Principal Fibre Bundles with Structural Lie Groupoid

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Abstract

This paper deals with transformation Lie groupoids and fibre bundles with structural groupoid. The aim of this paper is to prove the theorem of construction of a principal fibre bundle with structural Lie groupoid.

Mathematics Subject Classification: 22A22, 55R99, 57S99
Key words: Lie groupoid, transformation Lie groupoid, principal fibre bundle with structural Lie groupoid

Introduction

The principal fibre bundle with structural Lie groupoid introduced by A. Haefliger ([2]), is a generalization of the concept of principal fibre bundle with structural Lie group.

In the first two sections we reffer to Lie groupoids, Lie subgroupoids and to the action of a Lie groupoid on a manifold.

In the third section we study the principal fibre bundles with stuctural Lie groupoid. Also, we construct a principal fibre bundle with stuctural Lie groupoid Γ, associated of a given manifold M and of a familly of transition functions with values in Γ (theorem 3.1.).

In the four section is introduced the fibre bundle with type fibre and structural Lie groupoid.

Let us we shall assume that the manifolds are finite dimensional, smooth of class C^∞, paracompact, connected and without boundary.

1 Lie groupoids

Definition 1.1. ([7]) A *groupoid* Γ *on* Γ_0 is a pair of sets $(\Gamma;\Gamma_0)$ equipped with:

(i) surjections $\alpha,\beta:\Gamma\to\Gamma_0$, (*source* and *target* maps);

Balkan Journal of Geometry and Its Applications, Vol.6, No.2, 2001, pp. 39-48

(ii) map $\mu : \Gamma_{(2)} \to \Gamma, (x,y) \to \mu(x,y) = x \cdot y$ or xy *(partial) multiplication*, where $\Gamma_{(2)} = \{(x,y) \in \Gamma \times \Gamma \mid \beta(x) = \alpha(y)\}$; each pair (x,y) in $\Gamma_{(2)}$ is said to be *composable*;

(iii) injection $\epsilon : \Gamma_0 \to \Gamma,\ u \to \epsilon(u) = \widetilde{u}$ (*inclusion map*);

(iv) map $i : \Gamma \longrightarrow \Gamma,\ x \to i(x) = x^{-1}$ (*inversion map*).

These maps must satysfy the following algebraic axioms generalizing those of group:

(G1) (*associative law*) $(xy)z = x(yz)$ (if one triple product is defined then so is the other);

(G2) (*identities*) For each $x \in \Gamma$ we have $(\epsilon(\alpha(x)), x)$; $(x, \epsilon(\beta(x)) \in \Gamma_{(2)}$ and $\epsilon(\alpha(x)) \cdot x = x \cdot \epsilon(\beta(x)) = x$;

(G3) (*inverses*) For each $x \in \Gamma$ we have $(x, i(x))$; $(i(x), x) \in \Gamma_{(2)}$ and $\mu(x, i(x)) = \epsilon(\beta(x))$, $\mu(i(x), x) = \epsilon(\alpha(x))$. $\triangle$

We denote sometimes a groupoid Γ on Γ_0 by $(\Gamma, \alpha, \beta; \Gamma_0)$ or $(\Gamma; \Gamma_0)$.

The subset $\epsilon(\Gamma_0)$ of Γ is called the *unity set* of Γ .

For any $u \in \Gamma_0$, the sets $\alpha^{-1}(u)$ and $\beta^{-1}(u)$ will be called a α-*fibre* and β-*fibre* over u.

A groupoid Γ on Γ_0 is said to be *transitive* if the map $\alpha \times \beta : \Gamma \longrightarrow \Gamma_0 \times \Gamma_0$, defined by $(\alpha \times \beta)(x) = (\alpha(x), \beta(x)), (\forall) x \in \Gamma$ is surjective.

We deduce from the axioms that, for any groupoid Γ on Γ_0 the following rules hold:

(1.1) $\alpha \circ \epsilon = \beta \circ \epsilon = id_{\Gamma_0}$ and $\epsilon(u) \cdot \epsilon(u) = \epsilon(u)$, $(\forall)\ u \in \Gamma_0$.

