariant under this group [10], which included the Rosenau-Hyman PDE

$$uu_{xxx} + uu_x + 3u_x u_{xx} - u_y = 0$$

also. In the class of second order PDEs invariant under the symmetry group associated with CH PDE can be included: the nonlinear wave PDE

$$u_y = uu_x,$$

the Liouville-Ţiţeica PDE

$$uu_{xy} - u_x u_y = u^3,$$

and respectively the particular Monge-Ampère-Ţiţeica PDEs

$$u_{xx}u_{yy} - u_{xy}^2 = u^4 f\left(\frac{u_x}{u}, \frac{u_y}{u^2}\right).$$

In the paper [12], Blair studied the vector equation

$$rot\ B = |B| \cdot B.$$

A solution of it gives a conformally flat contact metric structure on $\mathbf{R}^3$. Moreover if the vector field B satisfies the equation

$$div\ B = 0,$$

then the solutions of the system represent "force-free" models of solar physics. The system of these two equations is called Blair PDEs system. Blair found a solution of it by using the method of succesive aproximations. The vector fields

$$B_1 = \sin z \frac{\partial}{\partial x} + \cos z \frac{\partial}{\partial y},$$

and

$$B_2 = \frac{8(xz-y)}{(1+x^2+y^2+z^2)^2}\frac{\partial}{\partial x} + \frac{8(x+yz)}{(1+x^2+y^2+z^2)^2}\frac{\partial}{\partial y} + \frac{4(1+z^2-x^2-y^2)}{(1+x^2+y^2+z^2)^2}\frac{\partial}{\partial z},$$

satisfy the vector equation, but only the first is a solenoidal vector field. The finding of the solutions of this PDEs system is still an open problem.

In 4.2 are studied the PDEs

$$rot\ B = f \cdot B,$$

which arise from solar physics, in the case $f = f(u,v,w)$ and $f = |B|$ respectively. One finds the determining equations of the symmetry groups. One finds again the symmetry group associated with the Blair PDEs system [12] and one shows that the known solutions are group-invariant solutions, so using these we can find new solutions.

The annex of the thesis presentes some symbolic programs for determining the symmetry group of certain PDEs systems. Also it contains the results produced by the LIE51 and BIGLIE Programs, author Head [22], in the case of some PDEs and PDEs systems studied in the thesis.

4 Original results of thesis

In this last part we shall present several original results found by using the symmetry group theory, as follows:

Theorem 2.1.1. *The Lie algebra of the infinitesimal symmetries associated to the Monge-Ampère PDE*

$$det(u_{ij}) = f(x,u), \tag{2.1.1}$$

is described by the vector field

$$X = \sum_{i=1}^{n} \zeta^i(x)\frac{\partial}{\partial x^i} + \phi(x,u)\frac{\partial}{\partial u},$$

with the property that the components

$$\zeta^i(x) = x^i \sum_{j=1}^{n} a_j x^j + \sum_{j=1}^{n} b^i_j x^j + c^i, \quad i = 1,...,n,$$

and

$$\phi(x,u) = u\left(\sum_{j=1}^{n} a_j x^j + b\right) + \sum_{j=1}^{n} c_j x^j + c,$$

where $a_j,\ b^i_j,\ b,\ c_j,\ c^i,\ c \in \boldsymbol{R}$, are the solutions of the first order PDE

$$-\sum_{i=1}^{n} \zeta^i f_{x^i} - \phi f_u + nf\left(\phi_u - \frac{2}{n}\sum_{i=1}^{n} \zeta^i_{x^i}\right) = 0.$$

Theorem 2.1.2 *The Lie algebra of the infinitesimal symmetries associated to the Monge-Ampère PDE*

$$det(u_{ij}) = 1 \tag{2.1.2}$$

is described by the vector fields

$$X_{ij} = x^i \frac{\partial}{\partial x^j} + \delta^i_j \frac{2u}{n} \frac{\partial}{\partial u}, \quad 1 \le i, j \le n, \tag{2.1.3}$$

$$Y_i = x^i \frac{\partial}{\partial u}, \quad 1 \le i \le n, \quad Z_i = \frac{\partial}{\partial x^i}, \quad 1 \le i \le n, \quad V = \frac{\partial}{\partial u}.$$

