A Weierstrass - Type Representation for Harmonic Maps from Riemann Surfaces to General Lie Groups

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Dedicated to Prof.Dr. Constantin UDRIŞTE on the occasion of his sixtieth birthday

Abstract

The first part of the paper describes the harmonicity equations for harmonic maps from Riemann surfaces to Lie groups which carry a left-invariant pseudo-Riemannian structure; §3 includes basic facts about loop groups and their factorizations; §4 presents the formalism of [10] and an extension of Wu's formula to the case of generalized harmonic maps into arbitrary Lie groups. Finally, §5 includes three examples which outline the developed theory.

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1 Introduction

This paper continues the study of [2]. We consider an arbitrary real Lie group G admitting a faithful finite-dimensional representation, and discuss harmonic maps

$$\varphi : \ M \to G,$$

from connected, simply-connected Riemann surfaces to G. This generalizes to some extent the work of K. Uhlenbeck [29] and provides Weierstrass data for all such harmonic maps in the spirit of [10]. The first part of the paper describes the harmonicity equations for harmonic maps from arbitrary Riemann surfaces to G, where G carries a left-invariant pseudo-Riemannian structure. If we specialize, as in the classical case

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[29], to bi-invariant pseudo-metrics, then the equations specialize to the Uhlenbeck equations [29]. It is remarkable that these equations do not depend on the bi-invariant structure chosen. In section 3 we recall the basic facts about loop groups and their factorizations, and specialize to the based loop group. In section 4 we recall the formalism of [10] and present an extension of Wu's formula to the case of generalized harmonic maps into arbitrary Lie groups. It is here where [2] enters in an essential way. The final section illustrates with selected examples what types of "potentials" one obtains and what type of solutions they provide to the equations listed in section 2. We illustrate in section 5 the equations obtained by three typical examples.

2 The harmonicity equations

2.1 Let hereafter G be a real Lie group admitting a faithful finite-dimensional representation, and $g =< .,. >$ a left-invariant pseudo-Riemannian metric on G. Moreover, let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ or all of $\mathbb{C}$ and $\varphi : \mathbb{D} \to G$ an immersion such that the pull-back metric $\varphi^* g$ on $\mathbb{D}$ is of the form

$$\varphi^* g = (\gamma_{ij}) = \lambda(x,y)(\delta_{ij}), \tag{2.1.1}$$

where $\lambda(x,y) > 0$, for all $(x,y) \in \mathbb{D}$. Then *the energy* of φ [12, 13, 30]

$$E(\varphi) = \int_{\mathbb{D}} \epsilon(\varphi)\, \mathrm{d}x\, \mathrm{d}y, \tag{2.1.2}$$

is obtained by integrating *the energy-density*

$$\epsilon(\varphi) = \frac{1}{2} \mid \mathrm{d}\varphi \mid^2 = \frac{1}{2} h_{ij} \frac{\mathrm{d}\varphi^i}{\partial x^a} \frac{\partial \varphi^j}{\partial x^b} \gamma^{ab},\ (x,y) \equiv (x^1, x^2) \in \mathbb{D}, \tag{2.1.3}$$

of φ, where (γ^{ab}) is the inverse of γ and the Einstein summation convention is used. Note that the integral (2.1.2) is considered over every open bounded set with compact closure in $\mathbb{D}$ and needs to be minimized for every variation with support in the open set $\mathbb{D}$ [12], [13]. It is known that $E(\varphi)$ is conformally invariant (e.g., [30, 10.2, p. 42], [20, Lemma 1.32, p. 20]); moreover, [29, 3.3, p. 216] one can write the energy integral as

$$E(\varphi) = \frac{1}{2} \int_{\mathbb{D}} (\mid A_{(x)} \mid^2 + \mid A_{(y)} \mid^2)\, \mathrm{d}x\, \mathrm{d}y, \tag{2.1.4}$$

where

$$A_{(x)} = \varphi^{-1}\varphi_x,\ A_{(y)} = \varphi^{-1}\varphi_y \tag{2.1.5}$$

are $\mathbf{g}$-valued 1-forms on $\mathbb{D}$. Here we use $\mathbf{g} = Lie(G)$. Also note that subscripts x, y, xx, etc denote the partial differentiation with respect to the corresponding variable(s). The subscript (x) or (y), on the other hand, has no independent meaning. The map

φ is called *harmonic* if φ is a critical point of $E(\varphi)$, i.e., if for any *smooth variation* [30, 1.2, p124] with compact support $\varphi_t \in \mathcal{C}^\infty(\mathbb{D}, G),\ t \in (-\varepsilon, \varepsilon),\ \varphi_0 = \varphi$, we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} E(\varphi_t) = 0. \tag{2.1.6}$$

The harmonicity (Euler-Lagrange) equations for φ, are provided by

Theorem. *Let G be a real Lie group admitting a left-invariant pseudo-Riemannian metric g. Then the map $\varphi : \mathbb{D} \to G$ is harmonic iff the associated $\boldsymbol{g}$-valued 1-forms $A_{(x)}$ and $A_{(y)}$ of (2.1.5) satisfy the equation*

$$(adA_{(x)})^* A_{(x)} + (adA_{(y)})^* A_{(y)} - (\partial_x A_{(x)} + \partial_y A_{(y)}) = 0, \tag{2.1.7}$$

where the star superscript indicates the adjoint w.r.t. the nondegenerate bilinear form $\langle\ ,\ \rangle = g_e$ induced by g on the Lie algebra $\boldsymbol{g}$.

Proof. Let $\tilde{\varphi}$ be a smooth variation of φ, and $\eta = \frac{\mathrm{d}}{\mathrm{dt}}\big|_{t=0} \varphi^{-1}\tilde{\varphi}$ the variation field; assume that $\eta : \mathbb{D} \to \mathbf{g}$ has compact support contained in $\mathbb{D}$. Remark also that we have the relation

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} \varphi^{-1}\frac{\partial\tilde{\varphi}}{\partial x} = \eta_x + [A_{(x)}, \eta]. \tag{2.1.8}$$

By direct computation, using (2.1.8) and applying the Stokes formula, the derivative inside the harmonicity condition 2.1.6 for φ writes succesively

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} E(\tilde{\varphi}) &= \int_{\mathbb{D}} \{\langle \frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} \varphi^{-1}\frac{\partial\tilde{\varphi}}{\partial x}, A_{(x)}\rangle + \langle \frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} \varphi^{-1}\frac{\partial\tilde{\varphi}}{\partial y}, A_{(y)}\rangle\}\mathrm{d}x\mathrm{d}y = \\
(2.1.9) \qquad &= \int_{\mathbb{D}} \{\langle \eta_x, A_{(x)}\rangle + \langle [A_{(x)}, \eta], A_{(x)}\rangle + \langle \eta_y, A_{(y)}\rangle + \langle [A_{(y)}, \eta], A_{(y)}\rangle\}\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{D}} \{-\langle \eta, \partial_x A_{(x)}\rangle + \langle \mathrm{ad}A_{(x)}(\eta), A_{(x)}\rangle - \\
&\qquad -\langle \eta, \partial_y A_{(y)}\rangle + \langle \mathrm{ad}A_{(y)}(\eta), A_{(y)}\rangle\}\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{D}} \{\langle \eta, (\mathrm{ad}A_{(x)})^* A_{(x)} - \partial_x A_{(x)}\rangle + \langle \eta, (\mathrm{ad}A_{(y)})^* A_{(y)} - \partial_y A_{(y)}\rangle\}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Since the equation (2.1.9) holds true for any variation field η with compact support, the claim follows. □

Remark. The proof above is provided for completeness. It is similar to the one in [23] and for the positive definite case it appears also in [7, Corollary 2.4, p. 145].

2.2 As consequence, we have the following result of Urakawa [30, Chap.6, Sec.3]

Corollary. *If the pseudo-metric g is bi-invariant, then the harmonicity condition becomes:*

(2.2.1) $$\partial_x A_{(x)} + \partial_y A_{(y)} = 0,$$

with $A_{(x)},\ A_{(y)}$ *defined in (2.1.5).*

Proof. At the Lie algebra level, *ad* is antisymmetric with respect to the bilinear non-degenerate form $\langle\ \cdot\ ,\ \cdot\ \rangle = g_e$ associated to g [16, p. 125], hence we have for every $\eta \in \mathbf{g}$

(2.2.2) $$\begin{aligned}\langle(\mathrm{ad}A_{(x)})^*\mathrm{ad}A_{(x)}, \eta\rangle &= \langle \mathrm{ad}A_{(x)}(\eta), A_{(x)}\rangle = \\ &= -\langle \eta, \mathrm{ad}A_{(x)}(A_{(x)})\rangle = -\langle \eta, [A_{(x)}, A_{(x)}]\rangle = 0.\end{aligned}$$

Therefore $(\mathrm{ad}A_{(x)})^*\mathrm{ad}A_{(x)} = 0$. Similarly one establishes $(\mathrm{ad}A_{(y)})^*\mathrm{ad}A_{(y)} = 0$. □

Remarks. 1. For complex coordinates $z = x + iy, \bar{z} = x - iy$ on $\mathbb{D}$, the equations (2.2.1) rewrite

(2.2.3) $$\partial_{\bar{z}} A_{(z)} + \partial_z A_{(\bar{z})} = 0,$$

where $\partial_z = (\partial_x - i\partial_y)/2,\ \partial_{\bar{z}} = (\partial_x + i\partial_y)/2$, and $A_{(z)} = \varphi^{-1}\varphi_z,\ A_{(\bar{z})} = \varphi^{-1}\varphi_{\bar{z}}$.
2. Since (2.2.1) does not explicitely depend on the bi-invariant metric chosen, we can consider (2.2.1) *also for Lie groups not carrying a bi-invariant metric. In this case the equation does not correspond to a known variational problem* associated with a functional of type (2.1.2). It would be interesting to find geometric interpretations [30, Chapter 6, Section 3] for (2.2.1) also in this case.

3 Loop groups and factorization theorems

3.1 Let G be a connected Lie group, (or, as well, the connected component of a given Lie group), admitting a faithful finite dimensional representation. We assume that G and its Lie algebra $\mathbf{g}$ are realized by $N \times N$ matrices. As in [2] we consider various loop groups associated with G. To this end, let

(3.1.1) $$\mathcal{A} = \mathcal{A}_r = \{\ f(\lambda) = \sum_{n\in\mathbb{Z}} \lambda^n f_n \mid f_n \in \mathbb{C},\ \sum_{n\in\mathbb{Z}} w_r(n)\cdot|f_n| < \infty\},$$

where

(3.1.2) $$w_r(n) = (1 + |n|)^r,\ n \in \mathbb{Z},$$

with $r \in \mathbb{N}\backslash\{0\}$ fixed. Then $\mathcal{A}$ is organized as a complex Banach algebra with the norm

(3.1.3) $$||f||_r = ||f|| = \sum_{n\in\mathbb{Z}} w_r(n)|f_n|.$$

On the set of mappings

$$\Lambda Mat(N,\mathcal{A}) = \{A : S^1 \to \mathcal{M}_{n\times n}(\mathbb{C}) \mid (A(\lambda))_{ij} \in \mathcal{A} \text{ for all } i,j = 1,\ldots,N\},$$

we introduce the norm

$$|||A|||_r = \max_{j\in\{1,\ldots,N\}} \left\{ \sum_{i=1}^{N} ||A(\lambda)_{ij}||_r \right\}. \tag{3.1.4}$$

