Torsion, Curvature and Deflection d-Tensors on $J^1(T, M)$

Mircea Neagu and Constantin Udrişte

Abstract

The aim of this paper is twofold. On the one hand, to study the local representations of d-connections, d-torsions and d-curvatures with respect to adapted bases produced by a nonlinear connection Γ on the jet fibre bundle of order one $J^1(T, M) \to T \times M$. On the other hand, to open the problem of prolongations of tensors and connections from a product of two manifolds to 1-jet fibre bundle associated to these manifolds.

Section 2 defines the Γ-linear connection on $J^1(T, M)$, and determines its nine local components. Section 3 studies the main twelve components of torsion d-tensor field, and Section 4 describes the eighteen components of curvature d-tensor field, induced by the Γ-linear connection. Section 5 introduce the deflection d-tensors, and, using the Ricci identities, describes the relations that they must satisfy. Section 6 studies the problem of prolongation of vector fields from $T \times M$ to 1-jet space $J^1(T, M)$.

Mathematics Subject Classification: 53C07, 53C43, 53C99
Key words: 1-jet fibre bundle, nonlinear connection, Γ-linear connection, torsion and curvature d-tensor fields, deflection d-tensors.

1 Introduction

It is well known the jet fibre bundle of order one $J^1(T, M) \to T \times M$ is a basic object in the study of classical and quantum fields [6]. For that reason, many researchers were studying the differential geometry of this space, in the sense of connections, torsions and curvatures. In this direction, the book [7] developes the geometry of jet fibre bundles of arbitrary orders, which is characterized by a global approach of geometrical objects that are involved. In other way, using as a pattern the geometrical methods from the theory of Lagrange spaces, the book [2] studies the geometry of the particular jet fibre bundle $J^1(\mathbb{R}, M) \equiv \mathbb{R} \times \boldsymbol{T}M \to M$, in the sense of d-connections, d-torsions and d-curvatures. Also, some interesting geometrical aspects of the jet fibre bundle $J^1(\mathbb{R}, M) \equiv \mathbb{R} \times \boldsymbol{T}M \to \mathbb{R} \times M$ are exposed in [9].

Balkan Journal of Geometry and Its Applications, Vol.6, No.1, 2001, pp. 29-44

Extending geometrical results from [2] and [9], and using Lagrangian geometrical methods, our paper analyses the particular local features of the geometrical objects on 1-jet fibre bundle $J^1(T,M)$, produced by a nonlinear connection. This study was imposed by the construction of the contravariant geometrization of multi-time Lagrangians [5], required by important physical domains: theory of bosonic strings [1] or elasticity [6]. At the same time, the authors believe that differential geometry of 1-jet bundles is a very fruitful domain of mathematics, not only because this geometry allows a clear insight to mathematical physics concepts, but also provides many new ideas, suitable for the geometrical theory of PDEs [8].

2 Components of Γ-linear connections on $J^1(T,M)$

Let us consider T (resp. M) a "*temporal*" (resp. "*spatial*") manifold of dimension p (resp. n), coordinated by $(t^\alpha)_{\alpha=\overline{1,p}}$ (resp. $(x^i)_{i=\overline{1,n}}$). Let $J^1(T,M)\to T\times M$ be the jet fibre bundle of order one associated to these manifolds. The *bundle of configurations* $J^1(T,M)$ has the coordinates $(t^\alpha,x^i,x^i_\alpha)$, where $\alpha=\overline{1,p}$ and $i=\overline{1,n}$. Note that, throughout this paper, the indices $\alpha,\beta,\gamma,\dots$ run from 1 to p, and the indices $i,j,k,\dots$ run from 1 to n.

Let us fix a nonlinear connection Γ on $E=J^1(T,M)$, defined by the *temporal* components $M^{(i)}_{(\alpha)\beta}$ and the *spatial* components $N^{(i)}_{(\alpha)j}$. The transformation rules of the local components of the nonlinear connection Γ are expressed by [4]

$$(2.1)\qquad \begin{aligned} \tilde{M}^{(j)}_{(\beta)\mu}\frac{\partial\tilde{t}^\mu}{\partial t^\alpha} &= M^{(k)}_{(\gamma)\alpha}\frac{\partial\tilde{x}^j}{\partial x^k}\frac{\partial t^\gamma}{\partial\tilde{t}^\beta}-\frac{\partial\tilde{x}^j_\beta}{\partial t^\alpha}\ ,\\ \tilde{N}^{(j)}_{(\beta)k}\frac{\partial\tilde{x}^k}{\partial x^i} &= N^{(k)}_{(\gamma)i}\frac{\partial\tilde{x}^j}{\partial x^k}\frac{\partial t^\gamma}{\partial\tilde{t}^\beta}-\frac{\partial\tilde{x}^j_\beta}{\partial x^i}\ . \end{aligned}$$

Let $\left\{\frac{\delta}{\delta t^\alpha},\frac{\delta}{\delta x^i},\frac{\partial}{\partial x^i_\alpha}\right\}\subset\mathcal{X}(E)$ and $\{dt^\alpha,dx^i,\delta x^i_\alpha\}\subset\mathcal{X}^*(E)$ be the dual adapted bases produced by the nonlinear connection $\Gamma=(M^{(i)}_{(\alpha)\beta},N^{(i)}_{(\alpha)j})$, via the formulas

$$(2.2)\qquad \begin{aligned} \frac{\delta}{\delta t^\alpha} &= \frac{\partial}{\partial t^\alpha}-M^{(j)}_{(\beta)\alpha}\frac{\partial}{\partial x^j_\beta}\ ,\\ \frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i}-N^{(j)}_{(\beta)i}\frac{\partial}{\partial x^j_\beta}\ ,\\ \delta x^i_\alpha &= dx^i_\alpha+M^{(i)}_{(\alpha)\beta}dt^\beta+N^{(i)}_{(\alpha)j}dx^j\ . \end{aligned}$$

These bases are suitable for the description of geometrical objects on E, because their transformation laws have the following simple form [4]:

$$(2.3)\qquad \begin{aligned} &\frac{\delta}{\delta t^\alpha}=\frac{\partial\tilde{t}^\beta}{\partial t^\alpha}\frac{\delta}{\delta\tilde{t}^\beta}\ ,\quad \frac{\delta}{\delta x^i}=\frac{\partial\tilde{x}^j}{\partial x^i}\frac{\delta}{\delta\tilde{x}^j}\ ,\quad \frac{\partial}{\partial x^i_\alpha}=\frac{\partial\tilde{x}^j}{\partial x^i}\frac{\partial t^\alpha}{\partial\tilde{t}^\beta}\frac{\partial}{\partial\tilde{x}^j_\beta}\ ,\\ &dt^\alpha=\frac{\partial t^\alpha}{\partial\tilde{t}^\beta}d\tilde{t}^\beta,\qquad dx^i=\frac{\partial x^i}{\partial\tilde{x}^j}d\tilde{x}^j,\qquad \delta x^i_\alpha=\frac{\partial x^i}{\partial\tilde{x}^j}\frac{\partial\tilde{t}^\beta}{\partial t^\alpha}\delta\tilde{x}^j_\beta\ . \end{aligned}$$

In order to develope the theory of Γ-linear connections on $E = J^1(T,M)$, we need the following

Proposition 2.1 *i) The Lie algebra $\mathcal{X}(E)$ of vector fields decomposes as*

$$\mathcal{X}(E) = \mathcal{X}(\mathcal{H}_T) \oplus \mathcal{X}(\mathcal{H}_M) \oplus \mathcal{X}(\mathcal{V}),$$

where

$$\mathcal{X}(\mathcal{H}_T) = Span\left\{\frac{\delta}{\delta t^\alpha}\right\}, \quad \mathcal{X}(\mathcal{H}_M) = Span\left\{\frac{\delta}{\delta x^i}\right\}, \quad \mathcal{X}(\mathcal{V}) = Span\left\{\frac{\partial}{\partial x^i_\alpha}\right\}.$$

ii) The Lie algebra $\mathcal{X}^*(E)$ of covector fields decomposes as

$$\mathcal{X}^*(E) = \mathcal{X}^*(\mathcal{H}_T) \oplus \mathcal{X}^*(\mathcal{H}_M) \oplus \mathcal{X}^*(\mathcal{V}),$$

where

$$\mathcal{X}^*(\mathcal{H}_T) = Span\{dt^\alpha\}, \quad \mathcal{X}^*(\mathcal{H}_M) = Span\{dx^i\}, \quad \mathcal{X}^*(\mathcal{V}) = Span\{\delta x^i_\alpha\}.$$

Let us consider h_T, h_M (horizontal) and v (vertical) as the canonical projections of the above decompositions. In this context, we have

Corollary 2.2 *i) Any vector field X can be written in the form*

$$X = h_T X + h_M X + vX.$$

ii) Any covector field ω can be written in the form

$$\omega = h_T\omega + h_M\omega + v\omega.$$

Definition 2.1 A linear connection $\nabla : \mathcal{X}(E) \times \mathcal{X}(E) \to \mathcal{X}(E)$ is called *a Γ-linear connection on E* if $\nabla h_T = 0$, $\nabla h_M = 0$ and $\nabla v = 0$.