In the case $n = 2$, the PDE (2.1.2) turns in

$$u_{xx}u_{yy} - u_{xy}^2 = 1. \tag{2.1.4}$$

This PDE is contained in the class,

$$u_{xx}u_{yy} - u_{xy}^2 = H(x, y, u, u_x, u_y),$$

so-called *Monge-Ampère-Ţiţeica PDEs.*

The Theorem 2.1.2 implies that the Lie algebra of the infinitesimal symmetries associated to the PDE (2.1.4) is spaned by following vector fields

$$X_1 = \frac{\partial}{\partial x}, \ X_2 = \frac{\partial}{\partial y}, \ X_3 = \frac{\partial}{\partial u}, \ X_4 = y\frac{\partial}{\partial x}, \ X_5 = x\frac{\partial}{\partial y},$$

$$X_6 = x\frac{\partial}{\partial u}, \ X_7 = y\frac{\partial}{\partial u}, \ X_8 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \tag{2.1.5}$$

$$X_9 = y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u},$$

where we denote $x^1 = x, \ x^2 = y$, respectively $Z_1 = X_1, \ Z_2 = X_2, \ V = X_3, \ X_{21} = X_4, \ X_{12} = X_5, \ Y_1 = X_6, \ Y_2 = X_7, \ X_{11} = X_8$ and $X_{22} = X_9$.

Theorem 2.1.3. *The second order PDE*

$$u_{xx} = H(x, y, u, u_x, u_y, u_{xy}, u_{yy}),$$

invariant under the Lie group G with the infinitesimal generators given by the relation (2.1.5), is reduced to the Monge-Ampère PDE (2.1.4).

Theorem 2.2.1. *The vector fields*

$$X_1 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, \ X_2 = y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}, \ X_3 = y\frac{\partial}{\partial x}, \ X_4 = u\frac{\partial}{\partial x} \tag{2.2.5}$$

$$X_5 = x\frac{\partial}{\partial y}, \ X_6 = u\frac{\partial}{\partial y}, \ X_7 = x\frac{\partial}{\partial u}, \ X_8 = y\frac{\partial}{\partial u}$$

describe the Lie algebra of the infinitesimal symmetries associated to the Ţiţeica surface PDE

$$u_{xx}u_{yy} - u_{xy}^2 = \alpha(xu_x + yu_y - u)^4. \tag{2.2.4}$$

Theorem 2.2.2. *The second order Monge-Ampère-Ţiţeica PDE of maximal rank,*

$$u_{xx}u_{yy} - u_{xy}^2 = H(x, y, u, u_x, u_y), \tag{2.2.3}$$

which admits the group G' (with X_1, X_2, X_3, X_7 as infinitesimal generators) as symmetry group, is the Ţiţeica surfaces PDE.

Theorem 2.3.1. *The operator T associated to the Ţiţeica surfaces PDE is equivalent to an Euler-Lagrange operator.*

Theorem 2.3.2. *A second order Lagrangian associated to the Ţiţeica surfaces PDE (2.2.4) is*

$$L(x, y, u^{(2)}) = \frac{u(u_{xy}^2 - u_{xx}u_{yy})}{(xu_x + yu_y - u)^4} - \alpha u. \tag{2.3.1}$$

Theorem 2.3.3. *The class of second order PDEs*

$$u_{xx}u_{yy} - u_{xy}^2 = \alpha(xu_x + yu_y - u)^m, \tag{2.3.2}$$

with $m \neq 0, 2, 3$ and $\alpha \neq 0$, is equivalent to a class of Euler-Lagrange PDEs.

Theorem 2.3.4. *A second order Lagrangian associated to the PDE (2.3.2) is*

$$L_m(x, y, u^{(2)}) = \frac{u(u_{xy}^2 - u_{xx}u_{yy})}{(xu_x + yu_y - u)^m} - \alpha(m - 3)u. \tag{2.3.3}$$

Theorem 2.4.1. *The Lie algebra of the variational symmetry group for the functional (2.4.1) is described by the vectors fields*

$$Y_1 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, \quad Y_2 = y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}, \tag{2.4.2}$$

$$Y_3 = y\frac{\partial}{\partial x}, \quad Y_4 = x\frac{\partial}{\partial y}.$$