Then $Mat(N,\mathcal{A})$ with the norm (3.1.4) becomes an associative Banach algebra. The complexification $\mathbf{g}^{\mathbb{C}} = \mathbf{g} + i\mathbf{g}$ of the real Lie algebra $\mathbf{g}$ of G can also be regarded as a Lie algebra of complex matrices. Then

$$\Lambda\mathbf{g}^{\mathbb{C}} = \mathbf{g}^{\mathbb{C}} \otimes \mathcal{A} = \{f : S^1 \to \mathbf{g}^{\mathbb{C}} \mid (f(\lambda))_{ij} \in \mathcal{A}, \text{ for all } i,j \in \{1,\ldots,n\}\}, \tag{3.1.5}$$

together with the norm (3.1.4) is a Banach Lie algebra, and the group

$$\Lambda G^{\mathbb{C}} = \{f : S^1 \to G^{\mathbb{C}} \mid (f(\lambda))_{ij} \in \mathcal{A}, \text{ for all } i,j \in \{1,\ldots,n\}\}, \tag{3.1.6}$$

becomes a Banach Lie group with the Lie algebra $\Lambda\mathbf{g}^{\mathbb{C}}$. Moreover, the group

$$\Lambda G = \{f \in \Lambda G^{\mathbb{C}} \mid f(\lambda) \in G, \text{ for all } \lambda \in S^1\} \tag{3.1.7}$$

is a real Banach subgroup of $\Lambda G^{\mathbb{C}}$, and has the Lie algebra

$$\Lambda\mathbf{g} = \{f \in \Lambda\mathbf{g}^{\mathbb{C}} \mid f(\lambda) \in \mathbf{g}, \text{ for all } \lambda \in S^1\}. \tag{3.1.8}$$

In the following we shall consider *based loops*, i.e., the subset

$$\Lambda_1 G^{\mathbb{C}} = \{f \in \Lambda G^{\mathbb{C}} \mid f(1) = I\} \subset \Lambda G^{\mathbb{C}}, \tag{3.1.9}$$

where we denoted by I the identity matrix of order $N \times N$. Similarly, one can define $\Lambda_1 G$. Then the corresponding Lie algebras $\Lambda_1\mathbf{g}^{\mathbb{C}}$ and $\Lambda_1\mathbf{g}$ are defined by the condition $f(1) = 0$. The elements of $\Lambda_1\mathbf{g}^{\mathbb{C}}$ have expansions of the form

$$f(\lambda) = \sum_{n\in\mathbb{Z}\setminus\{0\}} (\lambda^n - 1) f_n, \quad f_n \in \mathbf{g}^{\mathbb{C}}. \tag{3.1.10}$$

and each loop $f \in \Lambda_1 G^{\mathbb{C}}$ has an expansion of the form

$$f(\lambda) = I + \sum_{n\in\mathbb{Z}\setminus\{0\}} (\lambda^n - 1) f_n, \quad f_n \in \mathcal{M}_{N\times N}(\mathbb{C}). \tag{3.1.11}$$

3.2 Loop group splittings are essential tools in the DPW method [10]. Such splittings were used in the study of harmonic maps by Uhlenbeck [29], later by Pressley, Segal [27], and recently by Guest and Ohnita [14]. Also, in a slightly different differential geometric context, such splittings have been used in [11] and [15].

The splittings involve at the Lie algebra level the subalgebras $\Lambda_1\mathbf{g}$, $\Lambda_1^+\mathbf{g}^{\mathbb{C}}$ and $\Lambda_1^-\mathbf{g}^{\mathbb{C}}$, where

(3.2.1) $$\Lambda_1^+\mathbf{g}^{\mathbb{C}} = \{f \in \Lambda_1\mathbf{g}^{\mathbb{C}} \mid f_n = 0, \text{ for } n < 0\}.$$

In particular, every element $f \in \Lambda_1^+\mathbf{g}^{\mathbb{C}}$ is of the form

(3.2.2) $$f(\lambda) = \sum_{n=1}^{\infty}(\lambda^n - 1)f_n,$$

and similarly one can describe the elements of $\Lambda_1^-\mathbf{g}^{\mathbb{C}}$. The following result is straightforward

Lemma. *The three algebras above have pairwise trivial intersections and*

(3.2.3) $$\Lambda_1\mathbf{g}^{\mathbb{C}} = \Lambda_1\mathbf{g} + \Lambda_1^-\mathbf{g}^{\mathbb{C}} = \Lambda_1^-\mathbf{g}^{\mathbb{C}} + \Lambda_1^+\mathbf{g}^{\mathbb{C}}.$$

We define the following subgroups of $\Lambda G^{\mathbb{C}}$:

$\Lambda^+ G^{\mathbb{C}} = \{g \in \Lambda G^{\mathbb{C}} \mid g$ and g^{-1} extend holomorphically to $\mathbb{D}\}$,
$\Lambda^- G^{\mathbb{C}} = \{g \in \Lambda G^{\mathbb{C}} \mid g$ and g^{-1} extend holomorphically to $\overline{\mathbb{C}} \setminus \mathbb{D}\}$,
where $\mathbb{D} = \{z \in \mathbb{C} \mid \mid z \mid < 1\}$ and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere. Then the Lie subalgebras $\Lambda_1^+\mathbf{g}^{\mathbb{C}}$ and $\Lambda_1^-\mathbf{g}^{\mathbb{C}}$ above determine respectively the based subgroups

(3.2.4) $$\Lambda_1^+ G^{\mathbb{C}} = \Lambda^+ G^{\mathbb{C}} \cap \Lambda_1 G^{\mathbb{C}}, \qquad \Lambda_1^- G^{\mathbb{C}} = \Lambda^- G^{\mathbb{C}} \cap \Lambda_1 G^{\mathbb{C}}.$$

Proposition. *The groups $\Lambda_1^+ G^{\mathbb{C}}$ and $\Lambda_1^- G^{\mathbb{C}}$ are connected Banach subgroups of $\Lambda_1 G^{\mathbb{C}}$.*

Proof. For $g \in \Lambda_1^+ G^{\mathbb{C}}$ and $r \in [0,1]$, we have that also $g_r(\lambda) = g(r)^{-1} \cdot g(r\lambda)$ is in $\Lambda_1^+ G^{\mathbb{C}}$ and the mapping $r \to g_r$ is continuous.

□

Similarly one verifies that $\Lambda_1^- G^{\mathbb{C}}$ is connected. From [2, Theorem 4.5] we recall

Theorem. **(Birkhoff Factorization Theorem).**

Any element $g \in \Lambda G^{\mathbb{C}}$ can be written as

(3.2.5) $$g = g_- D g_+$$

where $g_\pm \in \Lambda^\pm G^{\mathbb{C}}$, $D = sb_-^+ \in \Lambda^d G^{\mathbb{C}}$, and

(3.2.6) $$\Lambda^d G^{\mathbb{C}} = \bigcup_{s \in \Lambda^d H^{\mathbb{C}}} s(\Lambda^- B^{\mathbb{C}})_s^+$$

and

(3.2.7) $$(\Lambda^- B^{\mathbb{C}})_s^- = \{b \in \Lambda^- B^{\mathbb{C}} \mid sbs^{-1} \in \Lambda^- B^{\mathbb{C}}\}$$

(3.2.8) $$(\Lambda^- B^{\mathbb{C}})_s^+ = \{b \in \Lambda^- B^{\mathbb{C}} \mid sbs^{-1} \in \Lambda_*^+ B^{\mathbb{C}}\}.$$

The expression (3.2.5) will be called a Birkhoff decomposition*, of* g.

Corollary. *If* $g \in \Lambda_1 G^{\mathbb{C}}$ *is contained in the big cell* $\Lambda^- G^{\mathbb{C}} \cdot \Lambda^+ G^{\mathbb{C}}$ *of* $\Lambda G^{\mathbb{C}}$*, then* $g = g_-\, g_+$ *with uniquely determined* $g_- \in \Lambda_1^- G^{\mathbb{C}}$ *and* $g_+ \in \Lambda_1^+ G^{\mathbb{C}}$.

Moreover, the big cell $\Lambda_1^- G^{\mathbb{C}} \cdot \Lambda_1^+ G^{\mathbb{C}}$ *is open and dense in* $\Lambda_1^+ G^{\mathbb{C}}$ *and the map*

$$\Lambda_1^- G^{\mathbb{C}} \times \Lambda_1^+ G^{\mathbb{C}} \to \Lambda_1^- G^{\mathbb{C}} \cdot \Lambda_1^+ G^{\mathbb{C}}, \qquad (g_-, g_+) \to g_- g_+$$

is an analytic diffeomorphism.

Proof. By assumption $g = \tilde{g}_- \tilde{g}_+$ with $\tilde{g}_- \in \Lambda^- G^{\mathbb{C}}$ and $\tilde{g}_+ \in \Lambda^+ G^{\mathbb{C}}$. Then $g_- = \tilde{g}_- \cdot \tilde{g}_-(\lambda = 1)^{-1} \in \Lambda_1^- G^{\mathbb{C}}$, $g_+ = \tilde{g}_-(\lambda = 1)\tilde{g}_+ \in \Lambda^+ G^{\mathbb{C}}$ and $g = g_+ g_-$. Since $g \in \Lambda_1 G^{\mathbb{C}}$, also $g_+ \in \Lambda_1^+ G^{\mathbb{C}}$. Assume now $g_- g_+ = \tilde{g}_- \tilde{g}_+$ with $g_-, \tilde{g}_- \in \Lambda_1^- G^{\mathbb{C}}$ and $g_+, \tilde{g}_+ \in \Lambda_1^+ G^{\mathbb{C}}$. Then $g_-^{-1} \tilde{g}_- = g_+ (\tilde{g}_+)^{-1} = A$ is independent of λ. But evaluating the left side at $\lambda = 1$ yields I, whence $A = I$ and the decomposition is unique.

To show the last statements it suffices to prove that the big cell is dense in $\Lambda_1 G^{\mathbb{C}}$. Let $g \in \Lambda_1 G^{\mathbb{C}}$. Then, since the big cell of $\Lambda G^{\mathbb{C}}$ is dense, for every $\varepsilon > 0$ there exists some $\tilde{g} \in \Lambda G^{\mathbb{C}}$ such that $||g - \tilde{g}|| < \varepsilon$. Let $h = \tilde{g}_0^{-1} \tilde{g}$, where $\tilde{g}_0 = g_{(\lambda=1)}$. Then h is still in the big cell of $\Lambda G^{\mathbb{C}}$ and we have

$$||g - h|| \leq ||g - \tilde{g}|| + ||\tilde{g} - \tilde{g}_0^{-1} \tilde{g}|| < \varepsilon + ||\tilde{g}|| \cdot ||I - \tilde{g}_0^{-1}||.$$

Since evaluation at $\lambda = 1$ is a continuous map,

$$(3.2.9) \qquad || \sum_{i,j \in \{1,\dots,N\}} (g_{ij} - \tilde{g}_{ij}) || \leq \sum_{i,j \in \{1,\dots,N\}} ||g_{ij} - \tilde{g}_{ij}|| = ||g - \tilde{g}|| < \varepsilon,$$

we conclude that we can choose h arbitrarily close to g. □

3.3 The second splitting we are using in this paper is a generalized Iwasawa splitting. From [2, Theorem 6.5] we recall

Theorem. (Iwasawa Factorization Theorem).

Let G *be a connected real Lie group, which admits a finite-dimensional faithful representation. Then*

$$(3.3.1) \qquad \Lambda G^{\mathbb{C}} = \Lambda G \cdot \Lambda^m G^{\mathbb{C}} \cdot \Lambda^+ G^{\mathbb{C}},$$

is a disjoint union of double cosets indexed by the middle terms

$$(3.3.2) \qquad \Lambda^m G^{\mathbb{C}} = \underset{s \in \Lambda^m H^{\mathbb{C}}}{\cup} (\Lambda B)_s^{\#} \cdot s,$$

where $(\Lambda B)_s^{\#} = s.\Lambda B.s^{-1}$.

More precisely, every $g \in \Lambda G^{\mathbb{C}}$ *has a unique representation of the form*

$$g = hb.wsb_+h_+ \tag{3.3.3}$$

where $g = \tilde{h}\tilde{b}$ *and* $\tilde{h} = hsh_+$ *is the unique representation of* $\tilde{h} \in \Lambda H^{\mathbb{C}} = \Lambda H \cdot \Lambda^d H^{\mathbb{C}} \cdot \Lambda^+ H^{\mathbb{C}}$ *as described in the Appendix of [2].*

Corollary. *If* $g \in \Lambda_1 G^{\mathbb{C}}$ *is contained in* $\Lambda G \cdot \Lambda^+ G^{\mathbb{C}}$*, then*

$$g = hv_+, \tag{3.3.4}$$

with uniquely determined $h \in \Lambda_1 G$ *and* $v_+ \in \Lambda_1 G^{\mathbb{C}}$*.*

Moreover, if the big cell $\Lambda G \cdot \Lambda^+ G^{\mathbb{C}}$ *is open and dense in* $\Lambda G^{\mathbb{C}}$*, then the big cell* $\Lambda_1 G \cdot \Lambda_1^+ G^{\mathbb{C}}$ *is open and dense in* $\Lambda_1 G^{\mathbb{C}}$ *and the map*

$$\Lambda_1 G \times \Lambda_1^+ G^{\mathbb{C}} \to \Lambda_1 G \cdot \Lambda_1^+ G^{\mathbb{C}}, \qquad (h, v_+) \to hv_+, \tag{3.3.5}$$

is an analytic diffeomorphism.

