On Weak Symmetries of Kaehler Manifolds

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Dedicated to Prof.Dr. Constantin UDRIŞTE
on the occasion of his sixtieth birthday

Abstract

Weakly symmetric Riemannian manifolds are generalizations of the locally symmetric manifolds, spaces of recurrent curvature and pseudo symmetric manifolds. These are manifolds in which the covariants derivative ∇R of the curvature tensor R is a linear expression in R. The appearing coefficients of this expression are called associated 1-forms. They satisfy in the specified types of manifolds gradually weaker conditions. Weakly Ricci-symmetric Riemannian or Kaehler manifolds are defined by a similar representation of ∇S in place of ∇R, where S is the Ricci tensor.

We prove several relations that exist between the properties of the weakly symmetric or weakly Ricci-symmetric Kaehler manifolds and the associated 1-forms of these spaces. In these relations the Ricci tensor and its eigenvalues play the decisive role.

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1 Introduction

The notions of weakly symmetric and weakly Ricci symmetric manifolds were introduced by the first an third authors [7], [8]. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called *weakly symmetric* (denoted by $(WS)_n$) if the curvature tensor R of type $(0,4)$ satisfies the condition

$$\begin{aligned} (\nabla_X R)(Y,Z,U,V) &= \alpha(X)R(Y,Z,U,V) + \beta(Y)R(X,Z,U,V)+ \\ &+ \gamma(Z)R(Y,X,U,V) + \delta(U)R(Y,Z,X,V)+ \\ &+ \rho(V)R(Y,Z,U,X),\ \forall X,Y,Z,U,V \in \mathcal{X}(M), \end{aligned} \tag{1}$$

where α, β, γ, δ, ρ are 1-forms called the *associated 1-forms* which are not zero simultaneously and ∇ denotes covariant differentiation.

A non-flat Riemannian manifold is called *weakly Ricci-symmetric* and denoted by $(WRS)_n$ if the Ricci tensor S is non-zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = \alpha(X)S(Y,Z) + \beta(Y)S(X,Z) + \gamma(Z)S(Y,X), \tag{2}$$

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where α, β, γ are again 1-forms, not zero simultaneously. Weakly symmetric manifolds have been studied by M. Prvanović [6], T.Q. Binh [2], U.C. De and S. Bandyopadhyay [5] and others. If in (1) the 1-form α is replaced by 2α and ρ is equal to α, then the manifold is called a *generalized pseudo symmetric manifold* introduced and investigated by M. C. Chaki [3], and if in (2) the 1-form α is replaced by 2α, then the manifold is called a *generalized pseudo Ricci symmetric* manifold introduced by Chaki and Koley [4]. So the defining conditions of weakly symmetric and weakly Ricci symmetric manifolds are a litte weaker than the generalized pseudo symmetric and generalized pseudo Ricci symmetric manifolds.

In a recent paper [5] U.C. De and S. Bandyopadhyay gave an example of $(WS)_n$ and showed that in (1) necessarily $\gamma = \beta$ and $\varrho = \delta$. So (1) takes the form:

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= \alpha(X)R(Y,Z,U,V) + \beta(Y)R(X,Z,U,V)\\ &+\beta(Z)R(Y,X,U,V) + \delta(U)R(Y,Z,X,V) + \delta(V)R(Y,Z,U,X).\end{aligned} \tag{3}$$

Let A, B and P be the vector fields associated with the 1-forms α, β and δ respectively i.e; $g(X,A) = \alpha(X)$, $g(X,B) = \beta(X)$ and $g(X,P) = \delta(X)$ for all X. A, B and P are called the *associated vector fields* corresponding to the 1-forms α, β and δ respectively.