Theorem 3.1.3. *The Lie algebra g_1 of the Lie symmetry group G_1, which acts on the space of the dependent variables of the PDEs system*

$$\begin{cases} x_{uu} &= ax_u + bx_v \\ x_{uv} &= hx \\ x_{vv} &= a''x_u + b''x_v \\ y_{uu} &= ay_u + by_v \\ y_{uv} &= hy \\ y_{vv} &= a''y_u + b''y_v \\ z_{uu} &= az_u + bz_v \\ z_{uv} &= hz \\ z_{vv} &= a''z_u + b''z_v, \end{cases} \tag{3.1.11}$$

$$(y_u z_v - z_u y_v)x - (x_u z_v - x_v z_u)y + (x_u y_v - x_v y_u)z = f, \tag{3.1.12}$$

where the functions a, b, h, a'', b'' satisfy the relations

$$ah = h_u,\ \ a_v = ba'' + h,\ \ b_v + bb'' = 0, \tag{3.1.5}$$

$$h_v = b''h,\ \ a''_u + aa'' = 0,\ \ h = b''_u + a''b,$$

and $f = f(u, v)$ is a nonidentically zero function, is described by the vector fields

$$X_1 = x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z},\ \ X_2 = y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z},\ \ X_3 = y\frac{\partial}{\partial x},\ \ X_4 = z\frac{\partial}{\partial x} \tag{3.1.13}$$

$$X_5 = x\frac{\partial}{\partial y},\ \ X_6 = z\frac{\partial}{\partial y},\ \ X_7 = x\frac{\partial}{\partial z},\ \ X_8 = y\frac{\partial}{\partial z}.$$

The Lie symmetry group G_1 is the unimodular subgroup of the centroaffine group.
Theorem 3.1.4. *The general vector field of the Lie algebra of infinitesimals symetries of the group G_2 associated with the second order PDEs system (3.1.11)+(3.1.12), is*

$$Z = \zeta(u)\frac{\partial}{\partial u} + \eta(v)\frac{\partial}{\partial v},$$

where the components ζ and η satisfy the following PDEs

$$\begin{cases} \zeta a_u + \eta a_v + a\zeta_u + \zeta_{uu} = 0 \\ \zeta b_u + \eta b_v - b\eta_v + 2b\zeta_u = 0 \\ \zeta h_u + \eta h_v + h(\zeta_u + \eta_v) = 0 \\ \zeta a''_u + \eta a''_v - a''\zeta_u + 2a''\eta_v = 0 \\ \zeta b''_u + \eta b''_v + b''\eta_v + \eta_{vv} = 0. \end{cases} \tag{3.1.14}$$

Proposition 3.1.3. *The function*

$$h(u, v) = \frac{3}{2\mathrm{sh}^2\frac{(u+v)\sqrt{3}}{2}} + 1 \tag{3.1.16}$$

gives a Ţiţeica revolution surface. Moreover, there is a Ţiţeica ruled surface associated to it.
Proposition 3.1.4. *The Ţiţeica's solution marked in [2] by (3.1.17) is invariant under the subgroup G_2, for which*

$$Z = C_1\frac{\partial}{\partial u} + C_2\frac{\partial}{\partial v},$$

is an infinitesimal generator, in the case of Ţiţeica surfaces which are not ruled.
Theorem 3.2.2. *The general vector field which describes the algebra of infinitesimal symmetries associated to Liouville-Ţiţeica equation*

$$\omega_{uv} = e^{\omega}, \tag{3.2.1}$$

is

$$W = f\frac{\partial}{\partial u} + g\frac{\partial}{\partial v} - (f' + g')\frac{\partial}{\partial \omega}, \tag{3.2.5}$$

where $f = f(u)$ and $g = g(v)$.
Theorem 3.2.3. *The vector fields which describe the Lie algebra of infinitesimal symmetries associated to Ţiţeica surfaces PDE*

$$\omega_{uv} = e^{\omega} - e^{-2\omega}, \tag{3.2.2}$$

are the following

$$U_1 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \ \ U_2 = \frac{\partial}{\partial u}, \ \ U_3 = \frac{\partial}{\partial v}. \tag{3.2.6}$$

Theorem 3.2.4. *A second order PDE invariant under the symmetry group associated to Ţiţeica PDE (3.2.2) takes the form*

$$H(\omega, \omega_u\omega_v, \omega_{uv}, \omega_{uu}\omega_{vv}) = 0. \tag{3.2.7}$$

In particular, if the PDE is a Monge-Ampère-Ţiţeica PDE (2.2.3), then one finds the PDE

$$\omega_{uu}\omega_{vv} - \omega_{uv}^2 = f(\omega, \omega_u\omega_v). \tag{3.2.8}$$