In order to describe, in local terms, a Γ-linear connection ∇ on E, we need nine unique local components,

$$\nabla\Gamma = (\bar{G}^\alpha_{\beta\gamma},\ G^k_{i\gamma},\ G^{(i)(\beta)}_{(\alpha)(j)\gamma},\ \bar{L}^\alpha_{\beta j},\ L^k_{ij},\ L^{(i)(\beta)}_{(\alpha)(j)k},\ \bar{C}^{\alpha(\gamma)}_{\beta(k)},\ C^{j(\gamma)}_{i(k)},\ C^{(i)(\beta)(\gamma)}_{(\alpha)(j)(k)}), \tag{2.4}$$

which are defined by the relations

$$(h_T) \quad \nabla_{\frac{\delta}{\delta t^\gamma}}\frac{\delta}{\delta t^\beta} = \bar{G}^\alpha_{\beta\gamma}\frac{\delta}{\delta t^\alpha}\ ,\quad \nabla_{\frac{\delta}{\delta t^\gamma}}\frac{\delta}{\delta x^i} = G^k_{i\gamma}\frac{\delta}{\delta x^k}\ ,\quad \nabla_{\frac{\delta}{\delta t^\gamma}}\frac{\partial}{\partial x^i_\beta} = G^{(k)(\beta)}_{(\alpha)(i)\gamma}\frac{\partial}{\partial x^k_\alpha}\ ,$$

$$(h_M) \quad \nabla_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta t^\beta} = \bar{L}^\alpha_{\beta j}\frac{\delta}{\delta t^\alpha}\ ,\quad \nabla_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^i} = L^k_{ij}\frac{\delta}{\delta x^k}\ ,\quad \nabla_{\frac{\delta}{\delta x^j}}\frac{\partial}{\partial x^i_\beta} = L^{(k)(\beta)}_{(\alpha)(i)j}\frac{\partial}{\partial x^k_\alpha}\ ,$$

$$(v) \quad \nabla_{\frac{\partial}{\partial x^j_\gamma}}\frac{\delta}{\delta t^\beta} = \bar{C}^{\alpha(\gamma)}_{\beta(j)}\frac{\delta}{\delta t^\alpha}\ ,\quad \nabla_{\frac{\partial}{\partial x^j_\gamma}}\frac{\delta}{\delta x^i} = C^{k(\gamma)}_{i(j)}\frac{\delta}{\delta x^k}\ ,\quad \nabla_{\frac{\partial}{\partial x^j_\gamma}}\frac{\partial}{\partial x^i_\beta} = C^{(k)(\beta)(\gamma)}_{(\alpha)(i)(j)}\frac{\partial}{\partial x^k_\alpha}\ .$$

The transformation laws of the elements $\left\{\frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^i_\alpha}\right\}$, together with the properties of the Γ-linear connection ∇, imply

Theorem 2.3 *i) The components of the* Γ*-linear connection* ∇ *modify by the rules*

$$(h_T)\quad \begin{cases} \bar{G}^{\delta}_{\alpha\beta}\dfrac{\partial\tilde{t}^{\varepsilon}}{\partial t^{\delta}} = \tilde{\bar{G}}^{\varepsilon}_{\mu\gamma}\dfrac{\partial\tilde{t}^{\mu}}{\partial t^{\alpha}}\dfrac{\partial\tilde{t}^{\gamma}}{\partial t^{\beta}} + \dfrac{\partial^2\tilde{t}^{\varepsilon}}{\partial t^{\alpha}\partial t^{\beta}} \\ G^{k}_{i\gamma} = \tilde{G}^{m}_{j\beta}\dfrac{\partial x^k}{\partial\tilde{x}^m}\dfrac{\partial\tilde{x}^j}{\partial x^i}\dfrac{\partial\tilde{t}^{\beta}}{\partial t^{\gamma}} \\ G^{(k)(\beta)}_{(\gamma)(i)\alpha} = \tilde{G}^{(p)(\eta)}_{(\varepsilon)(j)\mu}\dfrac{\partial x^k}{\partial\tilde{x}^p}\dfrac{\partial\tilde{t}^{\varepsilon}}{\partial t^{\gamma}}\dfrac{\partial\tilde{x}^j}{\partial x^i}\dfrac{\partial t^{\beta}}{\partial\tilde{t}^{\eta}}\dfrac{\partial\tilde{t}^{\mu}}{\partial t^{\alpha}} + \delta^k_i\dfrac{\partial\tilde{t}^{\mu}}{\partial t^{\alpha}}\dfrac{\partial\tilde{t}^{\varepsilon}}{\partial t^{\gamma}}\dfrac{\partial^2 t^{\beta}}{\partial\tilde{t}^{\mu}\partial\tilde{t}^{\varepsilon}}, \end{cases}$$

$$(h_M)\quad \begin{cases} \bar{L}^{\gamma}_{\beta j}\dfrac{\partial x^j}{\partial\tilde{x}^l} = \tilde{\bar{L}}^{\eta}_{\mu l}\dfrac{\partial t^{\gamma}}{\partial\tilde{t}^{\eta}}\dfrac{\partial\tilde{t}^{\mu}}{\partial t^{\beta}} \\ L^{m}_{ij}\dfrac{\partial\tilde{x}^r}{\partial x^m} = \tilde{L}^{r}_{pq}\dfrac{\partial\tilde{x}^p}{\partial x^i}\dfrac{\partial\tilde{x}^q}{\partial x^j} + \dfrac{\partial^2\tilde{x}^r}{\partial x^i\partial x^j} \\ L^{(k)(\beta)}_{(\gamma)(i)j} = \tilde{L}^{(r)(\eta)}_{(\nu)(p)l}\dfrac{\partial x^k}{\partial\tilde{x}^r}\dfrac{\partial\tilde{t}^{\nu}}{\partial t^{\gamma}}\dfrac{\partial\tilde{x}^p}{\partial x^i}\dfrac{\partial t^{\beta}}{\partial\tilde{t}^{\eta}}\dfrac{\partial\tilde{x}^l}{\partial x^j} + \delta^{\beta}_{\gamma}\dfrac{\partial x^k}{\partial\tilde{x}^r}\dfrac{\partial^2\tilde{x}^r}{\partial x^i\partial x^j}, \end{cases}$$

$$(v)\quad \begin{cases} \bar{C}^{\gamma(\alpha)}_{\beta(i)} = \tilde{\bar{C}}^{\mu(\delta)}_{\varepsilon(j)}\dfrac{\partial t^{\gamma}}{\partial\tilde{t}^{\mu}}\dfrac{\partial\tilde{t}^{\varepsilon}}{\partial t^{\beta}}\dfrac{\partial\tilde{x}^j}{\partial x^i}\dfrac{\partial t^{\alpha}}{\partial\tilde{t}^{\delta}} \\ C^{k(\alpha)}_{i(j)} = \tilde{C}^{s(\beta)}_{p(r)}\dfrac{\partial x^k}{\partial\tilde{x}^s}\dfrac{\partial\tilde{x}^p}{\partial x^i}\dfrac{\partial\tilde{x}^r}{\partial x^j}\dfrac{\partial t^{\alpha}}{\partial\tilde{t}^{\beta}} \\ C^{(k)(\beta)(\alpha)}_{(\gamma)(i)(j)} = \tilde{C}^{(r)(\mu)(\nu)}_{(\varepsilon)(p)(q)}\dfrac{\partial x^k}{\partial\tilde{x}^r}\dfrac{\partial\tilde{t}^{\varepsilon}}{\partial t^{\gamma}}\dfrac{\partial\tilde{x}^p}{\partial x^i}\dfrac{\partial t^{\beta}}{\partial\tilde{t}^{\mu}}\dfrac{\partial\tilde{x}^q}{\partial x^j}\dfrac{\partial t^{\alpha}}{\partial\tilde{t}^{\nu}}. \end{cases}$$

ii) Conversely, to give a Γ*-linear connection* ∇ *on the 1-jet space* E *is equivalent to give a set of nine local components (2.4), whose local transformations laws are described in i.*

Using Theorem 2.3, we can construct a natural and interesting example of Γ-linear connection on the 1-jet space E.

Example 2.1 Suppose that $h_{\alpha\beta}(t)$ (resp. $\varphi_{ij}(x)$) is a pseudo-Riemannian metric on T (resp. M). We denote $H^{\gamma}_{\alpha\beta}$ (resp. γ^k_{ij}) the Christoffel symbols of the metric $h_{\alpha\beta}$ (resp. φ_{ij}). The *canonical nonlinear connection* Γ_0 *associated to these metrics* is defined by the local components [4]

$$M^{(i)}_{(\alpha)\beta} = -H^{\gamma}_{\alpha\beta}x^i_{\gamma}\ ,\quad N^{(i)}_{(\alpha)j} = \gamma^i_{jm}x^m_{\alpha}\ . \tag{2.5}$$

In this context, using the well known local transformation rules of the Christoffel symbols $H^{\gamma}_{\alpha\beta}$ and γ^i_{jk} and setting

$$\bar{G}^{\gamma}_{\alpha\beta} = H^{\gamma}_{\alpha\beta}\ ,\ \ G^{(k)(\beta)}_{(\gamma)(i)\alpha} = -\delta^k_i H^{\beta}_{\alpha\gamma}\ ,\ \ L^k_{ij} = \gamma^k_{ij}\ ,\ \ L^{(k)(\beta)}_{(\gamma)(i)j} = \delta^{\beta}_{\gamma}\gamma^k_{ij}\ , \tag{2.6}$$

we conclude that the set of local components

$$B\Gamma_0 = (H^{\gamma}_{\alpha\beta},\ 0,\ G^{(k)(\beta)}_{(\gamma)(i)\alpha},\ 0,\ \gamma^k_{ij},\ L^{(k)(\beta)}_{(\gamma)(i)j},\ 0,\ 0,\ 0)$$

defines a Γ_0-linear connection. This is called the *Berwald connection attached to the pair of metrics* $(h_{\alpha\beta},\varphi_{ij})$.

Remark 2.1 In the particular case $(T,h) = (I\!R,\delta)$, the Berwald connection is a natural generalization of the canonical nonlinear connection induced by the spray $2G^i = \gamma^i_{jk}y^jy^k$, from the classical theory of Lagrange spaces. For more details, the reader is invited to consult [2], [4].

Now, let ∇ be a Γ-linear connection on E, locally defined by (2.4). The linear connection ∇ induces a natural linear connection on the d-tensors set of the jet fibre bundle $E = J^1(T,M)$, in the following fashion: starting with a vector field X and a d-tensor field D locally expressed by

$$X = X^\alpha \frac{\delta}{\delta t^\alpha} + X^m \frac{\delta}{\delta x^m} + X^{(m)}_{(\alpha)} \frac{\partial}{\partial x^m_\alpha},$$

$$D = D^{\alpha i(j)(\delta)\dots}_{\gamma k(\beta)(l)\dots} \frac{\delta}{\delta t^\alpha} \otimes \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial x^j_\beta} \otimes dt^\gamma \otimes dx^k \otimes \delta x^l_\delta \dots,$$

we introduce the covariant derivative

$$\nabla_X D = X^\varepsilon \nabla_{\frac{\delta}{\delta t^\varepsilon}} D + X^p \nabla_{\frac{\delta}{\delta x^p}} D + X^{(p)}_{(\varepsilon)} \nabla_{\frac{\partial}{\partial x^p_\varepsilon}} D = \Big\{ X^\varepsilon D^{\alpha i(j)(\delta)\dots}_{\gamma k(\beta)(l)\dots/\varepsilon} + X^p D^{\alpha i(j)(\delta)\dots}_{\gamma k(\beta)(l)\dots|p} + X^{(p)}_{(\varepsilon)} D^{\alpha i(j)(\delta)\dots}_{\gamma k(\beta)(l)\dots}|^{(\varepsilon)}_{(p)} \Big\} \frac{\delta}{\delta t^\alpha} \otimes \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial x^j_\beta} \otimes dt^\gamma \otimes dx^k \otimes \delta x^l_\delta \dots,$$