Proof. The proof is almost verbatim identical with the proof of Corollary 3.2. □

Remark. We would like to point out that one can show that $\Lambda G \times \Lambda^+ G^{\mathbb{C}}$ is dense in $\Lambda G^{\mathbb{C}}$ if the semisimple part of a maximal compact subgroup of $G^{\mathbb{C}}$ is simply connected [2, Theorem 7.2]. This applies when the semisimple part of $\Lambda G^{\mathbb{C}}$ is simply conneceted. Moreover, $\Lambda G.\Lambda^+ G^{\mathbb{C}} = \Lambda G^{\mathbb{C}}$ iff the reductive part of G (i.e., the group H in [2, Section 2]) is compact.

3.4 Using [27, 8.6,8.7], we characterize the group elements *not* in the big cell by the vanishing of some function. This will be of importance in Theorem 4.7.

We can also prove a different splitting by [27, 8.6,8.7], and give a more direct proof using [8] to a more particular result.

We consider the "lexicographic" isomorphism [27, 6.6], [8, 3.1] of $\mathcal{L}^n$ with $\mathcal{L} = L^2(S^1)$. This induces an injection of $\Lambda G^{\mathbb{C}}$ into $GL_{\text{res}}(\mathcal{L})$. We first split $\mathcal{L} = \mathcal{L}_- \oplus \mathcal{L}_+$, where $\mathcal{L}_-$ is spanned by the functions λ^n, $n \leq 0$ and $\mathcal{L}_+$ by the λ^n, $n > 0$. This way the bounded operators of $\mathcal{L}$ decompose naturally into 2×2 blocks. An algebra $\mathcal{B}$ of such blocks has been presented for a large class of weights in [8, 1.10]. In our case, we use the weight $w_{r-1} = (1 + |n|)^{r-1}$, for $\mathbf{g}^{\mathbb{C}}$ defined by the weight w_r. From [8, Proposition 3.4] we know that the embedding of $\mathbf{g}^{\mathbb{C}}$ into the bounded operators $\mathcal{L}$ is contained in $\mathcal{B}$. Since the inclusion of $\mathcal{B}$ into the bounded operators with off-diagonal Hilbert-Schmidt operators is bounded, it follows that the map $\Lambda^0 G^{\mathbb{C}} \to GL_{\text{res}}(\mathcal{L})$ is holomorphic, where $\Lambda^0 G^{\mathbb{C}}$ denotes the connected component of $\Lambda G^{\mathbb{C}}$. Since $GL_{\text{res}}(\mathcal{L})$ acts holomorphically on the Grassmannian $Gr(\mathcal{L})$ as defined in [27, Chapter 7, Section 7], the map

$$g \in \Lambda^0 G^{\mathbb{C}} \to i(g).\mathcal{L}_- \in Gr(\mathcal{L}) \tag{3.4.1}$$

is holomorphic. Finally, since the τ-function defined in [28, Chapter 3, Paragraph 3], [27, Chapter 8, Paragraph 10] is holomorphic on $Gr(\mathcal{L})$, we obtain altogether that the map

(3.4.2) $$\varphi : \Lambda^0 G^{\mathbb{C}} \to \mathbb{C}, \;\; \varphi(g) = \tau(i(g).\mathcal{L}_-), \;\; \text{for all } g \in \Lambda^0 G^{\mathbb{C}}$$

is holomorphic. Hence we have

Theorem.

(3.4.3) $$\varphi(g) \neq 0 \Leftrightarrow g \in (\Lambda_1^- G^{\mathbb{C}}).(\Lambda_1^+ G^{\mathbb{C}}).$$

Proof. Since $G^{\mathbb{C}} \subset SL(n, \mathbb{C})$, one obtains the Birkhoff decomposition $g = a_- d a_+$ for all $g \in \Lambda^0 G^{\mathbb{C}}$. Then

(3.4.4) $$\varphi(g) = \tau\left(i(a_-)i(d)i(a_+)\right).\mathcal{L}_- = \tau\left(i(a_-)i(d).\mathcal{L}_-\right).$$

Hence $\varphi(g) \neq 0$ iff $\varphi(d) \neq 0$ and it follows directly that $\varphi(d) \neq 0$ iff $d = e$. □

4 Generalized Weierstrass representation

4.1 For a real Lie group G we can characterize the harmonicity of the map $\varphi : \mathbb{D} \to G$, in terms of its associated Maurer-Cartan form

$$\alpha \equiv \varphi^{-1} \mathrm{d}\varphi : \mathbb{D} \to \mathbf{g} \equiv Lie(G).$$

We assume in the following that G is endowed with a bi-invariant pseudo-Riemannian metric. First we call [29, p. 5]:

Proposition. *The following statements are equivalent:*
a) The map φ is harmonic.
b) The form α satisfies the integrability and harmonicity equations:

(4.1.1) $$\begin{cases} d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0 \\ \partial_{\bar{z}}\alpha' + \partial_z \alpha'' = 0, \end{cases}$$

where α' and α'' are the holomorphic and the antiholomorphic parts of the differential form α, respectively.

K. Uhlenbeck's proof of the Proposition above carries over without change to the present pseudo-metric situation.

4.2 Following a successful procedure in soliton theory and related geometric theories [29], [10, p. 647], [11, 2.5, p. 50], we introduce a parameter λ into our theory. We will use $\mathrm{d} = \partial_z + \partial_{\bar{z}}$ and split forms as

$$\alpha = \alpha' + \alpha'' \equiv A_{(z)}\mathrm{d}z + A_{(\bar z)}\mathrm{d}\bar z : T\mathbb{D}^{1,0} \oplus T\mathbb{D}^{0,1} \to \mathbf{g}^{\mathbb{C}},\ \text{ for all } \lambda \in S^1.$$

Then we introduce the $\mathbf{g}^{\mathbb{C}}$-valued "loopified form" α_λ defined on $\mathbb{D}$,

$$\alpha_\lambda = \frac{1-\lambda^{-1}}{2}\alpha' + \frac{1-\lambda}{2}\alpha'', \tag{4.2.1}$$

for all $\lambda \in S^1$. Then we see by a straightforward compuatation that the system of equations (4.1.1) can be replaced by one equation:

Proposition. *φ is harmonic iff the loopified form α_λ is integrable for all $\lambda \in S^1$, i.e., it satisfies the integrability condition:*

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0. \tag{4.2.2}$$

We would like to point out that equation (4.2.2) implies that we can find some map ("extended harmonic map")

$$\varphi : (x, y; \lambda) \in \mathbb{D} \times S^1 \to \varphi(x, y, \lambda) \in G,$$

such that

$$\alpha_\lambda = \varphi^{-1}\mathrm{d}\varphi : \mathbb{D} \times S^1 \to \mathbf{g},$$

for any $\lambda \in S^1$. This will be the basis for our group splitting method.

We will call the family of extended harmonic maps definied above the "associated family" for φ. We notice that though the parameter λ can be chosen arbitrary on $\mathbb{C}^*$, in certain cases, the condition $\alpha_\lambda(x, y; \lambda) \in \mathbf{g}$ requires $\lambda \in S^1$.

4.3 We prove the following statement.

Theorem. *Let $u : \mathbb{D} \to \Lambda_1 G$ be an extended harmonic map such that $u(0, 0, 1) = I$. Then there exists a map $v_+ : \mathbb{D} \to \Lambda_1^+ G^{\mathbb{C}}$ such that*

$$g = uv_+ : \mathbb{D} \to \Lambda_1 G^{\mathbb{C}} \tag{4.3.1}$$

is holomorphic and $g(0) \in \Lambda_1^+ G^{\mathbb{C}}$. Also, for certain holomorphic maps $Q, A_n : \mathbb{D} \to \mathbf{g}$ we can write

$$g^{-1}\,\mathrm{d}\,g = \left(\frac{1-\lambda^{-1}}{2}\,Q(z) + \sum_{n=0}^{\infty}(1-\lambda^n)A_n(z)\right)\mathrm{d}\,z. \tag{4.3.2}$$

Proof. Consider

$$u^{-1}\,\mathrm{d}u = \frac{1-\lambda^{-1}}{2}A\;d\,z + \frac{1-\lambda}{2}\bar A\;d\,\bar z. \tag{4.3.3}$$

Then we have

$$(uv_+)^{-1}\,\mathrm{d}(uv_+) = v_+^{-1}\cdot u^{-1}\cdot\ \mathrm{d}u\cdot v_+ + v_+^{-1}\,\mathrm{d}v_+. \tag{4.3.4}$$

Hence the $d\bar{z}$ part of this expression is

$$v_+^{-1}\,\frac{1-\lambda}{2}\,\bar{A}v_+ + v_+^{-1}\partial_{\bar{z}}v_+. \tag{4.3.5}$$

In order to make uv_+ holomorphic we must annihilate this expression with $v_+ \in \Lambda_1^+ G^{\mathbb{C}}$, i.e., we must satisfy the differential equation

$$\partial_{\bar{z}}v_+ = -\frac{1-\lambda}{2}\bar{A}v_+, \tag{4.3.6}$$

with $v_+ \in \Lambda_1^+ G^{\mathbb{C}}$.

Since $\bar{A}$ is real analytic, we can consider z and $\bar{z}$ as independent variables and solve (4.3.6) for $\bar{z} \in \mathbb{D}$. Since the coefficient matrix $-\frac{1-\lambda}{2}\bar{A} \in \Lambda_1^+ G^{\mathbb{C}}$, we can assume that the solution is in $\Lambda_1^+ G^{\mathbb{C}}$. Hence we have shown that exists some $v_+ : \mathbb{D} \to \Lambda_1^+ G^{\mathbb{C}}$, such that $g = uv_+$ is holomorphic. Since $u(0,0,\lambda) = I$, we obtain $g(0,\lambda) = v_+(0,0,\lambda) \in \Lambda_1^+ G^{\mathbb{C}}$. Finally, we have

$$g^{-1}\,\mathrm{d}g = v_+^{-1}\left(\frac{1-\lambda^{-1}}{2}A\,\mathrm{d}z + \frac{1-\lambda}{2}\bar{A}\,\mathrm{d}\bar{z}\right)v_+ + v_+^{-1}\,\mathrm{d}v_+ \tag{4.3.7}$$

which shows that $g^{-1}\,\mathrm{d}g$ has the required form (4.3.2). □

4.4 Consider a holomorphic map $R : \mathbb{D} \to \Lambda_1 G^{\mathbb{C}}$ of the form

$$R(z,\lambda) = \frac{1-\lambda^{-1}}{2}\,Q(z) + \sum_{n=0}^{\infty}(1-\lambda^n)A_n(z), \tag{4.4.1}$$

i.e., Q and all A_n are holomorphic maps from $\mathbb{D}$ into $\mathbf{g}^{\mathbb{C}}$. Next we consider the differential equation

$$g^{-1}\,\mathrm{d}g = R\ \mathrm{d}z, \quad g(0) = I. \tag{4.4.2}$$

This differential equation has a unique solution $g \in \Lambda_1 G^{\mathbb{C}}$ once $g(0)$ is given. We note that actually $g \in \Lambda_1^0 G^{\mathbb{C}}$ holds since $\Lambda_1^+ G^{\mathbb{C}}$ is connected. Since $g \in \Lambda_1 G^{\mathbb{C}}$ and $g(0,\lambda) = I$, we can apply Corollary 3.3 and we split

$$g(z,\lambda) = u(z,\bar{z},\lambda)v_+(z,\bar{z},\lambda), \tag{4.4.3}$$

where $u \in \Lambda_1 G$ and $v_+ \in \Lambda_1^+ G^{\mathbb{C}}$.