Theorem 3.3.1. *The operators*

$$T_1(\omega) = \omega_{uv} - e^{\omega},$$

and respectively

$$T_2(\omega) = \omega_{uv} - e^{\omega} - e^{-2\omega},$$

associated to the PDEs (3.2.1) and (3.2.2) respectively, are identically with the Euler-Lagrange operators, for which the associated Lagrangians are

$$L_1(u, v, \omega^{(1)}) = -\frac{1}{2}\omega_u\omega_v - e^{\omega}, \tag{3.3.1}$$

and respectively

$$L_2(u, v, \omega^{(1)}) = -\frac{1}{2}\omega_u\omega_v - e^{\omega} - \frac{1}{2}e^{-2\omega}. \tag{3.3.2}$$

Theorem 3.3.2. *The Lie algebras of the variational symmetry groups for the functionals*

$$\mathcal{L}_i[\omega] = \int\int_D L_i(u, v, \omega^{(1)})dudv, \quad i = 1, 2,$$

where L_1 and L_2 are given by (3.3.1) and (3.3.2) respectively are generated by the vector fields

$$W_1 = u\frac{\partial}{\partial u} - \frac{\partial}{\partial \omega}, \quad W_2 = v\frac{\partial}{\partial v} - \frac{\partial}{\partial \omega}, \quad W_3 = \frac{\partial}{\partial u}, \quad W_4 = \frac{\partial}{\partial v}, \tag{3.3.5}$$

respectively

$$U_1 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \quad U_2 = \frac{\partial}{\partial u}, \quad U_3 = \frac{\partial}{\partial v}. \tag{3.3.6}$$

Proposition 3.3.1. *The fluxes P^1 and respectively the conserved densities P^2 associated to the conservation laws for Liouville-Ţiţeica PDE (3.2.1), are contained in the following table*

$-W_i$	P^1	P^2
$-W_1$	$\frac{1}{2}\omega_v - ue^{\omega}$	$\frac{1}{2}\omega_u(1+u\omega_u)$
$-W_2$	$\frac{1}{2}\omega_v(1+v\omega_v)$	$\frac{1}{2}\omega_u - ve^{\omega}$
$-W_3$	$-e^{\omega}$	$\frac{1}{2}\omega_u^2$
$-W_4$	$\frac{1}{2}\omega_v^2$	$-e^{\omega}$

Proposition 3.3.2. *The flux P^1 and respectively the conserved density P^2 associated to the conservation laws for Ţiţeica PDE (3.2.2), are contained in the following table*

$-U_i$	P^1	P^2
$-U_1$	$-\frac{1}{2}ue^{-2\omega} - \frac{1}{2}v\omega_v^2 - ue^{\omega}$	$\frac{1}{2}u\omega_u^2 + ve^{\omega} + \frac{1}{2}ve^{-2\omega}$
$-U_2$	$-e^{\omega} - \frac{1}{2}e^{-2\omega}$	$\frac{1}{2}\omega_u^2$
$-U_3$	$\frac{1}{2}\omega_v^2$	$-e^{\omega} - \frac{1}{2}e^{-2\omega}$

Theorem 4.1.1. *The Lie algebra $\boldsymbol{g}$ of the infinitesimal symmetries associated to Camassa-Holm PDE*

$$uu_{xxx} + u_{xxy} + 2u_x u_{xx} - 3uu_x - u_y = 0 \tag{4.1.1}$$

is generated by the vector fields

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}. \tag{4.1.2}$$

Theorem 4.1.2. *A third order PDE*

$$H(x, y, u^{(3)}) = 0$$

for which the Lie algebra of associated infinitesimal symmetries is described by the vector fields (4.1.2) can be written in the following form

$$h\left(\frac{u_x}{u}, \frac{u_y}{u^2}, \frac{u_{xx}}{u}, \frac{u_{xy}}{u^2}, \frac{u_{yy}}{u^3}, \frac{u_{xxx}}{u}, \frac{u_{xxy}}{u^2}, \frac{u_{xyy}}{u^3}, \frac{u_{yyy}}{u^3}\right) = 0. \tag{4.1.3}$$

Theorem 4.2.1. *The Lie algebra of infinitesimal symmetries associated to PDEs system (4.2.4) is described by the vector field*

$$X = \zeta\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \theta\frac{\partial}{\partial z} + \phi\frac{\partial}{\partial u} + \lambda\frac{\partial}{\partial v} + \psi\frac{\partial}{\partial w},$$

for which the components $\zeta, \eta, \theta, \phi, \lambda, \psi$, functions of x, y, z, u, v, w, are the solutions of the folllowing PDEs system

$$\psi_y - \lambda_z - \phi(uf_u + f) - u\lambda f_v - u\psi f_w - vf\lambda_u + wf\zeta_z +$$