(1.2) $\alpha(xy) = \alpha(x)$ and $\beta(xy) = \beta(y)$ for all $(x,y) \in \Gamma_{(2)}$.

(1.3) $x^{-1} \cdot x = \epsilon(\beta(x))$ *and* $x \cdot x^{-1} = \epsilon(\alpha(x))$

(1.4) $\beta(x^{-1}) = \alpha(x)$ *and* $\beta(x) = \alpha(x^{-1})$

(1.5) The set $\Gamma(u) = \alpha^{-1}(u) \cap \beta^{-1}(u)$ is a group under the restriction of the partial multiplication, called the *isotropy group at* u of Γ.

The isotropy groups $\Gamma(\alpha(x))$ and $\Gamma(\beta(x))$ are isomorphic.

(1.6) If $(\Gamma; \Gamma_0)$ is transitive, then the isotropy groups $\Gamma(u)$, $(\forall)\ u \in \Gamma_0$ are isomorphes.

By *group bundle* we mean a groupoid Γ on Γ_0 such that $\alpha(x) = \beta(x)$ for each $x \in \Gamma$.

If Γ is a groupoid on Γ_0, then $Is(\Gamma) = \{x \in \Gamma \mid \alpha(x) = \beta(x)\}$ is a group bundle, called the *isotropy group bundle* of Γ.

Example 1.1. *Coarse groupoid associated to a set.* If B is a set, then $B \times B$ is a groupoid on B with the rules: $\alpha(x,y) = x; \beta(x,y) = y; \epsilon(x) = (x,x)$, the partial multiplication is given by: $\mu((x,y),(y',z)) = (x,z)$ iff $y = y'$ and the inverse of $(x,y) \in B \times B$ is defined by $(x,y)^{-1} = (y,x)$. $\triangle$

Definition 1.2. Let $(\Gamma, \alpha, \beta, \mu; \Gamma_0)$ and $(\Gamma', \alpha', \beta', \mu'; \Gamma'_0)$ be two groupoids.

(i) A *morphism of groupoids* or *groupoid morphism* is a pair (f, f_0) of maps $f : \Gamma \longrightarrow \Gamma'$ and $f_0 : \Gamma_0 \longrightarrow \Gamma'_0$ such that the following two conditions are satisfied:

(1.7) $f(\mu(x,y)) = \mu'(f(x), f(y))$, $(\forall)\ (x,y) \in \Gamma_{(2)}$;

(1.8) $\alpha' \circ f = f_0 \circ \alpha$ and $\beta' \circ f = f_0 \circ \beta$.

If $\Gamma_0 = \Gamma'_0$ and $f_0 = Id_{\Gamma_0}$, we say that f is Γ_0- *morphism*.

(ii) A groupoid morphism $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$ is be called *strong morphism of groupoids*, if for every $(f(x), f(y)) \in \Gamma'_{(2)}$ we have $(x,y) \in \Gamma_{(2)}$. $\triangle$

Example 1.2. Let $(\Gamma, \alpha, \beta; \Gamma_0)$ be a groupoid and $\Gamma_0 \times \Gamma_0$ the coarse groupoid associated to Γ_0.Then $\alpha \times \beta : \Gamma \longrightarrow \Gamma_0 \times \Gamma_0$, $(\alpha \times \beta)(x) = (\alpha(x), \beta(x))$ is a Γ_0-morphism. $\triangle$

A groupoid morphism $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$ is said to be *isomorphism* of groupoids if the maps f and f_0 are bijective.

We observe that a Γ_0- morphism of groupoids $f : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma_0)$ is an isomorphism iff the map $f : \Gamma \longrightarrow \Gamma'$ is bijective.