Theorem. *The mapping u is an extended harmonic map from $\mathbb{D}$ into G and $u(0,0,\lambda) = I$.*

Proof. Differentiating (4.4.3), we obtain

$$R = g^{-1}\,\mathrm{d}g = v_+^{-1}\cdot u^{-1}\cdot\,\mathrm{d}u\cdot v_+ + v_+^{-1}\,\mathrm{d}v_+. \tag{4.4.4}$$

We use (4.4.2) and conjugate by v_+. Then we have

$$v_+ R v_+^{-1}\,\mathrm{d}z = u^{-1}\,\mathrm{d}u + \,\mathrm{d}v_+\cdot v_+^{-1}. \tag{4.4.5}$$

From (4.4.5) we see that $u^{-1}\,\mathrm{d}u$ is of the form

$$u^{-1}\,\mathrm{d}u = \frac{1-\lambda^{-1}}{2}g_1 + \sum_{k=1}^{\infty}(1-\lambda^k)g_k. \tag{4.4.6}$$

But $u^{-1}\,\mathrm{d}u = \overline{u^{-1}\,\mathrm{d}u}$, whence only $1-\lambda^{-1}$ and $1-\lambda$ can occur in $u^{-1}\,\mathrm{d}u$. Proposition 4.2 now shows that (4.1.1) holds.

This shows that u is an extended harmonic map. Finally, since $g(0,0,\lambda)\in\Lambda_1^+G^{\mathbb{C}}$, from (4.4.3) and the uniqueness of the splitting we obtain $u(0,0,\lambda)=I$. □

4.5 Considering the previous results we conclude that the set $\mathcal{H}$ of (extended) harmonic maps $u:\mathbb{D}\to\Lambda_1 G$ satisfying $u(0,0,\lambda)=I$ and the set $\mathcal{R}$ of "potentials" R as in (4.4.1) are closely related. In the following we shall make this relation more precise.

First we note that in (4.4.3) we can obtain a unique solution g associated with R by requiring $g(0,\lambda)=I$. In this case v_+ defined in (4.4.2) then satisfies $v_+(0,0,\lambda)=I$ as well. This way we obtain a map $\psi:\mathcal{R}\to\mathcal{H}$. Following the proof of Theorem 4.3, we see that the requirement $v_+(0,0,\lambda)=I$ determines $v_+(z,\bar z,\lambda)$ uniquely up to a holomorphic factor $q_+(z,\lambda)$ satisfying $q_+(0,\lambda)=I$. Any choice of q_+ will produce some $R\in\mathcal{R}$ and $\psi(R)$ is the same given $u\in\mathcal{H}$ for all these R. This shows that ψ is surjective. To describe the fibers of ψ we consider the "generalized gauge group"

$$\mathcal{G}^+ = \{q_+ : \mathbb{D}\to\Lambda_1^+G^{\mathbb{C}} \mid q_+ \text{ holomorphic }, q_+(0,\lambda)=I\}. \tag{4.5.1}$$

We define the action of $\mathcal{G}^+$ on $\mathcal{R}$ as follows: let $R\in\mathcal{R}$ and define g as in (4.4.1) with $g(0,\lambda)=I$. Then split $g=uv_+$ and form $\hat g = uv_+q_+ = gq_+$. We consider $\hat g^{-1}\,\mathrm{d}\hat g = \hat R\,\mathrm{d}z$ and set $R.q_+ = \hat R$. It is straightforward to see that

$$R.q_+ = q_+^{-1}Rq_+ + q_+^{-1}\,\mathrm{d}q_+ \tag{4.5.2}$$

holds, and also $(R.q_+).p_+ = R.(q_+p_+)$. Hence we have the following

Theorem. *a) The group $\mathcal{G}^+$ acts on the right on $\mathcal{R}$ by gauge transformations.*

b) The map $\psi:\mathcal{R}\to\mathcal{H}$ is surjective.

c) The fibers of ψ are orbits of $\mathcal{G}^+$ in $\mathcal{R}$.

4.6 Further, we consider a group action on $\mathcal{H}$ by $\Lambda_1^+ G^{\mathbb{C}}$. Let $w_+ \in \Lambda_1^+ G^{\mathbb{C}}$. Note that $w_+ = w_+(\lambda)$ does not depend on z or $\bar{z}$. Let $u \in \mathcal{H}$. Then we consider $w_+(\lambda)u(z,\bar{z},\lambda)w_+(\lambda)^{-1}$ and split

$$w_+(\lambda)u(z,\bar{z},\lambda)w_+(\lambda)^{-1} = \hat{u}(z,\bar{z},\lambda)\hat{v}_+(z,\bar{z},\lambda). \tag{4.6.1}$$

We set

$$w_+.u = \hat{u}. \tag{4.6.2}$$

Then we have the following

Theorem. *The operation (4.6.2) defines an action of $\Lambda_1^+ G^{\mathbb{C}}$ on $\mathcal{H}$.*

Proof. Clearly, $\hat{u}(0,0,\lambda) = I$. To see that $\hat{u}$ is an extended harmonic map, it suffices to show that $\hat{u}^{-1}\,\mathrm{d}\hat{u}$ is of the form (4.2.1). For this it suffices to show that is of the form (4.4.1), since $\hat{u} \in \Lambda_1 G$. But

$$(w_+uw_+^{-1})^{-1}\,\mathrm{d}(w_+uw_+^{-1}) = w_+(u^{-1}\,\mathrm{d}u)w_+^{-1} = (w_+Rw_+^{-1}) \tag{4.6.3}$$

is of the form (4.4.1), whence $\hat{u} \in \mathcal{H}$. To see that the action (4.6.2) is a group action, we note that $w_+.u = \hat{u}$ is equivalent with $w_+u = \hat{u}v_+$ for some $v_+ : \mathbb{D} \to \Lambda_1^+ G^{\mathbb{C}}$. Hence

$$p_+.(w_+.u) = p_+(w_+.u)a_+ = p_+(w_+ub_+)a_+ = (p_+w_+)u(b_+a_+), \tag{4.6.4}$$

whence

$$p_+.(w_+.u) = (p_+w_+).u. \tag{4.6.5}$$

□

Comparing the two actions defined here and in section 4.5, we have

Proposition. *For every $R \in \mathcal{R}$ and every $w_+ \in \Lambda_1^+ G^{\mathbb{C}}$ we have*

$$\psi(w_+Rw_+^{-1}) = w_+.\psi(R). \tag{4.6.6}$$

Proof. Let $R \in \mathcal{R}$ and solve $g^{-1}\,\mathrm{d}g = R\,\mathrm{d}z,\ g(0,\lambda) = I$. Then

$$w_+Rw_+^{-1} = \hat{g}^{-1}d\hat{g},\ \hat{g}(0,\lambda) = I \tag{4.6.7}$$

has the solution $\hat{g} = w_+gw_+^{-1}$. Hence $\psi(w_+Rw_+^{-1}) = \hat{u}$ where $\hat{g} = \hat{u}\hat{v}_+$. On the other hand $\hat{g} = w_+gw_+^{-1}$ implies

$$\hat{g} = w_+uv_+w_+^{-1} = w_+uw_+^{-1}.w_+v_+w_+^{-1} = (w_+.u)q_+.w_+v_+w_+^{-1} \tag{4.6.8}$$

for some $v_+, q_+ : \mathbb{D} \to \Lambda_1^+ G^{\mathbb{C}}$. The uniqueness of the group splitting now shows $w_+.u = \hat{u}$, whence the claim.

Remarks. a) To know all the (extended) harmonic maps it suffices to know a representative for each orbit of $\Lambda_1^+ G^{\mathbb{C}}$ on $\mathcal{H}$. It is an open question to provide geometric conditions which point out such a representative in each orbit.

b) The composition of the two actions looks like

$$(4.6.9) \qquad w_+.(R.q_+) = w_+(q_+^{-1} R q_+ + q_+^{-1}\,\mathrm{d}q_+)w_+^{-1}, \text{ for } w_+ \in \mathcal{G}_0^+ \equiv \Lambda_1^+ G^{\mathbb{C}},$$

and respectively

$$(4.6.10) \qquad (w_+.R).q_+ = (w_+^{-1} q_+)^{-1} R (w_+^{-1} q_+) + q_+^{-1}\,\mathrm{d}q_+,$$

for $q_+ : \mathbb{D} \to \Lambda_1^+ G^{\mathbb{C}}$ holomorphic, where

$$(4.6.11) \qquad R : \mathbb{D} \to \Lambda_1 G^{\mathbb{C}}, R(z,\lambda) = \frac{1-\lambda^{-1}}{2}\, Q(z) + \sum_{n \geq 0} (1-\lambda^n) A_n(z),$$

and $Q, A_n : \mathbb{D} \to \mathbb{C}$ are holomorphic maps. We notice that the two (left resp. right) actions generally do not commute on the level of potentials. However, the induced actions on the level of harmonic maps trivially commute, since the gauge transformation acts trivially on the immersions.

4.7 We have seen in 4.5 that the group $\mathcal{G}^+$ acts on $\mathcal{R}$ by gauge transformations and that the $\mathcal{G}^+$ orbits correspond to the points of $\mathcal{H}$. We shall examine if it is possible to find a natural representative in each $\mathcal{G}^+$ orbit. We guess that it might be a potential R without any component in $\Lambda_1^+ G^{\mathbb{C}}$. This can be achieved, if we drop the requirement that R stays holomorphic, as shows the following

Theorem. *a) Let $u \in \mathcal{H}$; then there exists a discrete subset $S_u \subset \mathbb{D}$ and a map*

$$v_+ : \mathbb{D} \backslash S_u \to \Lambda_1^+ G^{\mathbb{C}}, v_+(0,0,\lambda) = I,$$

such that $g = uv_+$ is meromorphic on $\mathbb{D}$ and

$$(4.7.1) \qquad g^{-1}\, dg = R\, dz$$

is of the form $R = \dfrac{1-\lambda^{-1}}{2}\, Q(z)$ with $Q : \mathbb{D} \to \boldsymbol{g}^{\mathbb{C}}$ meromorphic. Moreover, the poles of g and Q are contained in S_u.

b) On the other hand, let $R = \dfrac{1-\lambda^{-1}}{2}\, Q(z)$ be meromorphic on $\mathbb{D}$ with discrete pole set S. Assume that

$$g^{-1}\, dg = R\, dz, \quad g(0,\lambda) = I$$

has a meromorphic solution in $\mathbb{D}$ with poles in S. Then we split $g = uv_+$ for $z \in \mathbb{D} \backslash S$ and obtain an extended harmonic map $u : \mathbb{D} \backslash S \to G$.