where

$$(h_T)\quad \begin{cases} D^{\alpha i(j)(\delta)\dots}_{\gamma k(\beta)(l)\dots/\varepsilon} = \dfrac{\delta D^{\alpha i(j)(\delta)\dots}_{\gamma k(\beta)(l)\dots}}{\delta t^\varepsilon} + D^{\mu i(j)(\delta)\dots}_{\gamma k(\beta)(l)\dots} \bar{G}^\alpha_{\mu\varepsilon} + \\ + D^{\alpha m(j)(\delta)\dots}_{\gamma k(\beta)(l)\dots} G^i_{m\varepsilon} + D^{\alpha i(m)(\delta)\dots}_{\gamma k(\mu)(l)\dots} G^{(j)(\mu)}_{(\beta)(m)\varepsilon} + \dots - \\ - D^{\alpha i(j)(\delta)\dots}_{\mu k(\beta)(l)\dots} \bar{G}^\mu_{\gamma\varepsilon} - D^{\alpha i(j)(\delta)\dots}_{\gamma m(\beta)(l)\dots} G^m_{k\varepsilon} - D^{\alpha i(j)(\mu)\dots}_{\gamma k(\beta)(m)\dots} G^{(m)(\delta)}_{(\mu)(l)\varepsilon} - \dots, \end{cases}$$

$$(h_M)\quad \begin{cases} D^{\alpha i(j)(\delta)\dots}_{\gamma k(\beta)(l)\dots|p} = \dfrac{\delta D^{\alpha i(j)(\delta)\dots}_{\gamma k(\beta)(l)\dots}}{\delta x^p} + D^{\mu i(j)(\delta)\dots}_{\gamma k(\beta)(l)\dots} \bar{L}^\alpha_{\mu p} + \\ + D^{\alpha m(j)(\delta)\dots}_{\gamma k(\beta)(l)\dots} L^i_{mp} + D^{\alpha i(m)(\delta)\dots}_{\gamma k(\mu)(l)\dots} L^{(j)(\mu)}_{(\beta)(m)p} + \dots - \\ - D^{\alpha i(j)(\delta)\dots}_{\mu k(\beta)(l)\dots} \bar{L}^\mu_{\gamma p} - D^{\alpha i(j)(\delta)\dots}_{\gamma m(\beta)(l)\dots} L^m_{kp} - D^{\alpha i(j)(\mu)\dots}_{\gamma k(\beta)(m)\dots} L^{(m)(\delta)}_{(\mu)(l)p} - \dots, \end{cases}$$

$$(v)\quad \begin{cases} D^{\alpha i(j)(\delta)\dots}_{\gamma k(\beta)(l)\dots}|^{(\varepsilon)}_{(p)} = \dfrac{\partial D^{\alpha i(j)(\delta)\dots}_{\gamma k(\beta)(l)\dots}}{\partial x^p_\varepsilon} + D^{\mu i(j)(\delta)\dots}_{\gamma k(\beta)(l)\dots} \bar{C}^{\alpha(\varepsilon)}_{\mu(p)} + \\ + D^{\alpha m(j)(\delta)\dots}_{\gamma k(\beta)(l)\dots} C^{i(\varepsilon)}_{m(p)} + D^{\alpha i(m)(\delta)\dots}_{\gamma k(\mu)(l)\dots} C^{(j)(\mu)(\varepsilon)}_{(\beta)(m)(p)} + \dots - \\ - D^{\alpha i(j)(\delta)\dots}_{\mu k(\beta)(l)\dots} \bar{C}^{\mu(\varepsilon)}_{\gamma(p)} - D^{\alpha i(j)(\delta)\dots}_{\gamma m(\beta)(l)\dots} C^{m(\varepsilon)}_{k(p)} - D^{\alpha i(j)(\mu)\dots}_{\gamma k(\beta)(m)\dots} C^{(m)(\delta)(\varepsilon)}_{(\mu)(l)(p)} - \dots. \end{cases}$$

Definition 2.2 The local operators "$_{/\varepsilon}$", "$_{|p}$" or "$|^{(\varepsilon)}_{(p)}$" are called the *T-horizontal covariant derivative, M-horizontal covariant derivative* or vertical covariant derivative or *h_T-, h_M- and v- covariant derivatives* of the Γ-linear connection ∇.

Remarks 2.2 i) In the particular case of a function $f(t^\gamma, x^k, x^k_\gamma)$ on $J^1(T,M)$, the above covariant derivatives reduce to

$$(2.7)\qquad \begin{aligned} f_{/\varepsilon} &= \frac{\delta f}{\delta t^\varepsilon} = \frac{\partial f}{\partial t^\varepsilon} - M^{(k)}_{(\gamma)\varepsilon}\frac{\partial f}{\partial x^k_\gamma}\ ,\\ f_{|p} &= \frac{\delta f}{\delta x^p} = \frac{\partial f}{\partial x^p} - N^{(k)}_{(\gamma)p}\frac{\partial f}{\partial x^k_\gamma}\ ,\\ f|^{(\varepsilon)}_{(p)} &= \frac{\partial f}{\partial x^p_\varepsilon}\ . \end{aligned}$$

ii) Particularly, starting with a d-vector field $D = Y$ on $J^1(T,M)$, locally expressed by

$$Y = Y^\alpha \frac{\delta}{\delta t^\alpha} + Y^i \frac{\delta}{\delta x^i} + Y^{(i)}_{(\alpha)}\frac{\partial}{\partial x^i_\alpha},$$

the following expressions of above covariant derivatives hold good:

$$(h_T)\qquad \begin{cases} Y^\alpha_{/\varepsilon} = \dfrac{\delta Y^\alpha}{\delta t^\varepsilon} + Y^\mu \bar{G}^\alpha_{\mu\varepsilon}\\[2mm] Y^i_{/\varepsilon} = \dfrac{\delta Y^i}{\delta t^\varepsilon} + Y^m G^i_{m\varepsilon}\\[2mm] Y^{(i)}_{(\alpha)/\varepsilon} = \dfrac{\delta Y^{(i)}_{(\alpha)}}{\delta t^\varepsilon} + Y^{(m)}_{(\mu)} G^{(i)(\mu)}_{(\alpha)(m)\varepsilon}\ , \end{cases}$$

$$(h_M)\qquad \begin{cases} Y^\alpha_{|p} = \dfrac{\delta Y^\alpha}{\delta x^p} + Y^\mu \bar{L}^\alpha_{\mu p}\\[2mm] Y^i_{|p} = \dfrac{\delta Y^i}{\delta x^p} + Y^m L^i_{mp}\\[2mm] Y^{(i)}_{(\alpha)|p} = \dfrac{\delta Y^{(i)}_{(\alpha)}}{\delta x^p} + Y^{(m)}_{(\mu)} L^{(i)(\mu)}_{(\alpha)(m)p}\ , \end{cases}$$

$$(v)\qquad \begin{cases} Y^\alpha|^{(\varepsilon)}_{(p)} = \dfrac{\partial Y^\alpha}{\partial x^p_\varepsilon} + Y^\mu \bar{C}^{\alpha(\varepsilon)}_{\mu(p)}\\[2mm] Y^i|^{(\varepsilon)}_{(p)} = \dfrac{\partial Y^i}{\partial x^p_\varepsilon} + Y^m C^{i(\varepsilon)}_{m(p)}\\[2mm] Y^{(i)}_{(\alpha)}|^{(\varepsilon)}_{(p)} = \dfrac{\partial Y^{(i)}_{(\alpha)}}{\partial x^p_\varepsilon} + Y^{(m)}_{(\mu)} C^{(i)(\mu)(\varepsilon)}_{(\alpha)(m)(p)}\ . \end{cases}$$

iii) The local covariant derivatives associated to the Berwald Γ_0-linear connection, will be denoted by "$_{//\varepsilon}$", "$_{\|p}$" and "$\|^{(\varepsilon)}_{(p)}$".

Denoting by "$_{:A}$" one of local covariant derivatives "$_{/\varepsilon}$", "$_{|p}$" or "$|^{(\varepsilon)}_{(p)}$", we easily deduce

Proposition 2.4 *If $D^{\cdots}_{\cdots}$ and $F^{\cdots}_{\cdots}$ are two d-tensor fields on E, then the following statements hold good:*

i) $D^{\cdots}_{\cdots:A}$ are the components of a new d-tensor field;

ii) $(D^{\cdots}_{\cdots} + F^{\cdots}_{\cdots})_{:A} = D^{\cdots}_{\cdots:A} + F^{\cdots}_{\cdots:A}$;

iii) $(D^{\cdots}_{\cdots} \otimes F^{\cdots}_{\cdots})_{:A} = D^{\cdots}_{\cdots:A} \otimes F^{\cdots}_{\cdots} + D^{\cdots}_{\cdots} \otimes F^{\cdots}_{\cdots:A}$;
iv) The operator "$_{:A}$" commutes with the contraction operation.

3 Components of d-torsion on $J^1(T,M)$

Let us consider a Γ-linear connection ∇ on $E = J^1(T,M)$, defined by the local components (2.4). Obviously, the torsion d-tensor field associated to ∇ is

$$\mathbf{T} : \mathcal{X}(E) \times \mathcal{X}(E) \to \mathcal{X}(E), \quad \mathbf{T}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \ \forall\, X,Y \in \mathcal{X}(E).$$

To characterize locally the torsion d-tensor $\mathbf{T}$ of the connection ∇, we need the next

Proposition 3.1 *The following bracket identities are true:*

$$\left[\frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta t^\beta}\right] = R^{(m)}_{(\mu)\alpha\beta}\frac{\partial}{\partial x^m_\mu}, \qquad \left[\frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^j}\right] = R^{(m)}_{(\mu)\alpha j}\frac{\partial}{\partial x^m_\mu},$$

$$\left[\frac{\delta}{\delta t^\alpha}, \frac{\partial}{\partial x^j_\beta}\right] = \frac{\partial M^{(m)}_{(\mu)\alpha}}{\partial x^j_\beta}\frac{\partial}{\partial x^m_\mu}, \quad \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = R^{(m)}_{(\mu)ij}\frac{\partial}{\partial x^m_\mu},$$

$$\left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^j_\beta}\right] = \frac{\partial N^{(m)}_{(\mu)i}}{\partial x^j_\beta}\frac{\partial}{\partial x^m_\mu}, \quad \left[\frac{\partial}{\partial x^i_\alpha}, \frac{\partial}{\partial x^j_\beta}\right] = 0,$$

where $M^{(m)}_{(\mu)\alpha}$ *and* $N^{(m)}_{(\mu)i}$ *are the local components of the nonlinear connection* Γ*, while the components* $R^{(m)}_{(\mu)\alpha\beta}$, $R^{(m)}_{(\mu)\alpha j}$, $R^{(m)}_{(\mu)ij}$ *are d-tensors expressed by*

$$R^{(m)}_{(\mu)\alpha\beta} = \frac{\delta M^{(m)}_{(\mu)\alpha}}{\delta t^\beta} - \frac{\delta M^{(m)}_{(\mu)\beta}}{\delta t^\alpha}, \ R^{(m)}_{(\mu)\alpha j} = \frac{\delta M^{(m)}_{(\mu)\alpha}}{\delta x^j} - \frac{\delta N^{(m)}_{(\mu)j}}{\delta t^\alpha}, \ R^{(m)}_{(\mu)ij} = \frac{\delta N^{(m)}_{(\mu)i}}{\delta x^j} - \frac{\delta N^{(m)}_{(\mu)j}}{\delta x^i}.$$