In the present paper we study weakly symmetric and weakly Ricci symmetric Kaehler manifolds. In Section 2 we prove that in a weakly symmetric Kaehler manifold (a) if the scalar curvature is a non-zero constant, then the sum of the associated 1-forms is zero, and (b) the vector fields A, JA, B, JB, P and JP, with the almost complex structure J, are eigenvectors of the Ricci tensor S with the same eigenvalue $r/2$, where r is the scalar curvature of (M^n, g). Finally, we prove that in dimension $n = 6$ if A, JA, B, JB, P and JP are linearly independent, then the manifold will be Ricci flat. In the last Section 3 we consider a weakly Ricci symmetric Kaehler manifold and prove that in a weakly Ricci symmetric Kaehler manifold of non-zero constant scalar curvature the associated 1-forms α, β, γ are all equal.

Before starting with our investigations we collect some properties of Kaehler manifolds which will be used in the sequel. A Kaehler manifold is an even-dimensional manifold M^{2k} with a complex structure J and a positive-definite metric g which satisfies the following conditions [1]

$$J^2 = -I, \quad g(\overline{X},\overline{Y}) = g(X,Y), \quad \overline{X} = JX$$

and

$$\nabla J = 0, \tag{4}$$

where ∇ means the covariant derivation according to the Levi–Civita connection.

The formulas [1]:

$$R(X,Y) = R(\overline{X},\overline{Y}), \tag{5}$$

$$S(X,Y) = S(\overline{X},\overline{Y}), \tag{6}$$

$$S(X,\overline{Y}) + S(\overline{X},Y) = 0 \tag{7}$$

are well known for a Kaehler manifold.

2 Weakly symmetric Kaehler manifolds

In this section we suppose that (M^n, g) is a $(WS)_n$ and Kaehler manifold. Then from (3), (4) and (5) we find

$$(\nabla_X R)(Y, Z, U, V) = (\nabla_X R)(\overline{Y}, \overline{Z}, U, V) \tag{2.1}$$

and

$$(\nabla_X R)(Y, Z, U, V) = (\nabla_X R)(Y, Z, \overline{U}, \overline{V}). \tag{2.2}$$

From (3) and (2.1) we obtain

$$\begin{aligned} \beta(Y)R(X, Z, U, V) &+ \beta(Z)R(Y, X, U, V) = \\ &= \beta(\overline{Y})R(X, \overline{Z}, U, V) + \beta(\overline{Z})R(\overline{Y}, X, U, V). \end{aligned} \tag{2.3}$$

Let $m \in M^n$, and in a neighbourhood N around m, let $e_i \in \mathcal{X}(M^n) : g(e_i, e_j)|_m = \delta_{ij}$, $\nabla e_i|_m = 0$. Letting $Z = U = e_i$ in (2.3) we have

$$\begin{aligned} \beta(Y)S(X, V) &+ g(B, e_i)g(R(Y, X)e_i, V) = \\ &= \beta(\overline{Y})g(R(X, \overline{e}_i)e_i, V) + g(B, \overline{e}_i)g(R(\overline{Y}, X)e_i, V) \end{aligned}$$

or

$$\beta(Y)S(X, V) + g(R(X, Y)V, B) = \beta(\overline{Y})g(R(V, e_i)X, \overline{e}_i) + g(B, \overline{e}_i)g(R(\overline{Y}, X)e_i, V).$$

Putting $V = X = e_j$ in the above equation we obtain

$$\beta(Y)r - S(Y, B) = -\beta(\overline{Y})S(e_i, \overline{e}_i) - g(B, \overline{e}_i)S(\overline{Y}, e_i), \tag{2.4}$$

where r is the scalar curvature of (M^n, g). From (7) it follows that $S(e_i, \overline{e}_i) = 0$. Hence, from (2.4) it follows

$$\beta(Y)r - S(Y, B) = g(\overline{B}, e_i)g(L\overline{Y}, e_i) = g(\overline{B}, L\overline{Y}) = S(\overline{B}, \overline{Y}) = S(B, Y),$$

where L, defined by the relation $S(X, Y) = g(LX, Y)$, is the symmetric endomorphism corresponding to the Ricci tensor S, which implies that

$$\beta(Y)r = 2S(Y, B). \tag{2.5}$$

Similarly, the formulas (3) and (2.2) imply

$$\delta(Y)r = 2S(Y, P), \quad \delta(X) = g(X, P). \tag{2.6}$$

Now from (3) we find

$$\begin{aligned} (\nabla_X S)(Z, V) &= \alpha(X)S(Z, V) + \beta(R(X, Z)V) + \\ &+ \beta(Z)S(X, V) + \delta(V)S(Z, X) + \delta(R(X, V)Z). \end{aligned}$$