Proof. a) Let $u \in \mathcal{H}$. Then Theorem 4.3 shows that there exists some $q_+ : \mathbb{D} \to \Lambda_1^+ G^{\mathbb{C}}$ such that $\tilde{g} = uq_+$ is holomorphic and $\tilde{g}(0, \lambda) = I$. This shows that $\tilde{g}(z, \,\cdot\,) \in \Lambda_1^0 G^{\mathbb{C}}$ for every $z \in \mathbb{D}$. We consider the map $\sigma : \mathbb{D} \to \mathbb{C}$, $\sigma(z) = \varphi(\tilde{g}(z, \,\cdot\,))$, where φ has been defined in (3.4.2). Since φ and $\tilde{g}$ are holomorphic, σ is holomorphic as well. Moreover,

$$\sigma(0) = \varphi(\tilde{g}(0, \,\cdot\,)) = \varphi(I) = \tau(H_-) \neq 0. \tag{4.7.2}$$

From Theorem 3.4 we now conclude

$$\tilde{g}(z, \,\cdot\,) \in \Lambda_1^- G^{\mathbb{C}} \cdot \Lambda_1^+ G^{\mathbb{C}} \tag{4.7.3}$$

for all $z \in \mathbb{D}\backslash S_u$, where $S_u = \{z \in \mathbb{D} \mid \sigma(z) = 0\}$ is a discrete subset of $\mathbb{D}$. In particular,

$$u(z, \bar{z}, \lambda) = a_-(z, \bar{z}, \lambda) a_+(z, \bar{z}, \lambda) \tag{4.7.4}$$

for all $z \in \mathbb{D}\backslash S_u$. Moreover, from [10, Lemma 2.6] we obtain that the singularities of a_- and a_+ are only poles. Hence $g = a_- = u.q_+.a_+^{-1}$ for $z \in D\backslash S_u$. Now it is easy to see that $g^{-1}\,\mathrm{d}g$ is of the form $\dfrac{1-\lambda^{-1}}{2}\, Q(z)$ with Q holomorphic on $\mathbb{D}\backslash S_u$ and has only poles in S_u.

b) can be proved as in 4.4. $\square$

Remarks. a) The theorem above seems to suggest that one should consider "singular harmonic maps" rather than holomorphic harmonic maps. However, in this case, similar to [9], one needs to investigate the question when a singular R gives a nonsingular extended harmonic map.

b) If u is an extended harmonic map and $u = u_- u_+$ for $z \in \mathbb{D}\backslash S$ then R associated with u_-, $u_-^{-1}\,\mathrm{d}u_- = R\,\mathrm{d}z$ makes the choice of R independent of gauge transformations (i.e., R is unique). This is justified by the following argument: for

$$R = \frac{1-\lambda^{-1}}{2}\, Q(z), \qquad \hat{R} = \frac{1-\lambda^{-1}}{2}\, \hat{Q}(z),$$

one obtains $g, \hat{g} \in \Lambda_1^- G^{\mathbb{C}}$ which split $g = uv_+, \hat{g} = u\hat{v}_+$ whence $\hat{g}^{-1}g = \hat{v}_+^{-1}v_+$ is holomorphic, and hence $v_+ = \hat{v}_+^{-1}w_+(z, \lambda)$ implies $g = \hat{g}w_+$. Since $g, \hat{g} \in \Lambda_1^- G^{\mathbb{C}}, w_+ \in \Lambda_1^+ G^{\mathbb{C}}$, from the uniqueness of the splitting, we get $w_+ = I$, which infers $R = \hat{R}$.

c) The action of $\Lambda_1^+ G^{\mathbb{C}}$ on the general R was provided simply by the adjoint action. Fixing the form of R as $\dfrac{1-\lambda^{-1}}{2}\, Q(z)$, makes the $\Lambda_1^+ G^{\mathbb{C}}$ action more complicated.

4.8 In general it is fairly difficult to carry out the required group splittings. However, if one knows the harmonic map then it is not difficult to find the meromorphic potential. The result below extends Wu's formula to the case of generalized harmonic maps into arbitrary Lie groups.

Theorem. *Let $u(z,\bar z,\lambda) : \mathbb{D} \to G$ be an extended harmonic map. Then u has a meromorphic extension to $\mathbb{D}\times\overline{\mathbb{D}}$ and the meromorphic potential R associated with u satisfies*

(4.8.1) $$R(z,\lambda) = u(z,0,\lambda)^{-1}\partial_z u(z,0,\lambda).$$

Proof. We consider the based double loop group $\mathcal{X} = \Lambda_1 G^{\mathbb{C}} \times \Lambda_1 G^{\mathbb{C}}$ and its subgroups

(4.8.2) $$\begin{aligned}\mathcal{X}^{+} &= \Lambda_1^{+} G^{\mathbb{C}} \times \Lambda_1^{-} G^{\mathbb{C}}\\ \mathcal{X}^{-} &= \Lambda_1^{-} G^{\mathbb{C}} \times \Lambda_1^{+} G^{\mathbb{C}}\end{aligned}$$

and $\mathcal{X}^{\Delta} = \{(g,g) \mid g \in \Lambda_1 G\}$. Let σ denote complex conjugation in $\Lambda_1 G^{\mathbb{C}}$ relative to $\Lambda_1 G$. Then for every $h \in \Lambda_1 G^{\mathbb{C}}$ we set $i(h) = (h,\sigma(h))$. Splitting

(4.8.3) $$i(h) = (p,p)(h_+,h_-) \;\; p \in \Lambda_1 G,\; h_\varepsilon \in \Lambda^{\varepsilon} G^{\mathbb{C}},$$

we obtain $h = ph_+,\; \sigma(h) = ph_-$. (Note, if $G = SU(N)$, then this is the classical Iwasawa decomposition). Splitting

(4.8.4) $$i(h) = (q_+,q_-)(l_+,l_-)$$

we obtain $h = q_+ l_-$ and $\sigma(h) = q_- l_+$. If $h = u$ is an extended harmonic map, then $\sigma(h) = h$ and q_- leads to the holomorphic potential R, while $q_+ = \sigma(q_-)$ leads to the "complex conjugate" meromorphic potential $\sigma(R)$. Consider now the pair of potentials $\mathcal{R} = (\,R(z)\,\mathrm{d}z, \sigma(R)(\bar w)\,\mathrm{d}\bar w\,)$. Solving the ODE $\mathcal{B}^{-1}\,\mathrm{d}\mathcal{B} = \mathcal{R}$ yields a function $\mathcal{B} : \mathbb{D}\times\overline{\mathbb{D}} \to \mathcal{X}$ which is meromorphic in $(z,\bar w)$. Clearly $\mathcal{B} = (g(z),\sigma(g(w)))$. Splitting

(4.8.5) $$\mathcal{B} = (p,p)(h_+,h_-)$$

yields $g(z) = p(z,w)h_+(z,w)$ and $\sigma(g(w)) = p(z,w)h_+(z,w)$. Because of

(4.8.6) $$g^{-1}(z)\sigma(g(w)) = h_+^{-1}(z,w)h_-(z,w),$$

we have that h_+ and h_- are meromorphic in (z,w). Therefore also p is meromorphic in (z,w). Finally setting $w = \bar z$ we obtain $g(z) = p(z,\bar z)h_+(z)$, whence $p(z,\bar z) = u(z,\bar z)$ and $p(z,w)$ is a meromorphic extension of u.

Let now $v_+(z,\bar z,\lambda)$ be chosen as in Theorem 4.7. Then $g(z,\lambda) = u(z,\bar z,\lambda)v_+(z,\bar z,\lambda)$. Setting $\bar z = 0$ this yields

(4.8.7) $$g(z,\lambda) = u(z,0,\lambda)v_+(z,0,\lambda).$$

But the equation $u^{-1}\partial_z u = \frac{1-\lambda^{-1}}{2}\alpha'$ can be evaluated at $\bar{z} = 0$ and we obtain

$$u(z,0,\lambda)^{-1}\partial_z u(z,0,\lambda) = \frac{1-\lambda^{-1}}{2}\alpha'. \tag{4.8.8}$$

Since the right side is in $\Lambda_1^- G^{\mathbb{C}}$, this ODE has a unique solution $h \in \Lambda_1^- G^{\mathbb{C}}$ satisfying the initial condition $h(z = 0) = I$. Moreover, $h(z,\lambda) = c(\lambda)u(z,0,\lambda)$. But $I = u(0,0,\lambda)$ implies $c(\lambda) = I$ and $h(z,\lambda) = u(z,0,\lambda) \in \Lambda_1^- G^{\mathbb{C}}$. Therefore $g(z,\lambda) = u(z,0,\lambda)$ and the claim follows. □

Corollary. *Let $\varphi : \mathbb{D} \to G$ be a harmonic map and u its harmonic extension. Then*

$$u(z,\bar{z},\lambda = -1) = \varphi(z,\bar{z}) \tag{4.8.9}$$

and φ has a meromorphic extension to $\mathbb{D} \times \overline{\mathbb{D}}$ and the meromorphic potential R associated with u is of the form

$$R(z,\lambda) = u(z,0,\lambda)^{-1}\partial_z u(z,0,\lambda) = \frac{1-\lambda^{-1}}{2} \cdot \varphi(z,0)^{-1}\partial_z\varphi(z,0). \tag{4.8.10}$$

Proof. This follows from (4.8.8) if one sets $\lambda = -1$.

5 Applications

Below we list several Lie groups and determine their bi-invariant metrics. Moreover, we spell out the harmonic map equations (2.2.1) for these groups and illustrate these equations with some examples. We would like to recall, that in the cases, where there does exist a (non-degenerate) bi-invariant metric, these equations follow from a variational principle. In the other cases we can write down the equations, but do not know of any variational and/or geometric interpretation. We note that throughout the section, the loop group decompositions, as stated in the theorems, are in the based loop groups. Moreover, due to our conventions (see Section 4), for every harmonic map φ we assume $\varphi(0,0) = I$.

5.1 Let G be the nilpotent group

$$G = \left\{ \left(\begin{array}{cc} 1 & a \\ 0 & 1 \end{array} \right) \middle| \, a \in \mathbb{R} \right\}. \tag{5.1.1}$$

In this case $Lie(G)$ is one dimensional; therefore every nondegenerate symmetric bilinear form is determined by a non-zero scalar. Since G is abelian, the left-invariant metrics are also right-invariant.

Consider now a map $\varphi : \mathbb{D} \to G$,

$$\varphi(x,y) = \begin{pmatrix} 1 & a(x,y) \\ 0 & 1 \end{pmatrix}, \; a \in \mathcal{C}^2(\mathbb{D}, \mathbb{R}), \tag{5.1.2}$$

for all (x,y) in $\mathbb{D} \subset \mathbb{R}$. Applying (2.2.1) we obtain that φ is harmonic if and only if

$$\Delta\varphi \equiv (\varphi_x)_x + (\varphi_y)_y = 0, \tag{5.1.3}$$

which is the canonical harmonicity equation for maps with values in $(\mathbb{R}, +)$.

Let $\varphi : \mathbb{D} \to G$ be a harmonic map (5.1.2). Then $a =$ Re (a_h), with a_h a holomorphic function. The Maurer Cartan form associated to φ is then

$$\omega = \varphi^{-1}\varphi = \begin{pmatrix} 0 & \mathrm{d}a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \partial_z a \\ 0 & 0 \end{pmatrix} \mathrm{d}z + \begin{pmatrix} 0 & \partial_{\bar{z}} a \\ 0 & 0 \end{pmatrix} \mathrm{d}\bar{z}, \tag{5.1.4}$$

and, denoting $f = \frac{1-\lambda^{-1}}{2}$, the loopified form is,

$$\omega_\lambda = f \begin{pmatrix} 0 & \partial_z a \\ 0 & 0 \end{pmatrix} \mathrm{d}z + \bar{f} \begin{pmatrix} 0 & \partial_{\bar{z}} a \\ 0 & 0 \end{pmatrix} \mathrm{d}\bar{z}. \tag{5.1.5}$$

We note that $\partial_{\bar{z}} a = \overline{\partial_z a}$ and see that one can integrate the equation ${g_\lambda}^{-1}\,\mathrm{d}g_\lambda = \omega_\lambda$, with initial condition $g_\lambda(z, \bar{z}, \lambda)|_{z=\bar{z}=0} = I_2$ and obtains

$$g_\lambda(z, \bar{z}, \lambda) = \begin{pmatrix} 1 & (f a_h(z) + \bar{f}\overline{a_h(z)})/2 \\ 0 & 1 \end{pmatrix}. \tag{5.1.6}$$

The Birkhoff decomposition $g_\lambda = g_{\lambda-} \cdot g_{\lambda+}$ provides then

$$g_{\lambda-} = \begin{pmatrix} 1 & f a_h(z)/2 \\ 0 & 1 \end{pmatrix}, \tag{5.1.7}$$

whence the meromorphic potential is

$$\mathcal{R} = g_{\lambda-}^{-1} \partial_z g_{\lambda-} = \begin{pmatrix} 0 & f\partial_z a_h(z)/2 \\ 0 & 0 \end{pmatrix} = f \begin{pmatrix} 0 & \partial_z a \\ 0 & 0 \end{pmatrix}. \tag{5.1.8}$$

We note that Wu's extended formula (4.8.10) in Corollary 4.8 produces directly

$$\xi = \varphi^{-1}\partial_z\varphi \,|_{\bar{z}=0} = \begin{pmatrix} 0 & \partial_z a \\ 0 & 0 \end{pmatrix}, \tag{5.1.9}$$

which yields (5.1.8).

We have traced above explicitely all the steps of our general theory, starting with a harmonic map finally arriving at its potential. Conversely, it is easy to follow the splitting procedure outlined in Theorem 4.7 in this example: let a be some meromorphic function and $R = \dfrac{1-\lambda^{-1}}{2} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ the corresponding meromorphic potential.