As a consequence, by direct local computations, we deduce that the torsion d-tensor field $\mathbf{T}$ of the Γ-linear connection ∇ may be locally described by

Theorem 3.2 *The torsion d-tensor* $\mathbf{T}$ *of the* Γ*-linear connection* ∇ *is determined by the following local expressions:*

$$h_T\mathbf{T}\left(\frac{\delta}{\delta t^\beta}, \frac{\delta}{\delta t^\alpha}\right) = \bar{T}^\mu_{\alpha\beta}\frac{\delta}{\delta t^\mu}, \quad h_M\mathbf{T}\left(\frac{\delta}{\delta t^\beta}, \frac{\delta}{\delta t^\alpha}\right) = 0,$$

$$v\mathbf{T}\left(\frac{\delta}{\delta t^\beta}, \frac{\delta}{\delta t^\alpha}\right) = R^{(m)}_{(\mu)\alpha\beta}\frac{\partial}{\partial x^m_\mu},$$

$$h_T\mathbf{T}\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta t^\alpha}\right) = \bar{T}^\mu_{\alpha j}\frac{\delta}{\delta t^\mu}, \quad h_M\mathbf{T}\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta t^\alpha}\right) = T^m_{\alpha j}\frac{\delta}{\delta x^m},$$

$$v\mathbf{T}\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta t^\alpha}\right) = R^{(m)}_{(\mu)\alpha j}\frac{\partial}{\partial x^m_\mu},$$

$$h_T\mathbf{T}\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^i}\right)=0,\quad h_M\mathbf{T}\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^i}\right)=T^m_{ij}\frac{\delta}{\delta x^m},$$

$$v\mathbf{T}\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^i}\right)=R^{(m)}_{(\mu)ij}\frac{\partial}{\partial x^m_\mu},$$

$$h_T\mathbf{T}\left(\frac{\partial}{\partial x^j_\beta},\frac{\delta}{\delta t^\alpha}\right)=\bar{P}^{\mu(\beta)}_{\alpha(j)}\frac{\delta}{\delta t^\mu},\quad h_M\mathbf{T}\left(\frac{\partial}{\partial x^j_\beta},\frac{\delta}{\delta t^\alpha}\right)=0,$$

$$v\mathbf{T}\left(\frac{\partial}{\partial x^j_\beta},\frac{\delta}{\delta t^\alpha}\right)=P^{(m)\ (\beta)}_{(\mu)\alpha(j)}\frac{\partial}{\partial x^m_\mu},$$

$$h_T\mathbf{T}\left(\frac{\partial}{\partial x^j_\beta},\frac{\delta}{\delta x^i}\right)=0,\quad h_M\mathbf{T}\left(\frac{\partial}{\partial x^j_\beta},\frac{\delta}{\delta t^\alpha}\right)=P^{m(\beta)}_{i(j)}\frac{\delta}{\delta x^m},$$

$$v\mathbf{T}\left(\frac{\partial}{\partial x^j_\beta},\frac{\delta}{\delta x^i}\right)=P^{(m)\ (\beta)}_{(\mu)i(j)}\frac{\partial}{\partial x^m_\mu},$$

$$h_T\mathbf{T}\left(\frac{\partial}{\partial x^j_\beta},\frac{\partial}{\partial x^i_\alpha}\right)=0,\quad h_M\mathbf{T}\left(\frac{\partial}{\partial x^j_\beta},\frac{\partial}{\partial x^i_\alpha}\right)=0,$$

$$v\mathbf{T}\left(\frac{\partial}{\partial x^j_\beta},\frac{\partial}{\partial x^i_\alpha}\right)=S^{(m)(\alpha)(\beta)}_{(\mu)(i)(j)}\frac{\partial}{\partial x^m_\mu},$$

where $R^{(m)}_{(\mu)\alpha\beta}$, $R^{(m)}_{(\mu)\alpha j}$, $R^{(m)}_{(\mu)ij}$ *are the d-tensors from Proposition 3.1, and* $\bar{T}^\mu_{\alpha\beta}=\bar{G}^\mu_{\alpha\beta}-\bar{G}^\mu_{\beta\alpha}$, $\bar{T}^\mu_{\alpha j}=\bar{L}^\mu_{\alpha j}$, $\bar{P}^{\mu(\beta)}_{\alpha(j)}=\bar{C}^{\mu(\beta)}_{\alpha(j)}$, $T^m_{\alpha j}=-G^m_{j\alpha}$, $T^m_{ij}=L^m_{ij}-L^m_{ji}$, $P^{m(\beta)}_{i(j)}=C^{m(\beta)}_{i(j)}$, $S^{(m)(\alpha)(\beta)}_{(\mu)(i)(j)}=C^{(m)(\alpha)(\beta)}_{(\mu)(i)(j)}-C^{(m)(\beta)(\alpha)}_{(\mu)(j)(i)}$, $P^{(m)\ (\beta)}_{(\mu)\alpha(j)}=\dfrac{\partial M^{(m)}_{(\mu)\alpha}}{\partial x^j_\beta}-G^{(m)(\beta)}_{(\mu)(j)\alpha}$, $P^{(m)\ (\beta)}_{(\mu)i(j)}=\dfrac{\partial N^{(m)}_{(\mu)i}}{\partial x^j_\beta}-L^{(m)(\beta)}_{(\mu)(j)i}$.

In other words, the torsion $\mathbf{T}$ *of the* Γ*-linear connection* ∇ *is determined by twelve effective adapted d-tensor fields that we arrange in the following table:*

(3.1)

	h_T	h_M	v
h_Th_T	$\bar{T}^\mu_{\alpha\beta}$	0	$R^{(m)}_{(\mu)\alpha\beta}$
h_Mh_T	$\bar{T}^\mu_{\alpha j}$	$T^m_{\alpha j}$	$R^{(m)}_{(\mu)\alpha j}$
h_Mh_M	0	T^m_{ij}	$R^{(m)}_{(\mu)ij}$
vh_T	$\bar{P}^{\mu(\beta)}_{\alpha(j)}$	0	$P^{(m)\ (\beta)}_{(\mu)\alpha(j)}$
vh_M	0	$P^{m(\beta)}_{i(j)}$	$P^{(m)\ (\beta)}_{(\mu)i(j)}$
vv	0	0	$S^{(m)(\alpha)(\beta)}_{(\mu)(i)(j)}$

Remark 3.1 In the particular case of the Berwald Γ_0-linear connection associated to the metrics $h_{\alpha\beta}$ and φ_{ij}, all torsion d-tensors vanish, except $R^{(m)}_{(\mu)\alpha\beta}=-H^\gamma_{\mu\alpha\beta}x^m_\gamma$, $R^{(m)}_{(\mu)ij}=r^m_{ijl}x^l_\mu$, where $H^\gamma_{\mu\beta\gamma}$ (resp. r^m_{ijl}) are the curvature tensors of the metric $h_{\alpha\beta}$ (resp. φ_{ij}).

4 Components of d-curvature on $J^1(T,M)$

From the general theory of linear connections, we recall that the curvature d-tensor field associated to the Γ-linear connection ∇ is $\mathbf{R}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - -\nabla_{[X,Y]}Z, \quad \forall\, X,Y,Z \in \mathcal{X}(E)$.

Using an adapted basis, the properties of the Γ-linear connection ∇ and of the d-tensor $\mathbf{R}$, together with the expressions of the local T- , M- horizontal and vertical covariant derivatives attached to the Γ-linear connection ∇, by complicated local computations, we find

Theorem 4.1 *The curvature d-tensor* $\mathbf{R}$ *of the* Γ*-linear connection* ∇ *is determined by the following local expressions:*

$$\mathbf{R}\left(\frac{\delta}{\delta t^\gamma},\frac{\delta}{\delta t^\beta}\right)\frac{\delta}{\delta t^\alpha} = \bar{R}^\delta_{\alpha\beta\gamma}\frac{\delta}{\delta t^\delta}\ , \quad \mathbf{R}\left(\frac{\delta}{\delta t^\gamma},\frac{\delta}{\delta t^\beta}\right)\frac{\delta}{\delta x^i} = R^l_{i\beta\gamma}\frac{\delta}{\delta x^l}\ ,$$

$$\mathbf{R}\left(\frac{\delta}{\delta t^\gamma},\frac{\delta}{\delta t^\beta}\right)\frac{\partial}{\partial x^i_\alpha} = R^{(l)(\alpha)}_{(\delta)(i)\beta\gamma}\frac{\partial}{\partial x^l_\delta}\ ,$$

$$\mathbf{R}\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta t^\beta}\right)\frac{\delta}{\delta t^\alpha} = \bar{R}^\delta_{\alpha\beta k}\frac{\delta}{\delta t^\delta}\ , \quad \mathbf{R}\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta t^\beta}\right)\frac{\delta}{\delta x^i} = R^l_{i\beta k}\frac{\delta}{\delta x^l}\ ,$$

$$\mathbf{R}\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta t^\beta}\right)\frac{\partial}{\partial x^i_\alpha} = R^{(l)(\alpha)}_{(\delta)(i)\beta k}\frac{\partial}{\partial x^l_\delta}\ ,$$

$$\mathbf{R}\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta t^\alpha} = \bar{R}^\delta_{\alpha jk}\frac{\delta}{\delta t^\delta}\ , \quad \mathbf{R}\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^i} = R^l_{ijk}\frac{\delta}{\delta x^l}\ ,$$

$$\mathbf{R}\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial x^i_\alpha} = R^{(l)(\alpha)}_{(\delta)(i)jk}\frac{\partial}{\partial x^l_\delta}\ ,$$