Definition 1.3. A *subgroupoid* of $(\Gamma; \Gamma_0)$ is a pair (Γ', Γ'_0) of subsets, where $\Gamma' \subseteq \Gamma$, $\Gamma'_0 \subseteq \Gamma_0$ such that the following conditions are verified:

(i) $\alpha(\Gamma') \subseteq \Gamma'_0$, $\beta(\Gamma') \subseteq \Gamma'_0$, $\epsilon(\Gamma'_0) \subseteq \Gamma'$;

(ii) for every $x, y \in \Gamma'$ such that the product $x \cdot y$ is defined implies that $x \cdot y \in \Gamma'$, i.e. Γ' is closed under the partial multiplication.

(iii) $(\forall) x \in \Gamma' \implies x^{-1} \in \Gamma'$. $\triangle$

Example 1.3. Let $(\Gamma; \Gamma_0)$ be a groupoid and $u \in \Gamma_0$. Then:

(a) $\epsilon(\Gamma_0) = \{\widetilde{u} \mid u \in \Gamma_0\}$ is a subgroupoid of Γ over Γ_0, called the *nul subgroupoid* of Γ. .

(b) The isotropy group $\Gamma(u)$ at u and $Is(\Gamma) = \cup_{u \in \Gamma_0} \Gamma_u^u$ are subgroupoids of Γ. $\triangle$

Remark 1.1. If $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$ is a groupoid, then not always, $Imf = \{f(x) \mid x \in \Gamma\}$ is a subgroupoid of Γ'.

In the case when (f, f_0) is a strong morphism of groupoids, then Imf is a groupoid on Imf_0 (see, ([3])). $\triangle$

Definition 1.4. ([7]) (a) A groupoid Γ on Γ_0 is a *Lie groupoid* if Γ and Γ_0 are manifolds, α and β are differentiable submersions so that $\Gamma_{(2)}$ is a differentiable submanifold of the product manifold $\Gamma \times \Gamma$, ϵ is a differentiable embedding and μ and i are differentiable maps.

(b) A Lie groupoid Γ on Γ_0 is *transitive* if $\alpha \times \beta : \Gamma_0 \times \Gamma_0$ is a differentiable submersion.

(c) A morphism of groupoids $f : \Gamma \longrightarrow \Gamma^{\text{`}}$ will be called a *morphism of Lie groupoids* if f is a differentiable map. $\triangle$

We observe that any Lie groupoid $\mathcal{G}$ having e as unity may be considered to be a $\{e\}$ - Lie groupoid. Conversely, a Lie groupoid with one unit is a Lie group.

Example 1.4. (a) (the *coarse Lie groupoid*). Let Γ_0 be a manifold. The manifold $\Gamma_0 \times \Gamma_0$ endowed with the structure of coarse groupoid is a Lie groupoid.

(b) Let (P, π, M, G) be a principal bundle with the Lie group G as structural group. Let us consider the quotient Γ of $P \times P$ by the diagonal action (in other words $\Gamma = P \times P/\rho$ where ρ is the equivalence relation such that $(z, z') \sim (zs, z's)$ for any $s \in G$). Then Γ is a transitive Lie groupoid on M, whose α - fibres are principal bundles, called the *jauge groupoid* of given principal bundle.

(c) (the *fundamantal groupoid of a manifold*). Let B be a manifold and $\Pi(B) = \{(x, [\sigma], y) \mid x, y \in B,\ [\sigma]$ is the the homotopy classe of paths, $\sigma(0) = x, \sigma(1) = y\ \}$. Then $\Pi(B)$ is a groupoid on B with the rules: $\alpha(x, [\sigma], y) = x$, $\beta(x, [\sigma], y) = y$, $\mu((x, [\sigma], y), (y', [\tau], z)) = (x, [\sigma \circ \tau], z)$ iff $y = y'$, where $\sigma \circ \tau$ is the concatenation of paths σ and τ, $\epsilon(x) = (x, [constant], x)$ and $(x, [\sigma], y)^{-1} = (y, [\sigma^{-1}], x)$, where $\sigma^{-1}(t) = \sigma(1-t)$, $(\forall)\ t \in [0,1]$.