We solve the differential equation $g^{-1}\,\mathrm{d}g = R\,\mathrm{d}z$ (4.7.1) with the initial condition $g(0,\lambda) = I$. We see that in our case

$$g = \begin{pmatrix} 1 & \frac{1-\lambda^{-1}}{2}\int_0^z a(w)\,\mathrm{d}w \\ 0 & 1 \end{pmatrix} \tag{5.1.10}$$

and g is meromorphic if all residues of a vanish. The Iwasawa splitting $g = u \cdot v_+$ then is

$$g = \begin{pmatrix} 1 & \mathrm{Re}\,\left((1-\lambda^{-1})\int_0^z a(w)\,\mathrm{d}w\right) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{\lambda-1}{2}\int_0^z \bar{a}(w)\,\mathrm{d}w \\ 0 & 1 \end{pmatrix}. \tag{5.1.11}$$

This shows that in this case we recover the well known fact that every harmonic function is the real part of a holomorphic function.

5.2 Let G be the (Heisenberg) group of upper triangular unipotent matrices

$$G = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \; a, b, c \in \mathbb{R} \right\}. \tag{5.2.1}$$

From [24] it follows that the Heisenberg group does not have a bi-invariant metric. This can also easily be verified directly. Let now $\varphi : \mathbb{D} \to G$,

$$\varphi(x,y) = \begin{pmatrix} 1 & a(x,y) & c(x,y) \\ 0 & 1 & b(x,y) \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b, c \in \mathcal{C}^2(\mathbb{D}, \mathbb{R}) \tag{5.2.2}$$

for all (x, y) in $\mathbb{D} \subset \mathbb{R}^2$. Then, denoting

$$z = x + iy, \; a_z = (a_x - ia_y)/2, \; a_{\bar{z}} = (a_x + ia_y)/2,$$

the (formal) harmonicity equation (2.2.3) reads

$$\begin{cases} \Delta a \equiv 4a_{z\bar{z}} = 0, \quad b_{z\bar{z}} = 0 \\ c_{z\bar{z}} = (a_z b_{\bar{z}} + a_{\bar{z}} b_z)/2. \end{cases} \tag{5.2.3}$$

It is easy to see that (5.2.3) is equivalent with a, b and $c - \frac{1}{2}ab$ are harmonic functions.

From Wu's extended formula (4.8.10) we obtain that the meromorphic potential associated with any harmonic map φ of the form (5.2.2) is given by $R = \frac{1-\lambda^{-1}}{2} \cdot V^{-1} \cdot \partial_z V \mathrm{d}z$, where

$$V = \varphi(z/2, -iz/2). \tag{5.2.4}$$

A straightforward computation yields

$$R = \frac{1-\lambda^{-1}}{2} \begin{pmatrix} 0 & \partial_z a & \partial_z c - a\partial_z b \\ 0 & 0 & \partial_z b \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.2.5}$$

where $a = a(z/2, -iz/2),\ b = b(z/2, -iz/2),\ c = c(z/2, -iz/2)$.

As in section 5.1, also for the Heisenberg group every step of the general theory leading from a harmonic map to the potential can be made explicit.

For a given harmonic map $\varphi = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ as in (5.2.2), its Maurer-Cartan form is

$$\varphi^{-1}\,\mathrm{d}\varphi = \begin{pmatrix} 0 & \mathrm{d}a & \mathrm{d}c - a\,\mathrm{d}b \\ 0 & 0 & \mathrm{d}b \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.6}$$

whence by splitting $\mathrm{d}a = \partial_z a\,\mathrm{d}z + \partial_{\bar z} a\,\mathrm{d}\bar z$, and similar for $\mathrm{d}b$, $\mathrm{d}c$, one gets the loopified Maurer-Cartan form

$$\begin{aligned} \omega_\lambda \quad = \quad & \frac{1-\lambda^{-1}}{2}\begin{pmatrix} 0 & \partial_z a & \partial_z c - a\partial_z b \\ 0 & 0 & \partial_z b \\ 0 & 0 & 0 \end{pmatrix}\mathrm{d}z + \\ & + \frac{1-\lambda}{2}\begin{pmatrix} 0 & \partial_{\bar z} a & \partial_{\bar z} c - a\partial_{\bar z} b \\ 0 & 0 & \partial_{\bar z} b \\ 0 & 0 & 0 \end{pmatrix}\mathrm{d}\bar z. \end{aligned} \tag{5.2.7}$$

Then an extended frame $\varphi_\lambda = \begin{pmatrix} 1 & A & C \\ 0 & 1 & B \\ 0 & 0 & 1 \end{pmatrix}$ has the Maurer-Cartan form

$$\varphi_\lambda^{-1}\,\mathrm{d}\varphi_\lambda = \begin{pmatrix} 0 & \mathrm{d}A & \mathrm{d}C - A\,\mathrm{d}B \\ 0 & 0 & \mathrm{d}B \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.2.8}$$

To find its coefficients A, B, C requires to solve the system

$$\begin{cases} \mathrm{d}A = \frac{1-\lambda^{-1}}{2}\partial_z a\,\mathrm{d}z + \frac{1-\lambda}{2}\partial_{\bar z} a\,\mathrm{d}\bar z \\ \mathrm{d}B = \frac{1-\lambda^{-1}}{2}\partial_z b\,\mathrm{d}z + \frac{1-\lambda}{2}\partial_{\bar z} b\,\mathrm{d}\bar z \\ \mathrm{d}C = A\,\mathrm{d}B + \frac{1-\lambda^{-1}}{2}(\partial_z c - a\partial_z b)\,\mathrm{d}z + \frac{1-\lambda}{2}(\partial_{\bar z} c - a\partial_{\bar z} b)\,\mathrm{d}\bar z. \end{cases} \tag{5.2.9}$$

Remark that a, b and $k = c - \frac{ab}{2}$ are real parts of holomorphic functions, and hence

$$\begin{cases} a(z) = (a_h(z) + \overline{a_h(z)})/2 \\ b(z) = (b_h(z) + \overline{b_h(z)})/2, \\ k(z) = (k_h(z) + \overline{k_h(z)})/2, \end{cases} \tag{5.2.10}$$

where a_h, b_h, k_h are holomorphic functions which vanish at $z = 0$. As a consequence, we have the relations

$$\overline{\partial_z a} = \partial_{\bar z} a,\ \overline{\partial_z b} = \partial_{\bar z} b,\ \overline{\partial_z k} = \partial_{\bar z} k, \tag{5.2.11}$$

Setting $f = \frac{1-\lambda^{-1}}{2}$, the first two relations yield

$$\partial_z A = f\partial_z a,\quad \partial_z B = f\partial_z b, \tag{5.2.12}$$

and the equations can be rewritten as

$$\begin{cases} \mathrm{d}A = f\partial_z a\,\mathrm{d}z + \overline{f\partial_z a}\,\mathrm{d}\bar z, & A(0) = 0 \\ \mathrm{d}B = f\partial_z b\,\mathrm{d}z + \overline{f\partial_z b}\,\mathrm{d}\bar z, & B(0) = 0, \end{cases} \tag{5.2.13}$$

whence, using an argument similar to that of case 1, we integrate and obtain

$$A(z) = \text{ Re }(fa_h(z)),\quad B(z) = \text{ Re }(fb_h(z)). \tag{5.2.14}$$

The last equation of (5.2.9) writes

$$\mathrm{d}C = A\,\mathrm{d}B + f(\partial_z c - a\partial_z b)\,\mathrm{d}z + \bar f(\partial_{\bar z} c - a\partial_{\bar z} b)\,\mathrm{d}\bar z. \tag{5.2.15}$$

We assume $C(0) = 0$. Since $\mathrm{d}C = \partial_z C\,\mathrm{d}z + \partial_{\bar z} C\,\mathrm{d}\bar z$, $\partial_z B = f\partial_z b$, and $A = Re(fa_h)$ we have

$$\begin{aligned} \partial_z C &= A\partial_z B + f(\partial_z c - a\partial_z b) = \\ &= f\cdot[\,\partial_z c + (\text{Re }(fa_h) - a)\partial_z b\,], \end{aligned} \tag{5.2.16}$$

whence

$$\begin{aligned} C(z,\bar z) &= f(c(z) + u(z)) + c_p(\bar z) = \\ &= f\cdot(k(z) + v(z)) + c_p(\bar z), \end{aligned} \tag{5.2.17}$$

where we denoted

$$\begin{aligned} u &= \int_0^z [Re(fa_h(w)) - a(w)]\,\partial_z b(w)\,\mathrm{d}w \\ v &= \frac{ab}{2} + u = \int_0^z \frac{fa_h + \bar f\overline{a_h}}{2}\partial_z b + \left(\frac{ab}{2} - \int_0^z a(w)\partial_z b(w)\,\mathrm{d}w\right). \end{aligned} \tag{5.2.18}$$

Since $k = \bar k$ and $C = \bar C$, we have

$$0 = C - \bar C = (f - \bar f)k + (fv - \bar f\bar v) + c_p - \overline{c_p}, \tag{5.2.19}$$

whence

$$c_p - \overline{c_p} = (\bar f - f)k + (\bar f\bar v - fv). \tag{5.2.20}$$

Note that in view of (5.2.10), the factor k is the sum of a holomorphic and an anti-holomorphic function. Therefore also the last term on the right splits this way,

$$(\bar{f}\bar{v} - fv) = w_h + w_a. \tag{5.2.21}$$

Actually, the expansion of (5.2.21) in terms of holomorphic and antiholomorphic parts yields the given terms in the right, since the terms which are not purely holomorphic or antiholomorphic drop out by the harmonicity equation. Then explicit expressions of the two outcoming terms are

$$w_h = \frac{f - f^2}{4} \int_0^z a_h(w)\partial_z b_h(w)\,\mathrm{d}w + \frac{\bar{f} - f}{8} a_h b_h \tag{5.2.22}$$

$$w_a = \frac{\bar{f}^2 - \bar{f}}{4} \cdot \overline{\int_0^z a_h(w)\partial_z b_h(w)\,\mathrm{d}w} + \frac{\bar{f} - f}{8} \overline{a_h} \cdot \overline{b_h}, \tag{5.2.23}$$

and separate the holomorphic/antiholomorphic parts in (5.2.20), which implies

$$c_p - w_a - (\bar{f} - f)\overline{k_h}/2 = \overline{c_p} + (\bar{f} - f)k_h/2 + w_h. \tag{5.2.24}$$

Now, taking into account that $v(0) = k(0) = C(0) = c_p(0) = 0$, this yields that both terms in (5.2.24) vanish, and

$$c_p = w_a + (f - \bar{f})\overline{k_h}/2, \tag{5.2.25}$$

whence

$$C = \operatorname{Re}(fk_h) + fv + w_a. \tag{5.2.26}$$

Then the extended frame is

$$\varphi_\lambda = \begin{pmatrix} 1 & \operatorname{Re}(fa_h) & C \\ 0 & 1 & \operatorname{Re}(fb_h) \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.2.27}$$

Denoting $\varphi_\lambda = \begin{pmatrix} 1 & A & C \\ 0 & 1 & B \\ 0 & 0 & 1 \end{pmatrix}$, we perform its Birkhoff splitting

$$\begin{pmatrix} 1 & A & C \\ 0 & 1 & B \\ 0 & 0 & 1 \end{pmatrix} = \varphi_{\lambda-} \cdot \varphi_{\lambda+} = \begin{pmatrix} 1 & A_- & C_- \\ 0 & 1 & B_- \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & A_+ & C_+ \\ 0 & 1 & B_+ \\ 0 & 0 & 1 \end{pmatrix} \in \Lambda_1^- G^{\mathbb{C}} \cdot \Lambda_1^+ G^{\mathbb{C}}, \tag{5.2.28}$$

which leads to the relations

$$A = A_- + A_+ \in \Lambda^- \mathbb{C} + \Lambda^+ \mathbb{C} \tag{5.2.29}$$

$$B = B_- + B_+ \in \Lambda^- \mathbb{C} + \Lambda^+ \mathbb{C} \tag{5.2.30}$$

$$C = C_- + C_+ + A_- B_+ \tag{5.2.31}$$

where $A_\pm, B_\pm \in \Lambda^\pm \mathbb{C}$ are uniquely determined by the Birkhoff decomposition of the entries A and B,

$$\left\{\begin{array}{l} A_- = fa_h/2,\ A_+ = \overline{A_-} \\ B_- = fb_h/2,\ B_+ = \overline{B_-} \end{array}\right. . \tag{5.2.32}$$