$$\mathbf{R}\left(\frac{\partial}{\partial x^k_\gamma},\frac{\delta}{\delta t^\beta}\right)\frac{\delta}{\delta t^\alpha} = \bar{P}^{\delta\ (\gamma)}_{\alpha\beta(k)}\frac{\delta}{\delta t^\delta}\ , \quad \mathbf{R}\left(\frac{\partial}{\partial x^k_\gamma},\frac{\delta}{\delta t^\beta}\right)\frac{\delta}{\delta x^i} = P^{l\ (\gamma)}_{i\beta(k)}\frac{\delta}{\delta x^l}\ ,$$

$$\mathbf{R}\left(\frac{\partial}{\partial x^k_\gamma},\frac{\delta}{\delta t^\beta}\right)\frac{\partial}{\partial x^i_\alpha} = P^{(l)(\alpha)\ (\gamma)}_{(\delta)(i)\beta(k)}\frac{\partial}{\partial x^l_\delta}\ ,$$

$$\mathbf{R}\left(\frac{\partial}{\partial x^k_\gamma},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta t^\alpha} = \bar{P}^{\delta\ (\gamma)}_{\alpha j(k)}\frac{\delta}{\delta t^\delta}\ , \quad \mathbf{R}\left(\frac{\partial}{\partial x^k_\gamma},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^i} = P^{l\ (\gamma)}_{ij(k)}\frac{\delta}{\delta x^l}\ ,$$

$$\mathbf{R}\left(\frac{\partial}{\partial x^k_\gamma},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial x^i_\alpha} = P^{(l)(\alpha)\ (\gamma)}_{(\delta)(i)j(k)}\frac{\partial}{\partial x^l_\delta}\ ,$$

$$\mathbf{R}\left(\frac{\partial}{\partial x^k_\gamma},\frac{\partial}{\partial x^j_\beta}\right)\frac{\delta}{\delta t^\alpha} = \bar{S}^{\delta(\beta)(\gamma)}_{\alpha(j)(k)}\frac{\delta}{\delta t^\delta}\ , \quad \mathbf{R}\left(\frac{\partial}{\partial x^k_\gamma},\frac{\partial}{\partial x^j_\beta}\right)\frac{\delta}{\delta x^i} = S^{l(\beta)(\gamma)}_{i(j)(k)}\frac{\delta}{\delta x^l}\ ,$$

$$\mathbf{R}\left(\frac{\partial}{\partial x^k_\gamma},\frac{\partial}{\partial x^j_\beta}\right)\frac{\partial}{\partial x^i_\alpha} = S^{(l)(\alpha)(\beta)(\gamma)}_{(\delta)(i)(j)(k)}\frac{\partial}{\partial x^l_\delta}\ ,$$

whose eighteen adapted components we arrange in the table

(4.2)

	h_T	h_M	v
h_Th_T	$\bar{R}^{\delta}_{\alpha\beta\gamma}$	$R^{l}_{i\beta\gamma}$	$R^{(l)(\alpha)}_{(\delta)(i)\beta\gamma}$
h_Mh_T	$\bar{R}^{\delta}_{\alpha\beta k}$	$R^{l}_{i\beta k}$	$R^{(l)(\alpha)}_{(\delta)(i)\beta k}$
h_Mh_M	$\bar{R}^{\delta}_{\alpha jk}$	R^{l}_{ijk}	$R^{(l)(\alpha)}_{(\delta)(i)jk}$
vh_T	$\bar{P}^{\delta\ \ (\gamma)}_{\alpha\beta(k)}$	$P^{l\ \ (\gamma)}_{i\beta(k)}$	$P^{(l)(\alpha)\ (\gamma)}_{(\delta)(i)\beta(k)}$
vh_M	$\bar{P}^{\delta\ \ (\gamma)}_{\alpha j(k)}$	$P^{l\ \ (\gamma)}_{ij(k)}$	$P^{(l)(\alpha)\ (\gamma)}_{(\delta)(i)j(k)}$
vv	$\bar{S}^{\delta(\beta)(\gamma)}_{\alpha(j)(k)}$	$S^{l(\beta)(\gamma)}_{i(j)(k)}$	$S^{(l)(\alpha)(\beta)(\gamma)}_{(\delta)(i)(j)(k)}$

Moreover, the following expressions of the local adapted components of the curvature d-tensor $\mathbf{R}$ *hold good:*

$$(h_T)\quad\begin{cases}
1.\quad \bar{R}^{\delta}_{\alpha\beta\gamma}=\dfrac{\delta\bar{G}^{\delta}_{\alpha\beta}}{\delta t^{\gamma}}-\dfrac{\delta\bar{G}^{\delta}_{\alpha\gamma}}{\delta t^{\beta}}+\bar{G}^{\mu}_{\alpha\beta}\bar{G}^{\delta}_{\mu\gamma}-\bar{G}^{\mu}_{\alpha\gamma}\bar{G}^{\delta}_{\mu\beta}+\bar{C}^{\delta(\mu)}_{\alpha(m)}R^{(m)}_{(\mu)\beta\gamma}\\
2.\quad \bar{R}^{\delta}_{\alpha\beta k}=\dfrac{\delta\bar{G}^{\delta}_{\alpha\beta}}{\delta x^{k}}-\dfrac{\delta\bar{L}^{\delta}_{\alpha k}}{\delta t^{\beta}}+\bar{G}^{\mu}_{\alpha\beta}\bar{L}^{\delta}_{\mu k}-\bar{L}^{\mu}_{\alpha k}\bar{G}^{\delta}_{\mu\beta}+\bar{C}^{\delta(\mu)}_{\alpha(m)}R^{(m)}_{(\mu)\beta k}\\
3.\quad \bar{R}^{\delta}_{\alpha jk}=\dfrac{\delta\bar{L}^{\delta}_{\alpha j}}{\delta x^{k}}-\dfrac{\delta\bar{L}^{\delta}_{\alpha k}}{\delta x^{j}}+\bar{L}^{\mu}_{\alpha j}\bar{L}^{\delta}_{\mu k}-\bar{L}^{\mu}_{\alpha k}\bar{L}^{\delta}_{\mu j}+\bar{C}^{\delta(\mu)}_{\alpha(m)}R^{(m)}_{(\mu)jk}\\
4.\quad \bar{P}^{\delta\ \ (\gamma)}_{\alpha\beta(k)}=\dfrac{\partial\bar{G}^{\delta}_{\alpha\beta}}{\partial x^{k}_{\gamma}}-\bar{C}^{\delta(\gamma)}_{\alpha(k)/\beta}+\bar{C}^{\delta(\mu)}_{\alpha(m)}P^{(m)\ \ (\gamma)}_{(\mu)\beta(k)}\\
5.\quad \bar{P}^{\delta\ \ (\gamma)}_{\alpha j(k)}=\dfrac{\partial\bar{L}^{\delta}_{\alpha j}}{\partial x^{k}_{\gamma}}-\bar{C}^{\delta(\gamma)}_{\alpha(k)|j}+\bar{C}^{\delta(\mu)}_{\alpha(m)}P^{(m)\ \ (\gamma)}_{(\mu)j(k)}\\
6.\quad \bar{S}^{\delta(\beta)(\gamma)}_{\alpha(j)(k)}=\dfrac{\partial\bar{C}^{\delta(\beta)}_{\alpha(j)}}{\partial x^{k}_{\gamma}}-\dfrac{\partial\bar{C}^{\delta(\gamma)}_{\alpha(k)}}{\partial x^{j}_{\beta}}+\bar{C}^{\mu(\beta)}_{\alpha(j)}\bar{C}^{\delta(\gamma)}_{\mu(k)}-\bar{C}^{\mu(\gamma)}_{\alpha(k)}\bar{C}^{\delta(\beta)}_{\mu(j)}\ ,
\end{cases}$$

$$(h_M)\quad\begin{cases}
7.\quad R^{l}_{i\beta\gamma}=\dfrac{\delta G^{l}_{i\beta}}{\delta t^{\gamma}}-\dfrac{\delta G^{l}_{i\gamma}}{\delta t^{\beta}}+G^{m}_{i\beta}G^{l}_{m\gamma}-G^{m}_{i\gamma}G^{l}_{m\beta}+C^{l(\mu)}_{i(m)}R^{(m)}_{(\mu)\beta\gamma}\\
8.\quad R^{l}_{i\beta k}=\dfrac{\delta G^{l}_{i\beta}}{\delta x^{k}}-\dfrac{\delta L^{l}_{ik}}{\delta t^{\beta}}+G^{m}_{i\beta}L^{l}_{mk}-L^{m}_{ik}G^{l}_{m\beta}+C^{l(\mu)}_{i(m)}R^{(m)}_{(\mu)\beta k}\\
9.\quad R^{l}_{ijk}=\dfrac{\delta L^{l}_{ij}}{\delta x^{k}}-\dfrac{\delta L^{l}_{ik}}{\delta x^{j}}+L^{m}_{ij}L^{l}_{mk}-L^{m}_{ik}L^{l}_{mj}+C^{l(\mu)}_{i(m)}R^{(m)}_{(\mu)jk}\\
10.\quad P^{l\ \ (\gamma)}_{i\beta(k)}=\dfrac{\partial G^{l}_{i\beta}}{\partial x^{k}_{\gamma}}-C^{l(\gamma)}_{i(k)/\beta}+C^{l(\mu)}_{i(m)}P^{(m)\ \ (\gamma)}_{(\mu)\beta(k)}\\
11.\quad P^{l\ \ (\gamma)}_{ij(k)}=\dfrac{\partial L^{l}_{ij}}{\partial x^{k}_{\gamma}}-C^{l(\gamma)}_{i(k)|j}+C^{l(\mu)}_{i(m)}P^{(m)\ \ (\gamma)}_{(\mu)j(k)}\\
12.\quad S^{l(\beta)(\gamma)}_{i(j)(k)}=\dfrac{\partial C^{l(\beta)}_{i(j)}}{\partial x^{k}_{\gamma}}-\dfrac{\partial C^{l(\gamma)}_{i(k)}}{\partial x^{j}_{\beta}}+C^{m(\beta)}_{i(j)}C^{l(\gamma)}_{m(k)}-C^{m(\gamma)}_{i(k)}C^{l(\beta)}_{m(j)}\ ,
\end{cases}$$