Let again $Z = V = e_i$. Then we obtain

$$X(r)=\alpha(X)r+2S(X,B)+2(X,P). \tag{2.7}$$

So, by (2.5) and (2.6)

$$X(r)=[\alpha(X)+\beta(X)+\delta(X)]r. \tag{2.8}$$

(3) can be written as

$$\begin{aligned}(\nabla_X R)(Y,Z)V&=\alpha(X)R(Y,Z)V+\beta(Y)R(X,Z)V+\\&+\beta(Z)R(Y,X)V+\delta(V)R(Y,Z)X+g(R(Y,Z)V,X)P,\end{aligned} \tag{2.9}$$

where $g(X,P)=\delta(X)$, $\forall X$. Contracting, from (2.9) we derive

$$\begin{aligned}(divR)(Y,Z)V &= \alpha(R(Y,Z)V)+\beta(Y)S(Z,V)-\\&- \beta(Z)S(Y,V)+R(Y,Z,V,P).\end{aligned} \tag{2.10}$$

From the second Bianchi identity it follows that

$$(divR)(Y,Z)V=(\nabla_Y S)(Z,V)-(\nabla_Z S)(Y,V) \tag{2.11}$$

and

$$(divL)(Y)=\frac{1}{2}Y(r), \tag{2.12}$$

where $g(LX,Y)=S(X,Y)$. From (2.10) and (2.11) we deduce

$$\begin{aligned}(\nabla_Y S)(Z,V) &- (\nabla_Z S)(Y,V)=\alpha(R(Y,Z)V)+\beta(Y)S(Z,V)-\\&- \beta(Z)S(Y,V)+R(Y,Z,V,P).\end{aligned}$$

Letting $Y=V=e_i$ in the last equation, we obtain

$$(divL)(Z)-Z(r)=-S(Z,A)+S(Z,B)-B(Z)Y-S(Z,P). \tag{2.13}$$

Using (2.5), (2.6) and (2.12) in (2.13) we get

$$Z(r)=2S(Z,A)+2S(X,B)+2S(X,P). \tag{2.14}$$

From (2.7) and (2.14) it follows that

$$2S(Z,A)=\alpha(Z)r=g(Z,A)r, \tag{2.15}$$

$$\text{i.e.,} S(Z,A)=\frac{r}{2}g(Z,A),\ \forall Z, \tag{2.16}$$

which implies that A is an eigenvector of S corresponding to the eigenvalue $r/2$.

Letting $A=\overline{A}$ in (2.16) we obtain

$$S(Z,\overline{A})=\frac{r}{2}g(Z,\overline{A})$$

which implies that JA is also an eigenvector of S with the same eigenvalue $r/2$.

Similarly from (2.5) and (2.6) we find that B, JB, P and JP are eigenvectors of S corresponding to the same eigenvalue $r/2$.