The entry C rewrites as Birkhoff sum

$$C = 2\ \mathrm{Re}\ (C_m) = C_m + C_p \in \Lambda^-\mathbb{C} + \Lambda^+\mathbb{C}. \tag{5.2.33}$$

Also, in view of the equality

$$f \cdot \bar{f} = (f + \bar{f})/2, \tag{5.2.34}$$

it follows that the term A_-B_+ in the third relation of (5.2.29) splits

$$A_-B_+ = (fa_h \cdot \overline{fb_h})/4 = \frac{1}{8} fa_h\overline{b_h} + \frac{1}{8}\bar{f}a_h\overline{b_h} \in \Lambda^-\mathbb{C} + \Lambda^+\mathbb{C}, \tag{5.2.35}$$

which is exactly the Birkhoff splitting of A_-B_+. Hence (5.2.31) becomes

$$C = (C_- + \frac{1}{8} fa_h\overline{b_h}) + (C_+ + \frac{1}{8}\bar{f}a_h\overline{b_h}). \tag{5.2.36}$$

and, using the uniqueness of the Birkhoff decomposition (5.2.33), one gets

$$\left\{\begin{array}{l} C_- = C_m - fa_h\overline{b_h}/8 \\ C_+ = C_p - \bar{f}a_h\overline{b_h}/8. \end{array}\right. \tag{5.2.37}$$

So we have

$$\varphi_{\lambda-} = \begin{pmatrix} 1 & fa_h/2 & C_m - fa_h\overline{b_h}/8 \\ 0 & 1 & fb_h/2 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.2.38}$$

We use that

$$a = \frac{fa_h + \bar{f}\overline{a_h}}{2} \tag{5.2.39}$$

whence

$$\partial_z a = f\partial_z a_h/2 \tag{5.2.40}$$

and

$$\partial_z A_- = f\partial_z a, \tag{5.2.41}$$

and obtain the meromorphic potential $R = {\varphi_{\lambda-}}^{-1}\partial_z\varphi_{\lambda-}$,

$$R \ = \ \begin{pmatrix} 1 & A_- & C_- \\ 0 & 1 & B_- \\ 0 & 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 & \partial_z A_- & \partial_z C_- \\ 0 & 0 & \partial_z B_- \\ 0 & 0 & 0 \end{pmatrix} = \tag{5.2.42}$$

$$= \begin{pmatrix} 0 & \partial_z a & \partial_z C_- - A_-\partial_z B_- \\ 0 & 0 & \partial_z b \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.2.43}$$

The computation of the right-upper corner yields

$$\partial_z C_- - A_- \partial_z B_- = \partial_z C_m - \partial_z \left(f \frac{a_h \overline{b_h}}{8} \right) - f^2 \frac{a_h \partial_z b}{2}. \tag{5.2.44}$$

Comparing (5.2.43) to (5.2.5) we see that it suffices to show that the functions in the right upper corner of both matrices coincide. To this end we first note

$$\begin{aligned}
\partial_z C &= \partial_z (f(c+u)+c_p) = f\partial_z c + f\left(\frac{f a_h + \bar{f}\overline{a_h}}{2} - a\right)\partial_z b = && (5.2.45)\\
&= f\partial_z c + f^2 \frac{a_h \partial_z b}{2} + (f+\bar{f})\frac{\overline{a_h}\partial_z b}{4} - f a \partial_z b = && (5.2.46)\\
&= f^2 \frac{a_h \partial_z b}{2} + f\left(\partial_z c - a\partial_z b + \frac{\overline{a_h}\partial_z b}{4}\right) + \bar{f}\frac{\overline{a_h}\partial_z b}{4}. && (5.2.47)
\end{aligned}$$

This implies

$$\partial_z C_m = f^2 \frac{a_h \partial_z b}{2} + f\left(\partial_z c - a\partial_z b + \frac{\overline{a_h}\partial_z b}{4}\right). \tag{5.2.48}$$

Therefore

$$\partial_z C_- - A_- \partial_z B_- = f\left(\partial_z c - a\partial_z b + \frac{\overline{a_h}\partial_z b}{4} - \partial_z\left(\frac{a_h \overline{b_h}}{8}\right)\right). \tag{5.2.49}$$

From this we infer that $\partial_{\bar{z}}$ annihilates this expression which hence does not depend on $\bar{z}$, and is equal to the one obtained when setting $\bar{z}=0$. But $a_h(0)=b_h(0)=0$, and hence, for $\bar{z}=0$ we obtain

$$\partial_z C_- - A_- \partial_z B_- = f\left(\partial_z c - a\partial_z b\right). \tag{5.2.50}$$

Thus (5.2.5) and (5.2.43) coincide.

As shown in [7, Section 6, p. 158] maps of the form

$$\varphi : \mathbb{R}^2 \to G, \quad \varphi(x,y) = e^{xX}\cdot e^{yY}, \tag{5.2.51}$$

where $X, Y \in \mathbf{g}$, satisfy the harmonicity equations (5.2.3). In particular one can choose for the Heisenberg group X, Y from the generators of $\mathbf{g}$,

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.2.52}$$

More precisely, starting from a harmonic map of type (5.2.51)

$$\begin{aligned}
(5.2.53)\ \varphi &= exp\left\{ x\begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}\right\}\cdot exp\left\{ y\begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}\right\} = \\
&= \begin{pmatrix} 1 & ax+\alpha y & cx+\gamma y + a\beta xy + \frac{abx^2+\alpha\beta y^2}{2} \\ 0 & 1 & bx+\beta y \\ 0 & 0 & 1 \end{pmatrix}, \quad a,b,c,\alpha,\beta,\gamma \in \mathbb{R},
\end{aligned}$$

we apply Wu's formula and derive its meromorphic potential of the form

$$\xi = \frac{1}{4}\begin{pmatrix} 0 & 2(a-i\alpha) & iz(\alpha b - a\beta)+2(c-i\gamma) \\ 0 & 0 & 2(b-i\beta) \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.2.54}$$

As in 5.1 we also want to make explicit the converse chain of steps. From a given potential it is easy to follow the construction principle given in Theorem 4.7.

Let a, b, c provide the meromorphic potential $R = \frac{1-\lambda^{-1}}{2}\begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$. Then solving the differential equation $g^{-1}\,\mathrm{d}g = R\,\mathrm{d}z$ (4.7.1) with the initial condition $g(0,\lambda) = I$, we obtain

$$g = \begin{pmatrix} 1 & \frac{1-\lambda^{-1}}{2}\int_0^z a(w)\,\mathrm{d}w & \frac{1-\lambda^{-1}}{2}\int_0^z \left(c(w)+b(w)\int_0^w a(v)\,\mathrm{d}v\right)\,\mathrm{d}w \\ 0 & 1 & \frac{1-\lambda^{-1}}{2}\int_0^z b(w)\,\mathrm{d}w \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.2.55}$$

In order to make this meromorphic we need to assume that the residues of a, b and $c(z) + b(z)\int_0^z a(w)\,\mathrm{d}w$ all vanish.

The Iwasawa splitting of $g = \begin{pmatrix} 1 & A & C \\ 0 & 1 & B \\ 0 & 0 & 1 \end{pmatrix}$ can be carried out explicitely in this case:

$$g = u \cdot v_+ = \begin{pmatrix} 1 & A+\bar{A} & 2Re(C+\bar{A}\cdot B) \\ 0 & 1 & B+\bar{B} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -\bar{A} & \bar{A}\bar{B}-\bar{C}-B\bar{A} \\ 0 & 1 & -\bar{B} \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.2.56}$$

Thus every extended harmonic map from $\mathbb{D}$ to G is of the form

$$u = \begin{pmatrix} 1 & A+\bar{A} & 2Re(C+\bar{A}\cdot B) \\ 0 & 1 & B+\bar{B} \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.2.57}$$

where

$$\begin{cases} A = \frac{1-\lambda^{-1}}{2}\int_0^z a(w)\,\mathrm{d}w, B = \frac{1-\lambda^{-1}}{2}\int_0^z b(w)\,\mathrm{d}w \\ C = \frac{1-\lambda^{-1}}{2}\int_0^z \left(c(w)+b(w)\int_0^w a(v)\,\mathrm{d}v\right)\,\mathrm{d}w. \end{cases} \tag{5.2.58}$$

This implies that in terms of the coefficient functions of the potential every harmonic map from $\mathbb{D}$ to G is of the form

$$\varphi = \begin{pmatrix} 1 & 2\,\mathrm{Re}\int_0^z a(w)\,\mathrm{d}w & C_* \\ 0 & 1 & 2\,\mathrm{Re}\int_0^z b(w)\,\mathrm{d}w \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.2.59}$$

where

$$(5.2.60)\qquad C_* \;=\; 2\,\mathrm{Re}\left(\int_0^z \left(c(w)+b(w)\int_0^w a(v)\,\mathrm{d}v\right)\mathrm{d}w + \right.$$
$$\left. + \overline{\int_0^z a(w)\,\mathrm{d}w} \cdot \int_0^z b(w)\,\mathrm{d}w\right).$$

It is straightforward to verify that, indeed, starting from ξ of the form (5.2.54) formula (5.2.59) yields exactly (5.2.53).

5.3 Consider the special linear Lie group

$$(5.3.1)\qquad G \;=\; SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle|\; a,b,c,d \in \mathbb{R},\; ad-bc=1 \right\}.$$

For G, as for any simple real Lie group, there exists only one bi-invariant metric, up to scalar multiples, the one produced by the Killing form.

In this case the harmonicity equations (2.2.1) follow from applying the variational principle and rewrite

$$(5.3.2)\qquad \begin{cases} a\Delta c - c\Delta a = 0 \\ b\Delta d - d\Delta b = 0 \\ a\Delta d - d\Delta a = c\Delta b - b\Delta c, \end{cases}$$

or, at points where all four functions do not vanish, these have the equivalent form,

$$(5.3.3)\qquad \frac{\Delta a}{a} = \frac{\Delta b}{b} = \frac{\Delta c}{c} = \frac{\Delta d}{d}.$$

In the previous sections we have shown what meromorphic potentials one obtains from a given harmonic map, both by applying Wu's Formula and by direct computation of a Birkhoff splitting. We also followed explicitely the construction of the harmonic map from a potential, in particular carrying out an Iwasawa splitting.

In the case under consideration in this section the group splittings seem to be very difficult if not impossible to carry out "by hand" explicitely. In a sense this underlines the strength of the theory presented. We will show in several examples what harmonicity means in the case of $SL(2,\mathbb{R})$ and compute the corresponding potentials via Wu's Formula.