$$
(v)\quad\begin{cases}
13.\quad R^{(l)(\alpha)}_{(\delta)(i)\beta\gamma}=\dfrac{\delta G^{(l)(\alpha)}_{(\delta)(i)\beta}}{\delta t^\gamma}-\dfrac{\delta G^{(l)(\alpha)}_{(\delta)(i)\gamma}}{\delta t^\beta}+G^{(m)(\alpha)}_{(\mu)(i)\beta}G^{(l)(\mu)}_{(\delta)(m)\gamma}-\\
\qquad -G^{(m)(\alpha)}_{(\mu)(i)\gamma}G^{(l)(\mu)}_{(\delta)(m)\beta}+C^{(l)(\alpha)(\mu)}_{(\delta)(i)(m)}R^{(m)}_{(\mu)\beta\gamma}\\
14.\quad R^{(l)(\alpha)}_{(\delta)(i)\beta k}=\dfrac{\delta G^{(l)(\alpha)}_{(\delta)(i)\beta}}{\delta x^k}-\dfrac{\delta L^{(l)(\alpha)}_{(\delta)(i)k}}{\delta t^\beta}+G^{(m)(\alpha)}_{(\mu)(i)\beta}L^{(l)(\mu)}_{(\delta)(m)k}-\\
\qquad -L^{(m)(\alpha)}_{(\mu)(i)k}G^{(l)(\mu)}_{(\delta)(m)\beta}+C^{(l)(\alpha)(\mu)}_{(\delta)(i)(m)}R^{(m)}_{(\mu)\beta k}\\
15.\quad R^{(l)(\alpha)}_{(\delta)(i)jk}=\dfrac{\delta L^{(l)(\alpha)}_{(\delta)(i)j}}{\delta x^k}-\dfrac{\delta L^{(l)(\alpha)}_{(\delta)(i)k}}{\delta x^j}+L^{(m)(\alpha)}_{(\mu)(i)j}L^{(l)(\mu)}_{(\delta)(m)k}-\\
\qquad -L^{(m)(\alpha)}_{(\mu)(i)k}L^{(l)(\mu)}_{(\delta)(m)j}+C^{(l)(\alpha)(\mu)}_{(\delta)(i)(m)}R^{(m)}_{(\mu)jk}\\
16.\quad P^{(l)(\alpha)\ \ (\gamma)}_{(\delta)(i)\beta(k)}=\dfrac{\partial G^{(l)(\alpha)}_{(\delta)(i)\beta}}{\partial x^k_\gamma}-C^{(l)(\alpha)(\gamma)}_{(\delta)(i)(k)/\beta}+C^{(l)(\alpha)(\mu)}_{(\delta)(i)(m)}P^{(m)\ \ (\gamma)}_{(\mu)\beta(k)}\\
17.\quad P^{(l)(\alpha)\ \ (\gamma)}_{(\delta)(i)j(k)}=\dfrac{\partial L^{(l)(\alpha)}_{(\delta)(i)j}}{\partial x^k_\gamma}-C^{(l)(\alpha)(\gamma)}_{(\delta)(i)(k)|j}+C^{(l)(\alpha)(\mu)}_{(\delta)(i)(m)}P^{(m)\ \ (\gamma)}_{(\mu)j(k)}\\
18.\quad S^{(l)(\alpha)(\beta)(\gamma)}_{(\delta)(i)(j)(k)}=\dfrac{\partial C^{(l)(\alpha)(\beta)}_{(\delta)(i)(j)}}{\partial x^k_\gamma}-\dfrac{\partial C^{(l)(\alpha)(\gamma)}_{(\delta)(i)(k)}}{\partial x^j_\beta}+C^{(m)(\alpha)(\beta)}_{(\mu)(i)(j)}C^{(l)(\mu)(\gamma)}_{(\delta)(m)(k)}-\\
\qquad -C^{(m)(\alpha)(\gamma)}_{(\mu)(i)(k)}C^{(l)(\mu)(\beta)}_{(\delta)(m)(j)}\ .
\end{cases}
$$

Remark 4.1 In the case of the Berwald Γ_0-linear connection associated to the pair of metrics $(h_{\alpha\beta},\varphi_{ij})$, all curvature d-tensors vanish, except

$$R^\delta_{\alpha\beta\gamma}=H^\delta_{\alpha\beta\gamma},\quad R^l_{ijk}=r^l_{ijk},$$

where $H^\delta_{\alpha\beta\gamma}$ (resp. r^l_{ijk}) are the curvature tensors of the metric $h_{\alpha\beta}$ (resp. φ_{ij}).

5 Ricci and Bianchi identities on $J^1(T,M)$

Let us consider an arbitrary d-vector field X on the 1-jet space E, which is locally expressed by

$$X=X^\alpha\frac{\delta}{\delta t^\alpha}+X^i\frac{\delta}{\delta x^i}+X^{(i)}_{(\alpha)}\frac{\partial}{\partial x^i_\alpha}.$$

Taking again into account the local form of the T- ,M- horizontal and vertical covariant derivatives of the Γ-linear connection ∇, by a direct calculation, we obtain

Theorem 5.1 *The following Ricci identities associated to the Γ-linear connection ∇ and to the distinguished vector field X hold good:*

$$(h_T)\quad \begin{cases} X^\alpha_{/\beta/\gamma} - X^\alpha_{/\gamma/\beta} = X^\mu \bar{R}^\alpha_{\mu\beta\gamma} - X^\alpha_{/\mu}\bar{T}^\mu_{\beta\gamma} - X^\alpha|^{(\mu)}_{(m)} R^{(m)}_{(\mu)\beta\gamma} \\ X^\alpha_{/\beta|k} - X^\alpha_{|k/\beta} = X^\mu \bar{R}^\alpha_{\mu\beta k} - X^\alpha_{/\mu}\bar{T}^\mu_{\beta k} - X^\alpha_{|m} T^m_{\beta k} - X^\alpha|^{(\mu)}_{(m)} R^{(m)}_{(\mu)\beta k} \\ X^\alpha_{|j|k} - X^\alpha_{|k|j} = X^\mu \bar{R}^\alpha_{\mu jk} - X^\alpha_{|m} T^m_{jk} - X^\alpha|^{(\mu)}_{(m)} R^{(m)}_{(\mu)jk} \\ X^\alpha_{/\beta}|^{(\gamma)}_{(k)} - X^\alpha|^{(\gamma)}_{(k)/\beta} = X^\mu \bar{P}^{\alpha\ \ (\gamma)}_{\mu\beta(k)} - X^\alpha_{/\mu}\bar{C}^{\mu(\gamma)}_{\beta(k)} - X^\alpha|^{(\mu)}_{(m)} P^{(m)\ \ (\gamma)}_{(\mu)\beta(k)} \\ X^\alpha_{|j}|^{(\gamma)}_{(k)} - X^\alpha|^{(\gamma)}_{(k)|j} = X^\mu \bar{P}^{\alpha\ \ (\gamma)}_{\mu j(k)} - X^\alpha_{|m} C^{m(\gamma)}_{j(k)} - X^\alpha|^{(\mu)}_{(m)} P^{(m)\ \ (\gamma)}_{(\mu)j(k)} \\ X^\alpha|^{(\beta)}_{(j)}|^{(\gamma)}_{(k)} - X^\alpha|^{(\gamma)}_{(k)}|^{(\beta)}_{(j)} = X^\mu \bar{S}^{\alpha(\beta)(\gamma)}_{\mu(j)(k)} - X^\alpha|^{(\mu)}_{(m)} S^{(m)(\beta)(\gamma)}_{(\mu)(j)(k)}, \end{cases}$$

$$(h_M)\quad \begin{cases} X^i_{/\beta/\gamma} - X^i_{/\gamma/\beta} = X^m R^i_{m\beta\gamma} - X^i_{/\mu}\bar{T}^\mu_{\beta\gamma} - X^i|^{(\mu)}_{(m)} R^{(m)}_{(\mu)\beta\gamma} \\ X^i_{/\beta|k} - X^i_{|k/\beta} = X^m R^i_{m\beta k} - X^i_{/\mu}\bar{T}^\mu_{\beta k} - X^i_{|m} T^m_{\beta k} - X^i|^{(\mu)}_{(m)} R^{(m)}_{(\mu)\beta k} \\ X^i_{|j|k} - X^i_{|k|j} = X^m R^i_{mjk} - X^i_{|m} T^m_{jk} - X^i|^{(\mu)}_{(m)} R^{(m)}_{(\mu)jk} \\ X^i_{/\beta}|^{(\gamma)}_{(k)} - X^i|^{(\gamma)}_{(k)/\beta} = X^m P^{i\ \ (\gamma)}_{m\beta(k)} - X^i_{/\mu}\bar{C}^{\mu(\gamma)}_{\beta(k)} - X^i|^{(\mu)}_{(m)} P^{(m)\ \ (\gamma)}_{(\mu)\beta(k)} \\ X^i_{|j}|^{(\gamma)}_{(k)} - X^i|^{(\gamma)}_{(k)|j} = X^m P^{i\ \ (\gamma)}_{mj(k)} - X^i_{|m} C^{m(\gamma)}_{j(k)} - X^i|^{(\mu)}_{(m)} P^{(m)\ \ (\gamma)}_{(\mu)j(k)} \\ X^i|^{(\beta)}_{(j)}|^{(\gamma)}_{(k)} - X^i|^{(\gamma)}_{(k)}|^{(\beta)}_{(j)} = X^m S^{i(\beta)(\gamma)}_{m(j)(k)} - X^i|^{(\mu)}_{(m)} S^{(m)(\beta)(\gamma)}_{(\mu)(j)(k)}, \end{cases}$$