If $\Pi(B)$ is equipped with the quotient topology of the compact open topology on the space of paths of B, then $\alpha \times \beta : \Pi(B) \to B \times B$ is a covering map. It follows that $\Pi(B)$ is a Lie groupoid on B, called it the *fundamental* or *Poincaré groupoid associated to* B. Its isotropy groups are the fundamental groups $\pi_1(B,x)$, $(\forall)\, x \in B$.

(d) A *vector bundle* $E \xrightarrow{p} M$ is a group bundle on M. Here $\Gamma = E$ is the total space, $\Gamma_0 = M$ is the base space, $\alpha = \beta = \pi$ so that $\Gamma_{(2)} = \uplus_{x \in M} E_x \times E_x$ (E_x is the fibre at x) and the composition law is fibrewise addition. This groupoid is not transitive.

Proposition 1.1. ([7]) *If* Γ *is a transitive Lie groupoid over* Γ_0, *then*:

(i) *the isotropy groups of* Γ *are Lie groups isomorphes.*

(ii) *each* $u \in \Gamma_0$ *the fibre* $\alpha^{-1}(u)$, *is a differentiable principal bundle over* Γ_0 *with the projection* β *and the isotropy group* $\Gamma(u)$ *as structural group, i.e.:* $\xi_\alpha = (\alpha^{-1}(u), \beta, \Gamma_0; \Gamma(u))$ *is a differentiable principal bundle.*$\triangle$

Definition 1.5. Let $(\Gamma; \Gamma_0)$ be a Lie groupoid. A *Lie subgroupoid* of Γ is a Lie groupoid $(\Gamma'; \Gamma'_0)$ together with a morphism of Lie groupoids $(\varphi, \varphi_0) : (\Gamma'; \Gamma'_0) \to (\Gamma; \Gamma_0)$ which is an injective immersion.

A Lie subgroupoid (Γ', φ) is an *embedded differentiable subgroupoid* if φ is an embedding. $\triangle$

Proposition 1.2. ([1]) *Let* $(\Gamma, \alpha, \beta; \Gamma_0)$ *be a Lie groupoid. Then*:

(a) $Is(\Gamma)$ *is a closed embedded submanifold of* Γ *and it is a Lie subgroupoid of* Γ.

(b) *The* α*-connected component* Γ_α (i.e., *the union of connected components of the points of* Γ_0 *in the* α*-fibres of* Γ *is an open wide Lie subgroupoid of* Γ.

(c) *If* Γ *is a Lie groupoid with connected fibres, then the set* $\tilde{\Gamma}$ *of homotopy classes* α*-fibre to* α*-fibre of the paths contained in* α*-fibres having as origin the elements* u,*for all* $u \in \Gamma_0$, *has a naturally structure of Lie groupoid over* Γ_0, *with simply connected* α*-fibres, such that the projection* $\rho : \tilde{\Gamma} \longrightarrow \Gamma$, $[\sigma] \longrightarrow \rho([\sigma]) := \sigma(1)$ *is a morphism of Lie groupoids.*$\triangle$

2 The action of a Lie groupoid on a manifold

Let $(\Gamma, \alpha, \beta; \Gamma_0)$ be a Lie groupoid and M a manifold for which is given a differentiable map $\rho : M \to \Gamma_0$.

We denote by $M \times_{\Gamma_0} \Gamma = \{(x,a) \in M \times \Gamma \mid \rho(x) = \alpha(a)\}$ the corresponding fibre product.

Definition 2.1. (i) An *action* of Γ on M is a differentiable map $M \times_{\Gamma_0} \Gamma \to M$, $(x,a) \to x \cdot a$ which verify the following conditions:

(2.1) $\rho(x \cdot a) = \beta(a)$, for all $(x,a) \in M \times_{\Gamma_0} \Gamma$;

(2.2) $(x \cdot a) \cdot b = x \cdot (ab)$, for al $(a,b) \in \Gamma_{(2)}$;

(2.3) $x \cdot u = x$, for all $u \in \Gamma_0$.

