On (k, f, l)-Chordal Polygons

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Dedicated to Prof.Dr. Constantin UDRIŞTE
on the occasion of his sixtieth birthday

Abstract

The article generalizes the concept of the k-chordal polygons and the (k, λ, l)-chordal polygons as introduced in papers [3], [4] respectively. Namely, here we describe the positive real function f where $f(a_1), \cdots, f(a_n)$ are the side lengths of a l-chordal polygon and in the same time $a_1, \cdots, a_n$ are the lengths of the sides of a k-chordal polygon.

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1 Preliminary definitions and results

This article is an addendum to the papers [3], [4], where the concepts of the k-chordal and its generalized variant the (k, λ, l)-chordal polygons are introduced and discussed. The (k, λ, l) property means the following. Let $a_1, \cdots, a_n$ be the lengths of the sides of a k-chordal polygon $\underline{A}$, and $a_1^\lambda, \cdots, a_n^\lambda, \lambda \in R_+$ are the side lengths of a l-chordal polygon, then $\underline{A}$ is called (k, λ, l). The properties of these general geometrical objects are exposed in detail in [4]. Here we are interested in the following natural generalization of the main results of the paper [4].

Our main goal is now to describe a positive, real function f so that it maps the side lengths $a_1, \cdots, a_n$ of a k-chordal polygon into $f(a_1), \cdots, f(a_n)$, being these the lengths of the sides of a l-chordal polygon.

For convenience we first repeat briefly some definitions and the results of article [3].

Definition 1.1. Let $\underline{A} = A_1 A_2 \cdots A_n$ be a chordal polygon and let $\mathcal{C}^n_{\underline{A}}$ be its circumcirle. By S_{A_i} and $\widehat{S}_{A_i}$ we denote the semicircles so that

$$S_{A_i} \cup \widehat{S}_{A_i} = \mathcal{C}^n_{\underline{A}}, \qquad A_i \in S_{A_i} \cap \widehat{S}_{A_i}.$$

The polygon $\underline{A}$ is said to be of the first kind if it is fulfiled: **(i)** not all vertices $A_1, A_2, \cdots, A_n$ lie on the same semicircle; **(ii)** for every three consecutive vertices A_i, A_{i+1}, A_{i+2} it is valid that $A_i \in S_{A_{i+1}} \Longrightarrow A_{i+2} \in \widehat{S}_{A_i}$; **(iii)** any two consecutive vertices A_i, A_{i+1} do not lie on the same diameter.

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Definition 1.2. Let $\underline{A} = A_1A_2\cdots A_n$ be a chordal polygon and let k be a positive integer. The polygon $\underline{A}$ is said to be k-inscribed and called k-chordal polygon if it is of the first kind and if $\sum_{i=1}^{n} \angle A_iCA_{i+1} = 2k\pi$, where C denotes the centre of the circumscribed circle to the polygon $\underline{A}$.

In the sequel we will write β_i for $\angle CA_iA_{i+1}$ according to the notations introduced in [3],[4]. It follows that for a k-chordal polygon $\underline{A}$ it results that

$$\sum_{i=1}^{n} \beta_i = (n-2k)\pi. \tag{1}$$

The result that we need frequently throughout the paper and concerning the sum of the side lengths in the k-chordal polygons, is: *If $a_1, a_2, \cdots, a_n$ are the side lenghts of the k-chordal polygon $\underline{A}$, then*

$$\sum_{i=1}^{n} a_i > 2ka^\star, \tag{2}$$

where $a^\star = \max_{1\le j\le n} a_j$, cf. [3], Corrolary 1.2.

Finally we are ready to introduce the concept of the (k,f,l)-chordal polygons.

Definition 1.3. When $\underline{A}$ is k-chordal polygon with the sides of lenghts $a_1, \cdots, a_n$ and in the same time $f(a_1), \cdots, f(a_n)$ are the side lenghts of a l-chordal polygon, then $\underline{A}$ is called (k,f,l)-chordal polygon. The polygon that consists of the sides of the lengths $f(a_i)$, we denote as $f(\underline{A})$.

We also need the definition of the *convexity* and the definition of the *star-likeness with respect to the origin.* According to the references [1], [2] we give these definitions.

Definition 1.4. The real function $f : [a,b] \to R$ is said to be convex on $[a,b]$ when for all $x, y \in [a,b]$ and all $t \in [0,1]$ we have $f(tx+(1-t)y) \le tf(x) + (1-t)f(y)$.