$$(v)\quad \begin{cases} X^{(i)}_{(\alpha)/\beta/\gamma} - X^{(i)}_{(\alpha)/\gamma/\beta} = X^{(m)}_{(\mu)} R^{(i)(\mu)}_{(\alpha)(m)\beta\gamma} - X^{(i)}_{(\alpha)/\mu}\bar{T}^\mu_{\beta\gamma} - X^{(i)}_{(\alpha)}|^{(\mu)}_{(m)} R^{(m)}_{(\mu)\beta\gamma} \\ X^{(i)}_{(\alpha)/\beta|k} - X^{(i)}_{(\alpha)|k/\beta} = X^{(m)}_{(\mu)} R^{(i)(\mu)}_{(\alpha)(m)\beta k} - X^{(i)}_{(\alpha)/\mu}\bar{T}^\mu_{\beta k} - X^{(i)}_{(\alpha)|m} T^m_{\beta k} - \\ \qquad - X^{(i)}_{(\alpha)}|^{(\mu)}_{(m)} R^{(m)}_{(\mu)\beta k} \\ X^{(i)}_{(\alpha)|j|k} - X^{(i)}_{(\alpha)|k|j} = X^{(m)}_{(\mu)} R^{(i)(\mu)}_{(\alpha)(m)jk} - X^{(i)}_{(\alpha)|m} T^m_{jk} - X^{(i)}_{(\alpha)}|^{(\mu)}_{(m)} R^{(m)}_{(\mu)jk} \\ X^{(i)}_{(\alpha)/\beta}|^{(\gamma)}_{(k)} - X^{(i)}_{(\alpha)}|^{(\gamma)}_{(k)/\beta} = X^{(m)}_{(\mu)} P^{(i)(\mu)\ \ (\gamma)}_{(\alpha)(m)\beta(k)} - X^{(i)}_{(\alpha)/\mu}\bar{C}^{\mu(\gamma)}_{\beta(k)} - \\ \qquad - X^{(i)}_{(\alpha)}|^{(\mu)}_{(m)} P^{(m)\ \ (\gamma)}_{(\mu)\beta(k)} \\ X^{(i)}_{(\alpha)|j}|^{(\gamma)}_{(k)} - X^{(i)}_{(\alpha)}|^{(\gamma)}_{(k)|j} = X^{(m)}_{(\mu)} P^{(i)(\mu)\ \ (\gamma)}_{(\alpha)(m)j(k)} - X^{(i)}_{(\alpha)|m} C^{m(\gamma)}_{j(k)} - \\ \qquad - X^{(i)}_{(\alpha)}|^{(\mu)}_{(m)} P^{(m)\ \ (\gamma)}_{(\mu)j(k)} \\ X^{(i)}_{(\alpha)}|^{(\beta)}_{(j)}|^{(\gamma)}_{(k)} - X^{(i)}_{(\alpha)}|^{(\gamma)}_{(k)}|^{(\beta)}_{(j)} = X^{(m)}_{(\mu)} S^{(i)(\mu)(\beta)(\gamma)}_{(\alpha)(m)(j)(k)} - X^{(i)}_{(\alpha)}|^{(\mu)}_{(m)} S^{(m)(\beta)(\gamma)}_{(\mu)(j)(k)}. \end{cases}$$

Proof. Let (Y_A) and (ω^A), $A \in \{\alpha, i, {}^{(\alpha)}_{(i)}\}$, be the adapted bases of the nonlinear connection Γ, and let $X = X^M Y_M$ be an arbitrary d-vector field on $J^1(T,M)$. In this context, using the equalities

$$[\mathbf{R}(Y_C,Y_B)X]\cdot\omega^B\cdot\omega^C = \{\nabla_{Y_C}\nabla_{Y_B}X - \nabla_{Y_B}\nabla_{Y_C}X - \nabla_{[Y_B,Y_C]}X\}\cdot\omega^B\cdot\omega^C,$$

$$\mathbf{R}(Y_C,Y_B)Y_A = \mathbf{R}^M_{ABC}Y_M, \qquad \mathbf{T}(Y_C,Y_B) = \mathbf{T}^M_{BC} = \{\Gamma^M_{BC} - \Gamma^M_{CB} - R^M_{CB}\}Y_M,$$

$$\nabla_{Y_C}\omega^B = -\Gamma^B_{MC}\omega^M, \qquad [Y_B,Y_C] = R^M_{BC}Y_M, \qquad \nabla_{Y_C}Y_B = \Gamma^M_{BC}Y_M,$$

where " $\cdot$ " represents the tensorial product " $\otimes$ ", by a direct calcul, we obtain the relations

$$X^A_{:B:C} - X^A_{:C:B} = X^M \mathbf{R}^A_{MBC} - X^A_{:M}\mathbf{T}^M_{BC}, \tag{5.1}$$

where "$_{:D}$" represents one from the local covariant derivatives "$_{/\beta}$", "$_{|j}$" or "$|^{(\beta)}_{(j)}$" associated to the Γ-linear connection ∇.

Taking into account the indices $A, B, C, \ldots$ belong to set $\{\alpha, i, {}^{(\alpha)}_{(i)}\}$, after enough complicated computations, the identities (5.1) impliy the required Ricci identities. ■

Remark 5.1 For the arbitrary vector fields $X, Y, Z \in \mathcal{X}(E)$ and the arbitrary 1-form $\omega \in \mathcal{X}^*(E)$ on $J^1(T,M)$, the relations

$$\begin{aligned} &\mathbf{R}(X,Y)\omega = -\omega \circ \mathbf{R}(X,Y), \\ &\mathbf{R}(X,Y)(Z\otimes\omega) = \mathbf{R}(X,Y)Z\otimes\omega + Z\otimes\mathbf{R}(X,Y)\omega, \end{aligned} \tag{5.2}$$

are true. These relations allow us to generalize the Ricci identities to the d-tensors set of the 1-jet fibre bundle E. The generalization is a natural one, but the expressions of Ricci identities become extremely complicated. For that reason, we exemplify this generalization writing just one Ricci identity. For example, if $D = (D^{\delta i(k)(\varepsilon)\ldots}_{\alpha j(\eta)(l)\ldots})$ is an arbitrary d-tensor field on E, then the following Ricci identity

$$\begin{aligned} &D^{\delta i(k)(\varepsilon)\ldots}_{\alpha j(\eta)(l)\ldots/\beta/\gamma} - D^{\delta i(k)(\varepsilon)\ldots}_{\alpha j(\eta)(l)\ldots/\gamma/\beta} = D^{\mu i(k)(\varepsilon)\ldots}_{\alpha j(\eta)(l)\ldots}\bar{R}^{\alpha}_{\mu\beta\gamma} + D^{\delta m(k)(\varepsilon)\ldots}_{\alpha j(\eta)(l)\ldots}R^{i}_{m\beta\gamma} + \\ &+D^{\delta i(m)(\varepsilon)\ldots}_{\alpha j(\mu)(l)\ldots}R^{(k)(\mu)}_{(\eta)(m)\beta\gamma} + \ldots\ldots\ldots - D^{\delta i(k)(\varepsilon)\ldots}_{\mu j(\eta)(l)\ldots}\bar{R}^{\mu}_{\alpha\beta\gamma} - D^{\delta i(k)(\varepsilon)\ldots}_{\alpha m(\eta)(l)\ldots}R^{m}_{i\beta\gamma} - \\ &-D^{\delta i(k)(\mu)\ldots}_{\alpha j(\eta)(m)\ldots}R^{(m)(\varepsilon)}_{(\mu)(l)\beta\gamma} - \ldots\ldots\ldots - D^{\delta i(k)(\varepsilon)\ldots}_{\alpha j(\eta)(l)\ldots/\mu}\bar{T}^{\mu}_{\beta\gamma} - D^{\delta i(k)(\varepsilon)\ldots}_{\alpha j(\eta)(l)\ldots}|^{(\mu)}_{(m)}R^{(m)}_{(\mu)\beta\gamma} \end{aligned}$$

holds good.

Now, let us consider the Liouville canonical vector field $\mathbf{C}= x^i_\alpha \dfrac{\partial}{\partial x^i_\alpha}$ on $J^1(T,M)$, together with the *deflection d-tensors associated to the* Γ*-linear connection* ∇, which are defined by the local components

$$\bar{D}^{(i)}_{(\alpha)\beta} = x^i_{\alpha/\beta}\ , \quad D^{(i)}_{(\alpha)j} = x^i_{\alpha|j}\ , \quad d^{(i)(\beta)}_{(\alpha)(j)} = x^i_\alpha|^{(\beta)}_{(j)}\ .$$

By a direct calculation, we find the the following local expressions of the deflection d-tensors:

$$\begin{aligned} \bar{D}^{(i)}_{(\alpha)\beta} &= -M^{(i)}_{(\alpha)\beta} + G^{(i)(\mu)}_{(\alpha)(m)\beta}x^m_\mu \\ D^{(i)}_{(\alpha)j} &= -N^{(i)}_{(\alpha)j} + L^{(i)(\mu)}_{(\alpha)(m)j}x^m_\mu \\ d^{(i)(\beta)}_{(\alpha)(j)} &= \delta^i_j\delta^\beta_\alpha + C^{(i)(\mu)(\beta)}_{(\alpha)(m)(j)}x^m_\mu\ . \end{aligned} \tag{5.3}$$

Applying the v-set of the Ricci identities to the components of the Liouville vector field $\mathbf{C}$, we obtain

Theorem 5.2 *The deflection d-tensors of the* Γ*-linear connection* ∇ *satisfy the identities:*

$$
\begin{aligned}
&\bar{D}^{(i)}_{(\alpha)\beta/\gamma} - \bar{D}^{(i)}_{(\alpha)\gamma/\beta} = x^m_\mu R^{(i)(\mu)}_{(\alpha)(m)\beta\gamma} - \bar{D}^{(i)}_{(\alpha)\mu}\bar{T}^\mu_{\beta\gamma} - d^{(i)(\mu)}_{(\alpha)(m)} R^{(m)}_{(\mu)\beta\gamma}\ ,\\
&\bar{D}^{(i)}_{(\alpha)\beta|k} - D^{(i)}_{(\alpha)k/\beta} = x^m_\mu R^{(i)(\mu)}_{(\alpha)(m)\beta k} - \bar{D}^{(i)}_{(\alpha)\mu}\bar{T}^\mu_{\beta k} - D^{(i)}_{(\alpha)m}T^m_{\beta k} - d^{(i)(\mu)}_{(\alpha)(m)} R^{(m)}_{(\mu)\beta k}\ ,\\
&D^{(i)}_{(\alpha)j|k} - D^{(i)}_{(\alpha)k|j} = x^m_\mu R^{(i)(\mu)}_{(\alpha)(m)jk} - D^{(i)}_{(\alpha)m}T^m_{jk} - d^{(i)(\mu)}_{(\alpha)(m)} R^{(m)}_{(\mu)jk}\ ,\\
&\bar{D}^{(i)}_{(\alpha)\beta}|^{(\gamma)}_{(k)} - d^{(i)(\gamma)}_{(\alpha)(k)/\beta} = x^m_\mu P^{(i)(\mu)\ (\gamma)}_{(\alpha)(m)\beta(k)} - \bar{D}^{(i)}_{(\alpha)\mu}\bar{C}^{\mu(\gamma)}_{\beta(k)} - d^{(i)(\mu)}_{(\alpha)(m)} P^{(m)\ (\gamma)}_{(\mu)\beta(k)}\ ,\\
&D^{(i)}_{(\alpha)j}|^{(\gamma)}_{(k)} - d^{(i)(\gamma)}_{(\alpha)(k)|j} = x^m_\mu P^{(i)(\mu)\ (\gamma)}_{(\alpha)(m)j(k)} - D^{(i)}_{(\alpha)m}C^{m(\gamma)}_{j(k)} - d^{(i)(\mu)}_{(\alpha)(m)} P^{(m)\ (\gamma)}_{(\mu)j(k)}\ ,\\
&d^{(i)(\beta)}_{(\alpha)(j)}|^{(\gamma)}_{(k)} - d^{(i)(\gamma)}_{(\alpha)(k)}|^{(\beta)}_{(j)} = x^m_\mu S^{(i)(\mu)(\beta)(\gamma)}_{(\alpha)(m)(j)(k)} - d^{(i)(\mu)}_{(\alpha)(m)} S^{(m)(\beta)(\gamma)}_{(\mu)(j)(k)}\ .
\end{aligned}
$$