$$+uf(\psi_w - \eta_y) + vwf^2\zeta_u - u^2 f^2\eta_w = 0$$

$$\phi_z - \psi_x - vf_u\phi - (vf_v + f)\lambda - vf_w\psi + vf(\phi_u - \theta_z) -$$
$$-wf\psi_v + uf\eta_x - v^2 f^2\theta_u + uwf^2\eta_v = 0$$
$$\lambda_x - \phi_y - w\phi f_u - w\lambda f_v - (wf_w + f)\psi + vf\theta_y + wf(\lambda_v - \zeta_x) -$$
$$-uf\phi_w - w^2 f^2\zeta_v + uvf^2\theta_w = 0$$
$$\phi_v = -\eta_x + f(v\eta_w - w\eta_v)$$
$$\phi_w = -\theta_x + f(v\theta_w - w\theta_v) \tag{4.2.7}$$
$$\lambda_u = -\zeta_y + f(w\zeta_u - u\zeta_w)$$
$$\lambda_w = -\theta_y + f(w\theta_u - u\theta_w)$$
$$\psi_u = -\zeta_z + f(u\zeta_v - v\zeta_u)$$
$$\psi_v = -\eta_z + f(u\eta_v - v\eta_u)$$
$$\phi_u - \psi_w = -\zeta_x + \theta_z + f(2v\theta_u - w\zeta_v - u\eta_w)$$
$$\psi_w - \lambda_v = -\theta_z + \eta_y + f(2u\eta_w - v\theta_u - w\zeta_v)$$
$$\eta_u = \zeta_v \quad \theta_v = \eta_w \quad \zeta_w = \theta_u.$$

Theorem 4.2.2. *A Lie subalgebra of the Lie algebra of infinitesimal symmetries associated to the PDEs system (4.2.5) is described by the following vector fields:*

$$X_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}, \ \ X_2 = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} - w\frac{\partial}{\partial v} + v\frac{\partial}{\partial w},$$
$$X_3 = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z} - w\frac{\partial}{\partial u} + u\frac{\partial}{\partial w}, \ \ X_4 = \frac{\partial}{\partial x}, \ \ X_5 = \frac{\partial}{\partial y}, \ \ X_6 = \frac{\partial}{\partial z}.$$
$$X_7 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w}, \tag{4.2.8}$$
$$X_8 = 2xz\frac{\partial}{\partial x} + 2yz\frac{\partial}{\partial y} + (z^2 - x^2 - y^2)\frac{\partial}{\partial z} + 2(xw - zu)\frac{\partial}{\partial u} +$$
$$+2(yw - zv)\frac{\partial}{\partial v} - 2(xu + yv + zw)\frac{\partial}{\partial w},$$
$$X_9 = xy\frac{\partial}{\partial x} + \frac{1}{2}(y^2 - x^2 - z^2)\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z} + +(xv - yu)\frac{\partial}{\partial u} -$$
$$-(xu + yv + zw)\frac{\partial}{\partial v} + (zv - yw)\frac{\partial}{\partial w},$$
$$X_{10} = \frac{1}{2}(x^2 - y^2 - z^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z} - (xu + yv + zw)\frac{\partial}{\partial u} +$$
$$+(yu - xv)\frac{\partial}{\partial v} + (zu - xw)\frac{\partial}{\partial w}.$$

Theorem 4.2.3. *The following vector fields*

$$X_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}, \quad X_2 = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} - w\frac{\partial}{\partial v} + v\frac{\partial}{\partial w}, \tag{4.2.9}$$

$$X_3 = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z} - w\frac{\partial}{\partial u} + u\frac{\partial}{\partial w}, \quad X_4 = \frac{\partial}{\partial x}, \quad X_5 = \frac{\partial}{\partial y},$$

$$X_6 = \frac{\partial}{\partial z}, \quad X_7 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w},$$

described the Lie algebra $\mathfrak{g}$ of infinitesimal symmetries associated to Blair PDEs system

$$\begin{cases} w_y - v_z &= u\sqrt{u^2+v^2+w^2} \\ u_z - w_x &= v\sqrt{u^2+v^2+w^2} \\ v_x - u_y &= w\sqrt{u^2+v^2+w^2} \\ u_x + v_y + w_x &= 0. \end{cases} \tag{4.2.6}$$

Theorem 4.2.4. *Let $f = f(u, v, w)$. The only first order PDEs system*

$$\begin{cases} w_y - v_z &= uf \\ u_z - w_x &= vf \\ v_x - u_y &= wf \\ u_x + v_y + w_x &= 0, \end{cases} \tag{4.2.14}$$

invariant under the symmetry group G associated to the Blair PDEs system (4.2.6), is the Blair PDEs system.