Definition 1.5. The real function $f : [0,b] \to R$; $f(0) = 0$, that satisfies $f(tx) \le tf(x)$ for all $x \in [0,b]$ and $t \in [0,1]$ is called star-like with respect to the origin.

In the following section we give our principal results. Namely, we are focusing to the sufficient conditions for the existence of the (k,f,l)-chordal polygons and we are looking for some sufficient conditions upon the lengths $f(a_i); i = \overline{1,n}$ that ensures the existence of the chordal polygons $\underline{A}$ and $f(\underline{A})$.

2 Main results

Now, according to the already given definitions we are ready to give our principal results concerning the existence of the introduced (k,f,l)-chordal polygons. At first we give the sufficient conditions for the existence of this geometrical object, when we possess the information that $\underline{A}$ is k-chordal.

Theorem 2.1. *Let $\underline{A}$ be a k-chordal polygon with the sides of the lengths $a_1,\cdots,a_n$ where $k\in N$; $k\le\left[\frac{n-1}{2}\right]$. Then $f(a_1),\cdots,f(a_n)$ are the side lengths of a l-chordal polygon, i.e. $\underline{A}$ is (k,f,l)-chordal if*

1. *$f:R_+\to R_+$ is convex and monotonously nondecreasing,*
2. $$\arcsin\frac{f\left(\frac{2k}{n}a^\star\right)}{f(a^\star)}>\frac{l\pi}{n}. \tag{3}$$

Proof. It is clear that if we prove the existence of some angles $\beta_i,\ i=\overline{1,n}$ that satisfies

$$\beta_1+\beta_2+\cdots+\beta_n=(n-2l)\frac{\pi}{2} \tag{4}$$

$$\frac{\cos\beta_1}{f(a_1)}=\cdots=\frac{\cos\beta_n}{f(a_n)}, \tag{5}$$

then $f(\underline{A})$ is obviously l-chordal, i.e. $\underline{A}$ is (k,f,l)-chordal.

For this purpose consider

$$L:=\sum_{i=1}^{n}\arcsin\frac{f(a_i)}{f(a^\star)}.$$

As the main part of $\arcsin x$ is convex when $x\in(0,1)$, and f is convex as well, by the assumptions of the Theorem, it follows that

$$\begin{aligned} L &\ge n\arcsin\left\{\frac{1}{nf(a^\star)}\sum_{i=1}^{n}f(a_i)\right\}\ge n\arcsin\frac{f\left(\frac{1}{n}\sum_{i=1}^{n}a_i\right)}{f(a^\star)}\\ &> n\arcsin\frac{f\left(\frac{2k}{n}a^\star\right)}{f(a^\star)}>\frac{l\pi}{n}. \end{aligned} \tag{6}$$

Here we use the fact that $\underline{A}$ is k-chordal, that means it holds (2), and the arcsine functions are always well-defined, since f is nondecreasing. Therefore (6) ensures the existence of the unique $\theta\in(0,1)$ so that

$$\sum_{i=1}^{n}\arcsin\left\{\frac{f(a_i)}{f(a^\star)}\,\theta\right\}=l\pi. \tag{7}$$

But this is equivalent to

$$\sum_{i=1}^{n}\arccos\left\{\frac{f(a_i)}{f(a^\star)}\,\theta\right\}=(n-2l)\frac{\pi}{2}.$$

There is also an unique $\beta^\star \in (0, \frac{\pi}{2})$ corresponding to the maximal side length $f(a^\star)$ that $\theta = \cos\beta^\star$. Consecutively putting

$$\beta_i = \arccos\left\{\frac{f(a_i)}{f(a^\star)}\ \cos\beta^\star\right\},\ \ i = \overline{1,n},$$

we get the angles β_i of a l-chordal polygon the sides of which are of the lengths $f(a_i)$, $i = \overline{1,n}$ so that (4) and (5) are valid. Thus, the proof is complete. **Q.E.D.**

In the continuation we formulate a modest improvement of the previous Theorem, but with more weeker assumptions. Namely, we have not assumed the k -chordality of the initial polygon $\underline{A}$, but the price of this fact has to be paid by some additional conditions upon f.