At the end of this section, we point out the torsion $\mathbf{T}$ and the curvature $\mathbf{R}$ of the Γ-linear connection ∇ are not independent. They verify the general Bianchi identities

$$
\begin{cases}
\sum_{\{X,Y,Z\}}\{(\nabla_X\mathbf{T})(Y,Z) - \mathbf{R}(X,Y)Z + \mathbf{T}(\mathbf{T}(X,Y),Z)\} = 0, & \forall\ X,Y,Z \in \mathcal{X}(E)\\
\sum_{\{X,Y,Z\}}\{(\nabla_X\mathbf{R})(U,Y,Z) + \mathbf{R}(\mathbf{T}(X,Y),Z)U\} = 0, & \forall\ X,Y,Z,U \in \mathcal{X}(E),
\end{cases}
$$

where $\{X,Y,Z\}$ means cyclic sum.

In the adapted basis $(X_A) = \left(\dfrac{\delta}{\delta t^\alpha}, \dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial x^i_\alpha}\right)$, we have sixty-four effective local Bianchi identities attached to a Γ-linear connection, that can be obtained by the relations

$$
\begin{cases}
\sum_{\{A,B,C\}}\{R^F_{ABC} - T^F_{AB:C} - T^G_{AB}T^F_{CG}\} = 0\\
\sum_{\{A,B,C\}}\{R^F_{DAB:C} + T^G_{AB}R^F_{DAG}\} = 0,
\end{cases}
\tag{5.4}
$$

where $\mathbf{R}(X_A,X_B)X_C = R^D_{CBA}X_D$, $\mathbf{T}(X_A,X_B) = T^D_{BA}X_D$ and "$_{:A}$" represents one of the covariant derivatives "$_{/\alpha}$", "$_{|i}$" or "$|^{(\alpha)}_{(i)}$". The large number and the complicated form of the above Bianchi identities determined us to describe them effectively only for *h-normal* Γ*-linear connections*, which are studied in [3]. In that case, the number of Bianchi identities reduces to thirty.

6 Jet prolongation of vector fields

A general vector field X^* on $J^1(T,M)$ can be written under the form

$$
X^* = X^\alpha\frac{\partial}{\partial t^\alpha} + X^i\frac{\partial}{\partial x^i} + X^{(i)}_{(\alpha)}\frac{\partial}{\partial x^i_\alpha},
$$

where the components $X^\alpha,\ X^i,\ X^{(i)}_{(\alpha)}$ are functions of $(t^\alpha, x^i, x^i_\alpha)$.

The prolongation of a vector field X on $T\times M$ to a vector field on the 1-jet bundle $J^1(T,M)$ was solved by Olver, using [6]

Definition 6.1 Let X be a vector field on $T\times M$ with corresponding (local) one-parameter group $\exp(\varepsilon X)$. The *1-th prolongation* of X, denoted by $pr^{(1)}X$, is a vector field on the 1-jet space $J^1(T,M)$, which is defined as the infinitesimal generator of the corresponding prolonged one-parameter group $pr^{(1)}[\exp(\varepsilon X)]$. In other words, we have

$$[pr^{(1)}X](t^\alpha,x^i,x^i_\alpha)=\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}pr^{(1)}[\exp(\varepsilon X)](t^\alpha,x^i,x^i_\alpha). \tag{6.1}$$

In order to write the components of the 1-th prolongation of a vector field X on $T\times M$, Olver use the *α-th total derivative* D_α of an arbitrary function $f(t^\alpha,x^i)$ on $T\times M$, which is defined by the relation

$$D_\alpha f=\frac{\partial f}{\partial t^\alpha}+\frac{\partial f}{\partial x^i}x^i_\alpha. \tag{6.2}$$

In this direction, starting with $X=X^\alpha(t,x)\dfrac{\partial}{\partial t^\alpha}+X^i(t,x)\dfrac{\partial}{\partial x^i}$ like a vector field on $T\times M$, Olver conclude the 1-th prolongation of X is the vector field

$$pr^{(1)}X=X+X^{(i)}_{(\alpha)}(t^\beta,x^j,x^j_\beta)\frac{\partial}{\partial x^i_\alpha}, \tag{6.3}$$

where

$$X^{(i)}_{(\alpha)}=D_\alpha X^i-(D_\alpha X^\beta)x^i_\beta=\frac{\partial X^i}{\partial t^\alpha}+\frac{\partial X^i}{\partial x^j}x^j_\alpha-\left(\frac{\partial X^\beta}{\partial t^\alpha}+\frac{\partial X^\beta}{\partial x^j}x^j_\alpha\right)x^i_\beta.$$

Now, let us study certain geometrical aspects for obtaining the jet prolongation of a given vector field X on $T\times M$.

If we assume that is given a nonlinear connection $\Gamma=(M^{(i)}_{(\alpha)\beta},N^{(i)}_{(\alpha)j})$ on $J^1(T,M)$, we deduce the α-th total derivative used by Olver can be written as

$$D_\alpha f=\frac{\delta f}{\delta t^\alpha}+\frac{\delta f}{\delta x^i}x^i_\alpha\ , \tag{6.4}$$

and, consequently, $D_\alpha f$ represent the local components of the distinguished 1-form $Df=(D_\alpha f)dt^\alpha$ on $J^1(T,M)$.

In this geometrical context, supposing that $J^1(T,M)$ is also endowed with a Γ-linear connection (2.4), we may define the geometrical 1-th jet prolongation of X as the vector field

$$pr^{(1)}X=X^\alpha\frac{\delta}{\delta t^\alpha}+X^i\frac{\delta}{\delta x^i}+Y^{(i)}_{(\alpha)}(t^\beta,x^j,x^j_\beta)\frac{\partial}{\partial x^i_\alpha}\ , \tag{6.5}$$

where

$$Y^{(i)}_{(\alpha)}=X^i_{/\alpha}+X^i_{|j}x^j_\alpha-X^\beta_{/\alpha}x^i_\beta-X^\beta_{|j}x^j_\alpha x^i_\beta+$$
$$+X^\mu\left(M^{(i)}_{(\alpha)\mu}+\bar{G}^\beta_{\mu\alpha}x^i_\beta+\bar{L}^\beta_{\mu j}x^j_\alpha x^i_\beta\right)-X^m\left(N^{(i)}_{(\alpha)m}+G^i_{m\alpha}+L^i_{mj}x^j_\alpha\right).$$

Remarks 6.1 i) Taking into account the expressions of local covariant derivatives applied to a vector field (see Remarks 1.2 ii), we conclude our geometrical 1-th jet prolongation coincides with that constructed by Olver. Moreover, the following relations of conection are true:

$$Y^{(i)}_{(\alpha)}=X^{(i)}_{(\alpha)}+M^{(i)}_{(\alpha)\mu}X^\mu+N^{(i)}_{(\alpha)m}X^m, \tag{6.6}$$

where $M^{(i)}_{(\alpha)\mu}$ and $N^{(i)}_{(\alpha)m}$ are the components of the nonlinear connection Γ.

ii) In the particular case of the Berwald Γ_0-linear connection associated to the metrics $h_{\alpha\beta}$ and φ_{ij}, the expressions of $Y^{(i)}_{(\alpha)}$ reduce to

$$Y^{(i)}_{(\alpha)} = X^i_{//\alpha} + X^i_{\|j} x^j_\alpha - X^\beta_{//\alpha} x^i_\beta - X^\beta_{\|j} x^j_\alpha x^i_\beta - 2\gamma^i_{jm} x^j_\alpha X^m, \tag{6.7}$$

where γ^i_{jk} represent the Christoffel symbols of the metric φ_{ij}.

Open problem. – Study the prolongations of vectors, 1-forms, tensors, G-structures from $T \times M$ to $J^1(T, M)$.

Acknowledgements. It is a pleasure for us to thank Prof. Dr. D. Opriş, Prof. Dr. P. Olver and Conf. Dr. V. Balan for many helpful comments on this research.

A version of this paper was presented at the Third Conference of Balkan Society of Geometers, Workshop on Electromagnetic Flows and Dynamics, July 31 - August 3, 2000, University POLITEHNICA of Bucharest, Romania.

References

[1] M. J. Gotay, J. Isenberg, J. E. Marsden, *Momentum Maps and the Hamiltonian Structure of Classical Relativistic Fields*, http://xxx.lanl.gov/abs/hep/9801019, (1998).

[2] R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Academic Publishers, (1994).

[3] M. Neagu, *Upon h-normal Γ-linear connections on* $J^1(T, M)$, (2000), http://xxx.lanl.gov/abs/math.DG/0009070.

[4] M. Neagu, C. Udrişte, *Multi-Time Dependent Sprays and Harmonic Maps on* $J^1(T, M)$, Third Conference of Balkan Society of Geometers, Politehnica University of Bucharest, Romania, July 31-August 3, (2000); http://xxx.lanl.gov/abs/math.DG/0009049, (2000).

[5] M. Neagu, C. Udrişte, *The Geometry of Metrical Multi-Time Lagrange Spaces*, http://xxx.lanl.gov/abs/math.DG/0009071, (2000).

[6] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, (1986).

[7] D. Saunders, *The Geometry of Jet Bundles*, Cambridge University Press, New York, London, (1989).

[8] C. Udrişte, M. Neagu, *From PDE Systems and Metrics to Generalized Field Theories*, http://xxx.lanl.gov/abs/math.DG/0101207, (2001).

[9] A. Vondra, *Symmetries of Connections on Fibered Manifolds*, Archivum Mathematicum, Tomus 30, 97-115, (1994).

University POLITEHNICA of Bucharest, Department of Mathematics I
Splaiul Independentei 313, 77206 Bucharest, Romania
e-mail: mircea@mathem.pub.ro, e-mail: udriste@mathem.pub.ro