Acknowledgements. The ideas presented here comprise the major part of the author's doctoral thesis at University Politehnica of Bucharest (3rd December 1999), directed by Prof. Dr. Constantin Udrişte. The author is grateful to the advisor for his great helpful during this research work and to Prof. Dr. Peter J. Olver for his important suggestions and comments related to thesis.

A version of this paper was presented at the Workshop on Electromagnetic Flows and Dynamics, Oct. 15-19, 1999, University Politehnica of Bucharest, supported by MEN Grant No 21815/28.09.1998, CNCSU-31.

References

[1] V.Barbu -*Probleme la limită pentru ecuaţii cu derivate parţiale*, Editura Academiei, Bucureşti, 1993.

[2] N.Bîlă -*Symmetry groups and conservation laws of certain partial differential equations*, Ph.D. Thesis, University Politehnica of Bucharest, 1999.

[3] N.Bîlă -*Lie groups applications to minimal surfaces PDE*, Proceedings of the Workshop on Global Analysis, Differential Geometry and Lie Algebras, BSG Proceedings 3(1999), Geometry Balkan Press, Editor: Gr. Tsagas, 197-205; presented at The Second Conference of Balkan Society of Geometers, The Workshop on Global Analysis, Differential Geometry, Lie Algebras, Universitatea Aristotle, Thessaloniki, Greece, 24-27.06.1998.

[4] N.Bîlă -*Invariance of a second-order ordinary differential equation under a two-parameter Lie group*, Scientific Bulletin, University Politehnica of Bucharest, Series A, 60, 3-4(1998), 13-27; presented at First Conference of Balkan Society of Geometers, Universitatea Politehnica Bucureşti, 23-27.09.1996.

[5] N.Bîlă -*Invariance of n-th order ordinary differential equations under a one-parameter Lie group of transformations*, Scientific Bulletin, University Politehnica of Bucharest, Series A, 60, 3-4(1998), 57-71.

[6] N.Bîlă -*Symmetry Lie groups of PDE of surfaces with constant Gaussian curvature*, Scientific Bulletin, University Politehnica of Bucharest, Series A, 61, 1-2(1999), 123-136.

[7] N.Bîlă -*Symmetries of the Monge-Ampère-Ţiţeica, Camassa-Holm and Blair PDEs*, will be published in Proceeding of Fourth International Workshop on Differential Geometry and Its Applications, presented at The Fourth International Workshop on Differential Geometry and Its Applications, Universitatea Transilvania Braşov, 16-22.09.1999.

[8] N.Bîlă -*Symmetry Groups of Ţiţeica PDEs*, will be published in Proceeding of Workshop on Electromagnetic Flows and Dynamics; presented at Workshop on Electromagnetic Flows and Dynamics, Universitatea Politehnica Bucureşti, 15-19.10.1999.

[9] N.Bîlă -*Symmetries of PDEs systems appearing in solar physics and contact geometry*, sent to the Journal of Geometry and Physics; arXiv.org e-Print archive "http://xxx.lanl.gov/abs/math-ph/9910042".

[10] N.Bîlă, C.Udrişte -*Infinitesimal symmetries of Camassa-Holm equation*, Proceedings of the Conference of Geometry and Its Applications in Technology and Workshop on Global Analysis, Differential Geometry and Lie Algebras, BSG Proceedings 4(1999), Geometry Balkan Press, Editor: Gr. Tsagas, 149-160; presented at the Conference of Geometry and Its Applications in Technology, Workshop on Global Analysis, Differential Geometry, Lie Algeras, Universitatea Aristotle, Thessaloniki, Grecia, 24-28.06.1998.

[11] N.Bîlă -*Symmetry groups and Lagrangians associated to Ţiţeica surfaces*, sent to Nonlinear Differential Equations and Applications; arXiv.org e-Print archive "http://xxx.lanl.gov/abs/math.DG/9910138".

[12] D.E.Blair -*On the existence of conformally flat contact metric manifolds*, preprint.

[13] G.W.Bluman, J.D.Cole -*Similarity Methods for Differential Equations*, Applied Mathematical Sciences, 13, Springer Verlag, New York, 1974.

[14] A.I.Bobenko -*Surfaces in terms of 2 by 2 matrices: Old and new integrable cases*, in Harmonic maps and Integrable systems, Aspects of Mathematics, vol. E 23, A.P.Fordy, J.C.Wood, Friedr. Vieweg & Sohn Verlagsgesellschaft mbH, Braunschweig, Wiesbaden, 1994.