Theorem 2.2. *Let $a_1, \cdots, a_n$ be any given lengths with the property (2), i.e. so that $a_1 + \cdots + a_n > 2ka^\star$. Then $\underline{A}$ is (j, f, l)-chordal, $j = \overline{1,k}$ if*

1. *$f : R_+ \to R_+$, is convex, monotonously increasing,*
2. *$f(0) = 0$,*
3.

$$\arcsin\frac{f\left(\frac{2k}{n}a^\star\right)}{f(a^\star)} > \frac{l\pi}{n}. \tag{8}$$

Proof. The first step in proving procedure is completely the same as in the previous Theorem. Therefore $f(\underline{A})$ is clearly l-chordal. Now it remains to prove that $\underline{A}$ is k-chordal under the assumed conditions. For this purpose let us consider (8). This inequality is equivalent to

$$f\left(\frac{2k}{n}a^\star\right) > f(a^\star)\sin\frac{l\pi}{n}. \tag{9}$$

It is not hard to see that f is star-like with respect to the origin, because it is convex and $f(0) = 0$ [1]. Therefore it is

$$f\left(\frac{2k}{n}a^\star\right) > f(a^\star)\sin\frac{l\pi}{n} \geq f\left(a^\star\sin\frac{l\pi}{n}\right). \tag{10}$$

Furthermore, since f is bijective too, i.e. f is convex on a compact and it is increasing, by (10) we conlude that

$$\arcsin\frac{2k}{n} > \frac{l\pi}{n}. \tag{11}$$

Now, repeating the procedure for finding the exact angles β_i so that (1) holds and

$$\frac{\cos\beta_i}{a_i} = \frac{\cos\beta_j}{a_j};\ \ 1 \leq i < j \leq n,$$

[1] Speaking in the convexity hierarchy terminology, we assume that $f \in K_1(1)$. The $K_3(1)$ class consists of all functions being star-like with respect to the origin. So, as $K_1(1) \subset K_3(1)$consult e.g. [2], the star-likeness of f is obvious.

exposed in the proof of the Theorem 2.1., we finish the proof. **Q.E.D.**

Finally, we give a result in some way reverse to the Theorem 2.1. Namely, we are looking for the broadest possible class of functions so that a condition like (2) e.g.

$$a_1 + \cdots + a_n > \gamma(k,n)a^\star$$

suffices for the existence of the k-chordal polygon with the side lengths $f(a_i),\ i = \overline{1,n}$. The existence of such constant $\gamma(k,n)$ is shown in [4], and it has to be $\gamma(k,n) \geq n\sin\frac{k\pi}{n}$, cf. [4], Theorem 3.4.

Theorem 2.3. *Let $a_1, a_2, \cdots, a_n$ be any given lenghts so that*

$$f(a_1) + f(a_2) + \cdots + f(a_n) > \gamma(k,n)f(a^\star), \tag{12}$$

where $k \in N$ and $k \leq \left[\frac{n-1}{2}\right]$. Assume $\gamma(k,n) \geq n\sin\frac{k\pi}{n}$. If the positive increasing function f is star-like with respect to the origin, then there is a (k,f,k)-chordal polygon whose sides have lengths $a_1, a_2, \cdots, a_n$.

Proof. By the convexity of the arcsine function we conclude:

$$\sum_{i=1}^{n} \arcsin\frac{f(a_i)}{f(a^\star)} \geq n\arcsin\left\{\frac{1}{nf(a^\star)}\sum_{i=1}^{n} f(a_i)\right\} \geq n\arcsin\frac{\gamma(k,n)}{n} \geq k\pi. \tag{13}$$

Now, it follows that there is the unique $\theta_1 \in (0,1)$, so that by (13) we get

$$\sum_{i=1}^{n} \arcsin\left\{\frac{f(a_i)}{f(a^\star)}\,\theta_1\right\} = k\pi;$$

thus $f(\underline{A})$ is k-chordal.

In the same time it is

$$f(a_i) = f\left(\frac{a_i}{a^\star}\cdot a^\star\right) \leq \frac{a_i}{a^\star}f(a^\star), \tag{14}$$

henceforth f is star-like with respect to the origin. That means

$$\begin{aligned} \gamma(k,n) &< \frac{f(a_1) + \cdots + f(a_n)}{f(a^\star)} \\ &\leq \frac{\frac{a_1}{a^\star}f(a^\star) + \cdots + \frac{a_n}{a^\star}f(a^\star)}{f(a^\star)} \\ &= \sum_{i=1}^{n}\frac{a_i}{a^\star} < \sum_{i=1}^{n}\arcsin\frac{a_i}{a^\star}. \end{aligned} \tag{15}$$

Finally the k-chordality of $\underline{A}$ immediately follows from (15), or, what is the same, $\underline{A}$ is (k,f,k)-chordal. **Q.E.D.**

Remark 2.1 Taking $f(x) = x^\lambda$ the Theorems 2.1., 2.2. and 2.3. give the main results of the articles [3], [4] respectively as $\lambda = 1$ and $\lambda \geq 1$.

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