As a first class of examples we consider harmonic maps for which the coefficient functions a,b,c,d all are of the form $p(x)+q(y)$ and satisfy

$$(5.3.4)\qquad \frac{\Delta a}{a} = \frac{\Delta b}{b} = \frac{\Delta c}{c} = \frac{\Delta d}{d} = 1.$$

It follows that the solutions of (5.3.4) are of the form

$$\varphi(x,y) = \begin{pmatrix} a_1e^x + a_2e^{-x} + a_3e^y + a_4e^{-y} & b_1e^x + b_2e^{-x} + b_3e^y + b_4e^{-y} \\ c_1e^x + c_2e^{-x} + c_3e^y + c_4e^{-y} & d_1e^x + d_2e^{-x} + d_3e^y + d_4e^{-y} \end{pmatrix}, \tag{5.3.5}$$

where the condition $\det\varphi = 1$ is equivalent to the condition that the coefficients $a_i, b_i, c_i, d_i, i \in \{1,2,3,4\}$ satisfy the system

$$\begin{cases} a_id_i = b_ic_i, & \text{for all } i \in \{1,2,3,4\} \\ a_id_j + a_jd_i = b_ic_j + b_jc_i, & \text{for all } (i,j) \in \{1,2\} \times \{3,4\} \\ a_1d_2 + a_2d_1 + a_3d_4 + a_4d_3 - (b_1c_2 + b_2c_1 + b_3c_4 + b_4c_3) = 1. \end{cases} \tag{5.3.6}$$

Also, the condition $\varphi(0,0) = I$ implies

$$\begin{cases} a_1 + a_2 + a_3 + a_4 = d_1 + d_2 + d_3 + d_4 = 1 \\ b_1 + b_2 + b_3 + b_4 = c_1 + c_2 + c_3 + c_4 = 0. \end{cases} \tag{5.3.7}$$

The family of harmonic maps (5.3.5) provides via Wu's extended formula potentials $\xi = \varphi^{-1}\partial_z\varphi|_{\bar z=0}$ of the form

$$\begin{aligned} \xi \ &= \ (1-i)\cdot e^{(1+i)z/2}\begin{pmatrix} b_1c_4 - d_1a_4 & b_1d_4 - d_1b_4 \\ c_1a_4 - c_4a_1 & -(b_1c_4 - d_1a_4) \end{pmatrix} + \\ &+ \ (1+i)\cdot e^{(1-i)z/2}\begin{pmatrix} b_1c_3 - a_3d_1 & b_1d_3 - d_1b_3 \\ c_1a_3 - c_3a_1 & -(b_1c_3 - a_3d_1) \end{pmatrix} + \\ &+ \ (1+i)\cdot e^{(-1+i)z/2}\begin{pmatrix} a_4d_2 - b_2c_4 & b_4d_2 - d_4b_2 \\ c_4a_2 - c_2a_4 & -(a_4d_2 - b_2c_4) \end{pmatrix} + \\ &+ \ (1-i)\cdot e^{(-1-i)z/2}\begin{pmatrix} a_3d_2 - b_2c_3 & b_3d_2 - d_2b_3 \\ a_2c_3 - a_3c_2 & -(a_3d_2 - b_2c_3) \end{pmatrix} + \begin{pmatrix} a_* & b_* \\ c_* & d_* \end{pmatrix}, \end{aligned} \tag{5.3.8}$$

where

$$\begin{cases} a_* = -d_* = (a_1d_2 - a_2d_1) + (b_1c_2 - b_2c_1) + i[(a_4d_3 - a_3d_4) + (b_4c_3 - b_3c_4)] \\ b_* = 2[(b_1d_2 - b_2d_1) + i(b_4d_3 - b_3d_4)] \\ c_* = 2[(c_1a_2 - c_2a_1) + i(c_4a_3 - c_3a_4)]. \end{cases} \tag{5.3.9}$$

Note that for $b_1b_2b_3b_4d_1d_2d_3d_4 \neq 0$, the first equations in (5.3.6) rewrite

$$\frac{a_i}{b_i} = \frac{c_i}{d_i} = k_i, \quad i = \overline{1,4}, \tag{5.3.10}$$

and hence the second set in (5.3.6) yields

$$(k_i - k_j)\cdot(b_id_j - b_jd_i) = 0, \quad \text{for all } (i,j) \in \{1,2\} \times \{3,4\}. \tag{5.3.11}$$

In this case (5.3.8) rewrites

$$
\begin{aligned}
\xi &= (1-i)\cdot(b_1d_4-d_1b_4)e^{(1+i)z/2}\begin{pmatrix} k_4 & 1\\ -k_1k_4 & -k_4\end{pmatrix}+\\
&+ (1+i)\cdot(b_1d_3-d_1b_3)e^{(1-i)z/2}\begin{pmatrix} k_3 & 1\\ -k_1k_3 & -k_3\end{pmatrix}+\\
&+ (1+i)\cdot(b_4d_2-d_4b_2)e^{(-1+i)z/2}\begin{pmatrix} k_4 & 1\\ -k_2k_4 & -k_4\end{pmatrix}+\\
&+ (1-i)\cdot(b_3d_2-d_2b_3)e^{(-1-i)z/2}\begin{pmatrix} k_3 & 1\\ -k_2k_3 & -k_3\end{pmatrix}+\\
&+ (b_1d_2-b_2d_1)\begin{pmatrix} k_1+k_2 & 2\\ -2k_1k_2 & -(k_1+k_2)\end{pmatrix}+\\
&+ i(b_4d_3-b_3d_4)\begin{pmatrix} k_3+k_4 & 2\\ -2k_3k_4 & -(k_3+k_4)\end{pmatrix}.
\end{aligned}
$$

In particular, for $a_2=b_1=c_3=-d_4=\frac{1}{\sqrt{2}}$ and the rest of coefficients zero in (5.3.5), we obtain the following special example of a harmonic map $\varphi_*:\mathbb{R}^2\to SL(2,\mathbb{R})$

$$\varphi_*(x,y)=\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-x} & -e^{-y}\\ e^{y} & e^{x}\end{pmatrix}. \tag{5.3.12}$$

When multiplying φ_* by the constant matrix $\varphi_*|^{-1}_{x=y=0}=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1\\ -1 & 1\end{pmatrix}$ on the left, which does not change harmonicity, we obtain the harmonic map φ which satisfies the condition $\varphi(0,0)=I$,

$$\varphi(x,y)=\frac{1}{2}\begin{pmatrix} e^{-x}+e^{y} & e^{x}-e^{-y}\\ -e^{-x}+e^{y} & e^{x}+e^{-y}\end{pmatrix}. \tag{5.3.13}$$

This is a harmonic map given by J.Inoguchi, which can be obtained from (5.3.5) for $a_2=a_3=c_3=b_1=d_1=d_4=1/2$ and $c_2=b_4=-1/2$ and the rest of coefficients zero. For $\bar z=0$ this writes

$$\varphi|_{\bar z=0}=\frac{1}{2}\begin{pmatrix} e^{-z/2}+e^{-iz/2} & e^{z/2}-e^{iz/2}\\ -e^{-z/2}+e^{-iz/2} & e^{z/2}+e^{iz/2}\end{pmatrix} \tag{5.3.14}$$

and this leads to the holomorphic potential - a particular form of (5.3.8),

$$\varphi^{-1}\partial_z\varphi|_{\bar z=0} = \frac{1}{2}\begin{pmatrix} -(1+i) & (1-i)e^{z(1+i)/2}\\ (1-i)e^{-z(1+i)/2} & 1+i\end{pmatrix}. \tag{5.3.15}$$

If a map $\varphi:\mathbb{D}\to SL(2,\mathbb{R})$ has all its values in the subgroup $H_0=SO(2)\subset SL(2,\mathbb{R})$, then

$$\varphi = \begin{pmatrix} \cos t(x,y) & -\sin t(x,y) \\ \sin t(x,y) & \cos t(x,y) \end{pmatrix} \tag{5.3.16}$$

and φ is harmonic, i.e., it satisfies (5.3.2), iff t satisfies the Laplace equation $\Delta t = 0$. We remark that for the subgroup

$$H_1 = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| \; ad = 1, \; a, b, d \in \mathbb{R} \right\} \subset SL(2, \mathbb{R}), \tag{5.3.17}$$

the equations (5.3.2) lead to

$$\begin{cases} b\Delta d - d\Delta b = 0 \\ d_x^2 + d_y^2 - d\Delta d = 0. \end{cases} \tag{5.3.18}$$

Moreover, for the subgroup

$$H_2 = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \; b \in \mathbb{R} \right\} \subset H_1 \subset SL(2, \mathbb{R}), \tag{5.3.19}$$

the system (5.3.18) reduces to $\Delta b = 0$.

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References

[1] V. Balan, *On the DPW Method for the Tangent Group*, Proceedings of the Conference of Differential Geometry, Lie Algebras and Global Analysis, Thessaloniki 1998, Geometry Balkan Press, Proceedings 4., Bucharest 2000, 1-9.

[2] V. Balan, J. Dorfmeister, *Birkhoff Decomposition and Iwasawa Decomposition for General Loop Groups*, submitted.

[3] R. W.Brockett and H. J. Sussmann, *Tangent Bundles of Homogeneous Spaces are Homogeneous Spaces*, Proc.A.M.S., 35(1972), n.2, 550-551.

[4] L. Bungart, *On Analytic Fiber Bundles*, Topology, 7(1967), 55-68.

[5] F. E. Burstall and J. H. Rawnsley, *Twistor Theory for Riemannian Symmetric Spaces*, Springer-Verlag, 1990.

[6] C. Chevalley, *Theory of Lie Groups*, Princeton University Press, 1946.

[7] Y.-J. Dai, M. Shoji, H. Urakawa, *Harmonic Maps into Lie Groups and Homogeneous Spaces*, Differential Geom. Appl. 7 (1997), no. 2, 143–160.

[8] J. Dorfmeister, *Weighted L_1-Grassmannians and Banach Manifolds of Solutions to the KP-Equation and the KdV-Equation*, Math. Nachrichten, 180 (1996), 43-73

[9] J. Dorfmeister and G. Haak, *Meromorphic potentials and Smooth CMC Surfaces*, Math.Zeitschrift, 224(1997), 603-640.

[10] J. Dorfmeister, F. Pedit, and H. Wu, *Weierstrass Type Representations of Harmonic Maps into Symmetric spaces*, Comm. in Analysis and Geom., 6(1998), n.4, 633-668.

[11] J. Dorfmeister and H. Wu, *Constant Mean Curvature Surfaces and Loop Groups*, J. reine angew. Math., 440 (1993), pp. 43–76.

[12] J. Eells and L. Lemaire, *A Report on Harmonic Maps*, Bull. London Math. Soc., 20(1988), 385-524.

[13] J. Eells and L. Lemaire, *Another Report on Harmonic Maps*, Bull. London Math. Soc., 10(1978), 1-68.

[14] M.A. Guest and Y. Ohnita, *Loop Group Actions on Harmonic Maps and Their Applications*, (in "Harmonic Maps and Integrable Systems, Eds: A.P. Fordy and J.C. Wood), 273–292, Aspects Math., E23, Vieweg, Braunschweig, 1994.

[15] F. Helein, *Willmore Immersions and Loop Groups*, Jour. Diff. Geom., 50(1998), 331-385.

[16] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York, San Francisco, London, 1978.

[17] J. Hilgert and K.-H. Neeb, *Lie Semigroups and their Applications*, Lecture Notes in Mathematics 1552, Springer, Berlin, 1993.

[18] G. Hochschild, *The Structure of Lie Groups*, Holden-Day, Inc., San Francisco, London, Amsterdam, 1965.

[19] K. H. Hoffman and V. S. Keith, *Invariant Quadratic Forms on Finite Dimensional Lie Algebras*, Bull.Austral.Math.Soc, 33(1986), n.1, 21-36.

[20] J. Jost, *Two-Dimensional Geometric Variational Problems*, John Wiley & Sons, 1991.

[21] Y.Katznelson, *An Introduction to Harmonic Analysis*, Dover Publ. Inc., New York, 1976.

[22] P.Kellersch, *Eine Verallgemeinerung der Iwasawa Zerlegung in Loop Gruppen*, Dissertation, TU-Munich, 1999.

[23] M. Kokubu, *Weierstrass Representation for Minimal Surfaces in Hyperbolic Space*, Tohoku Math. Jour., 49(1997), no. 3, 367–377.

[24] A. Medina and Ph. Revoy, *Algebres de Lie et produit scalaire invariant*, Ann. Scient. Ec. Norm. Sup., 18(1985), n.3, 553-561.

[25] A. Medina and Ph. Revoy, *Characterisation des groupes de Lie ayant une pseudo-metrique bi-invariante*, South Rhone seminar in geometry, III (Lyon, 1983), 149-166, Hermann, Paris, 1984.

[26] A. L. Onishchik and E. B. Vinberg, *Lie Groups and Algebraic Groups*, Springer-Verlag, 1990.

[27] A. N. Pressley and G. Segal, *Loop Groups*, Oxford Math. Monographs, Clarendon Press, Oxford, 1986.

[28] G. Segal, G. Wilson, *Loop Groups and Equations of KdV Type*, Inst. Hautes Etudes Sci. Publ. Math. No. 61, (1985), 5-65.

[29] K. Uhlenbeck, *Harmonic Maps into Lie Groups (classical solutions of the chiral model)*, J.Diff. Geom. 30(1989), 1-50.

[30] H. Urakawa, *Calculus of Variations and Harmonic Maps*, Translations of the AMS, vol. 132, 1990.

[31] V.S. Varadarajan, *Lie Groups, Lie Algebras and Their Representations*, Prentice Hall, 1974.

[32] P. G. Walczak, *Invariant Metrics on the Tangent Bundle of a Homogeneous Space*, Colloquium Mathematicum, 39(1978), n.2, 297-300.

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