Summing up, we can state the following theorem:

Theorem 1. *In a weakly symmetric Kaehler manifold,*

(a) *If the scalar curvature is a non-zero constant, then the sum of the associated 1-forms is zero.*

(b) *A, JA, B, JB, P and JP are the eigenvectors of the Ricci tensor S with the same eigenvalue $r/2$.*

Next we prove the following:

Theorem 2. *Let M be a weakly symmetric Kaehler manifold of dimension $n = 6$ and let A, JA, B, JP, P and JP be linearly independent. Then the manifold is Ricci flat.*

Proof.

$$Y = aA + a^*JA + bB + b^*JB + cP + c^*JP.$$

Now with appropriate scalars a, a^*, b, b^*, c, c^*

$$\begin{aligned} S(X,Y) &= g(X, L(aA + a^*JA + bB + b^*JB + cP + c^*JP) = \\ &= g\left(X, \frac{r}{2}(Aa + a^*JA + Bb + bJB + cP + c^*JP)\right) = \\ &\qquad \text{(by (2.15), (2.5) and (2.6))} \\ &= g\left(X, \frac{r}{2}Y\right) = \frac{r}{2}g(X,Y). \end{aligned}$$

So

$$S(X,Y) = \frac{r}{2}g(X,Y).$$

Letting $X = Y = e_i$ in the above equation, we get $r = 0$. Hence $S(X,Y) = 0$. This completes the proof.

3 Weakly Ricci symmetric Kaehler manifolds

In this section we suppose that the Kaehler manifold is a $(WRS)_n$. Then (2) holds. That is,

$$(\nabla_X S)(Y,Z) = \alpha(X)S(Y,Z) + \beta(Y)S(X,Z) + \gamma(Z)S(Y,X). \tag{3.1}$$

From (4) and (6) it follows that

$$(\nabla_X S)(\overline{Y},\overline{Z}) = (\nabla_X S)(Y,Z). \tag{3.2}$$

Letting $Y = \overline{Y}$ and $Z = \overline{Z}$ in (3.1) and using (3.2) and (6) we find

$$\beta(Y)S(X,Z) + \gamma(Z)S(Y,X) = \beta(\overline{Y})S(X,\overline{Z}) + \delta(\overline{Z})S(\overline{Y},X) \tag{3.3}$$

Letting $X = Z = e_i$ in (3.3) gives

$$\beta(Y)r + \gamma(LY) = \beta(\overline{Y})S(e_i,\overline{e}_i) + \gamma(\overline{e}_i)S(\overline{Y},e_i) = -\delta(LY),$$

$$\text{since } S(e_i,\overline{e}_i) = 0.$$

Hence

$$\beta(Y)r + 2\gamma(LY) = 0, \qquad S(X,Y) = g(LX,Y). \tag{3.4}$$

Again putting $X = Y = e_i$ in (3.3) and proceeding in the same way as above, we get

$$\gamma(Y)r + 2\beta(LY) = 0 \tag{3.5}$$

From (3.1) we obtain

$$(\nabla_X S)(Y,Z) - (\nabla_X S)(Z,Y) = [\beta(Y) - \gamma(Y)]S(X,Z) + [\gamma(Z) - \beta(Z)]S(X,Y),$$

which implies

$$[\beta(Y) - \gamma(Y)]S(X,Z) + [\gamma(Z) - \beta(Z)]S(X,Y) = 0. \tag{3.6}$$

Letting $X = Z = e_i$ in the above equation, it follows

$$[\beta(Y) - \gamma(Y)]r + [\gamma - \beta](LY) = 0. \tag{3.7}$$

Using (3.4) and (3.5) in (3.7) we have

$$(\beta - \gamma)r = 0.$$

Hence we can state the following

Theorem 3.*In a weakly Ricci symmetric Kaehler manifold with non-zero scalar curvature the 1-forms β and γ are equal.*

Putting $Y = Z = e_i$, the relation (3.1) gives

$$X(r) = \alpha(X)r + \beta(LX) + \gamma(LX).$$

Using (3.4) and (3.5) in the above equation we can write

$$X(r) = \alpha(X)r - \frac{r}{2}(\beta(X) + \gamma(X)) \tag{3.8}$$

From (3.8) and Theorem 3 we find

$$X(r) = [\alpha(X) - \beta(X)]r.$$

Hence we get the following

Theorem 4. *In a weakly Ricci symmetric Kaehler manifold with non-zero constant scalar curvature, the 1-forms of $(WRS)_n$ are all equal.*

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