[15] R.Camassa, D.D.Holm -*An integrable shallow water equation with peaked solitons*, Phys. Rev. Kett., 71(1993), 1661-1664.

[16] E.Cartan -*Oeuvres complètes*, Gauthiers-Villars, Paris, 1953.

[17] P.A.Clarkson, E.L.Mansfield -*Symmetry reductions and exact solutions of shallow water wave equations*, Acta Appl. Math., 39(1995), 245-276.

[18] P.A.Clarkson, E.L.Mansfield, T.J.Priestley -*Symmetries of a class of nonlinear third order partial differential equations*, Mathematical and Computer Modelling, 25, 8-9(1997), 195-212.

[19] Gh.Th.Gheorghiu -*O clasă particulară de spaţii cu conexiune afină*, St. Cerc. Mat., 21, 8(1969), 1157-1168.

[20] Gh.Th.Gheorghiu -*Hipersuprafeţe Ţiţeica*, Lucrările Ştiinţifice ale Institutului Pedagogic Timişoara, 1959, 45-60.

[21] B.K.Harrison, F.B.Estabrook -*Geometric approach to invariance groups and solution of partial differential systems*, J. Math. Phys., 12, 4(1971), 653-666.

[22] A.K.Head -*Program LIE for analysis of differential equations on IBM type PCs*, Melbourne, Australia, "http://archives.math.utk.edu/ software /msdos/adv.diff.equations/lie/.html".

[23] D.D.Holm, J.E.Marsden, T.S.Ratiu -*The Euler-Poincaré equations and semidirect products with applications to continuum theories*, Adv. in Math., 1998, to appear.

[24] K.Jörgens -*Über die Lösungen der Differentialgleichung* $rt-s^2=1$, Math. Ann., 127(1954), 130-134.

[25] N.Kamran, P.J.Olver -*Eqiuvalence problems for first order Lagrangians on the line*, J. Diff. Eq., 80(1989), 32-78.

[26] S.Kouranbaeva -*The Camassa-Holm equation as a geodesic flow on the diffeomorphism group*, 1998, preprint.

[27] E.L.Mansfield -*Computer algebra for the Lie symmetry classification problem*, preprint.

[28] O.Mayer -*Géométrie centroaffine différentielle des surfaces*, Annales scientifiques de l'Université de Jassy, 21, 1-4(1934), 1-77.

[29] O.Mayer -*Opera matematică-note şi memorii 1919-1941*, Editura Academiei RSR, Bucureşti, 1974.

[30] W.Miller Jr. -*Symmetry and separation of variables*, Addison-Wesley Publ. Co., London, 1977.

[31] G.Misiolek -*A shallow water equation as a geodesic flow on the Bott-Virasoro group*, J. Geom. Phys. 24(1998), 203-208.

[32] G.Mocică, C.Udrişte -*Asupra 3-structurilor Sasaki*, Buletinul Institutului Politehnic, 39, 1(1977), 3-8.

[33] G.Morosanu -*Ecuaţii neliniare de evoluţie şi aplicaţii*, Editura Academiei, Bucureşti, 1986.

[34] L.Nicolescu -*Grupuri şi algebre Lie*, Universitatea Bucureşti, 1988.

[35] V.Obădeanu, C.Vernic -*Systèmes Dynamiques sur des Éspaces de Riemann*, Balkan Journal of Geometry and Its Applications, 2, 1(1997), 73-82.

[36] V.Obădeanu -*Differential dynamical systems. Differential geometry*, Tipografia Univ. de Vest, Timişoara, 1998.

[37] P.J.Olver -*Applications of Lie Groups to Differential Equations*, Graduate Texts in Math., 107, Springer Verlag, New York, 1986.

[38] P.J.Olver -*Symmetry groups and group invariant solutions of partial differential equations*, J. Diff. Geom., 14(1979), 497-542.

[39] P.J.Olver, P.Rosenau -*Group invariant solutions of differential equations*, SIAM J. Appl. Math., 47, 2(1987), 263-278.

[40] D.Opriş, I.Butulescu -*Metode geometrice în studiul sistemelor de ecuaţii diferenţiale*, Editura Mirton, Timişoara, 1997.

[41] L.V.Ovsiannikov -*Group analysis of differential equation*, Academic Press, New York, 1982.

[42] A.V.Pogorelov -*On the regularity of generalized solutions of the equation* $det\left(\frac{\partial^2 u}{\partial x^i x^j}\right) = \Phi(x^1, x^2, ..., x^n) > 0$, Dokl. Akad. Nauk SSSR, 200(1971), 534-537.

[43] A.V.Pogorelov -*Monge-Ampère equation*, Mockba Hayka, 1988.

[44] Shiu-Yen Cheng, Shing-Tung Yau -*On the regularity of the Monge-Ampère equation* $det\left(\frac{\partial^2 u}{\partial x^i x^j}\right) = F(x,u)$, CPAM, 30(1977), 69-94.

[45] F.Takens -*Symmetries, conservation laws and variational principles*, Lecture Notes in Mathematics, 597, Springer Verlag, New York, 1977, 581-603.

[46] K.Teleman, M.Teleman -*Elemente de teoria grupurilor cu aplicaţii în topologie şi fizică*, Editura Ştiinţifică, Bucureşti, 1973.

[47] P.P.Teodorescu, N.N.Nicorovici-Porumbaru -*Aplicaţii ale teoriei grupurilor în mecanică şi fizică*, Editura Tehnică, Bucureşti, 1985.

[48] G.Ţiţeica -*Sur une nouvelle classe de surfaces*, Comptes Rendus, Acad. Sci. Paris, 144(1907), 1257-1259.

[49] G.Ţiţeica -*Géométrie différentielle projective des réseaux*, Cultura Naţională, Bucureşti, 1923.

[50] G.Ţiţeica -*Sur la géométrie différentielle de l'équation de Laplace*, C. R. du Congres Int. de Math. Oslo, 1936.

[51] G.Ţiţeica -*Opere-note şi memorii*, I, Imprimeria Naţională, Bucureşti, 1941.

[52] C.Udrişte -*Asupra varietăţilor* A^0_m, Buletinul Institutului Politehnic "Gheorghe Gheorghiu-Dej", Bucureşti, 37, 3(1975), 7-10.

[53] C.Udrişte -*Conexiuni afine invariante pe spaţii riemanniene* V_4, St. Cerc. Mat., 22, 6(1970), 935-946.

[54] C.Udrişte -*Linii de câmp*, Editura Tehnică, Bucureşti, 1988.

[55] C.Udrişte -*Convex functions and optimization methods on riemannian manifolds*, Mathematics and Its Applications, 247, Kluwer Academic Publishers, 1994.

[56] C.Udrişte -*Geometric Dynamics*, Mathematics and Its Applications, Kluwer Academic Publishers, 2000.

[57] C.Udrişte, N.Bîlă -*Symmetry Lie groups of the Monge-Ampère equation*, Balkan Journal of Geometry and Its Applications, 3, 2(1998), 121-134.

[58] C.Udrişte, C.Bucur, N.Bîlă -*The characters of the general linear group*, Scientific Bulletin, University Politehnica of Bucharest, Series A, 55, 3-4(1993), 267-271, presented at The International Workshop on Differential Geometry and Its Applications, Universitatea Politehnica Bucureşti, 25-30.07.1993.

[59] C.Udrişte, N.Bîlă -*Symmetry group of Ţiţeica surfaces PDE*, Balkan Journal of Geometry and Its Applications, 4, 2(1999), 123-140, presented at Conferinţa de Analiză Matematică şi Aplicaţii, Secţiunea Algebră şi Geometrie, Universitatea de Vest, Timişoara, 14-15.11.1998, arXiv.org e-Print archive "http://xxx.lanl.gov/abs/math.DG/9910142".

[60] A.M.Vinogradov -*Symmetries and conservation laws of partial differential equations: Basic notions and results*, Acta Appl. Math., 15(1989), 3-21.

[61] G.Vrănceanu -*Invariants centro-affines d'une surface*, Revue Roumaine de Math. Pures et Appl., 24, 6(1972), 979-982.

[62] G.Vrănceanu -*Gh. Tzitzeica fondateur de la Géometrie centro-affine*, Revue Roumaine de Math. Pures et Appl., 24, 6(1979), 983-988.

[63] T.Wolf -*A comparison of four approaches to the calculation of conservation laws*, 1999, preprint.

[64] K.Yano -*Generalization of the connection of Tzitzeica*, Kodai, Math. Sem. Rep., 21(1969).

[65] S.T.Yau -*Survey on partial differential equations in differential geometry*, Seminar on Differential Geometry, Annals of Mathematics Studies, 102(1982), Princeton University Press, 3-72.

University Politehnica of Bucharest, Department of Mathematics I
Splaiul Independenţei 313, Bucharest, 77206, Romania, e-mail:nbila@mathem.pub.ro