(ii) We say that a Lie groupoid Γ is a *Lie transformation groupoid on a manifold* M or that Γ *acts (differentiably) on* M *on the right*, if there exists an action $M \times_{\Gamma_0} \Gamma \to M$, $(x,a) \to x \cdot a$. $\triangle$

We also write $R_a x$ for $x \cdot a$.

If we write $a \cdot x$ and assume $\sigma(a \cdot x) = \alpha(a)$, for all $(a,x) \in \Gamma \times_{\Gamma_0} M$, where $\sigma : M \to \Gamma_0$ is a differentiable map and $a \cdot (b \cdot x) = (ab) \cdot x$, for all $(a,b) \in \Gamma_{(2)}$

instead (2.3), we say that Γ *acts on* M *on the left* and we use the notation $L_a x$ for $a \cdot x$ also.

An action of Γ on M on the left is denoted by: $\Gamma \times_{\Gamma_0} M \to M,\ (a,x) \to a \cdot x$, with $\Gamma \times_{\Gamma_0} M = \{\ (a,x) \in \Gamma \times M \mid \sigma(x) = \beta(a)\ \}$.

From definitions, it follows that every element $a \in \Gamma$ induces a transformation of M, denoted by $x \to x \cdot a$, where $x \in M$. In particular, R_u and L_u are identity transformations of M, for all $u \in \Gamma_0$.

We say that Γ acts *effectivelly (resp., freely) on* M if $R_a x = x$, for all $x \in M$ (resp., for some $x \in M$) implies $a = u \in \Gamma_0$.

We say that the action of Γ on M is *proper*, if the map

$M \times_{\Gamma_0} \Gamma \to \Gamma \times \Gamma,\quad (x,a) \to (x, x \cdot a)$ is proper.

Examples 2.1. Let $(\Gamma;\Gamma_0)$ be a Lie groupoid and H a closed wide subgroupoid of Γ.

(i) We let H acts on Γ on the right as follows.

Every $a \in H$ maps $x \in \Gamma$ into $x \cdot a$; $M = \Gamma$ is manifold for which $(H, \alpha, \beta; \Gamma_0$ is Lie groupoid and $\rho = \beta :\ M = \Gamma \to \Gamma_0$ is differentiable map.

In this case, $\Gamma \times_{\Gamma_0} H = \{(x,a) \in \Gamma \times H \mid \beta(x) = \alpha(a)\ \}$.

The conditions (2.1)-(2.3) from Definition 2.1. are easy verified and we obtain that $\Gamma \times_{\Gamma_0} H \to \Gamma,\ \ (x,a) \to x \cdot a$ is an action of H on Γ on the right.

(ii) We let Γ acts on Γ/H on the right as follows. a closed wide subgroupoid of Γ.

The quotient space Γ/H admits a structure of manifold and the projection $\Gamma \to \Gamma/H$ is a differentiable map.

Also, we have that $\rho : \Gamma/H \to \Gamma_0,\ \ \rho(xH) = \beta(x),\ \ (\forall)\ xH \in \Gamma\ /H$ is a differentiable map and the corresponding fibre product is $\Gamma/H \times_{\Gamma_0} \Gamma = \{(xH,a) \in \Gamma/H \times \Gamma \mid \rho(xH) = \alpha(a)\ \}$.

Then the mapping $(xH,a) \to xH \cdot a = xaH$ defines an action of Γ on Γ/H on the right. Indeed, we have:

- $\rho(xH,a) = \rho(xaH) = \beta(xa) = \beta(a),\ \ (\forall)\ (xH,a) \in \Gamma/H \times_{\Gamma_0} \Gamma$;
- $(xH \cdot a) \cdot b = (xa)H \cdot b = ((xa) \cdot b)H = x \cdot (ab)H = xH \cdot (ab),\ (\forall)\ (a,b) \in \Gamma_{(2)}$,
- $xH \cdot u = (xu) \cdot H = xH,\ \ (\forall)\ u \in \Gamma_0$,

i.e. the conditions (2.1) - (2.3) from Definition 2.1. hold.

Γ is a Lie transformation groupoid on Γ/H which is transitive.

The action of Γ on Γ/H is effective iff H does not contains any normal subgroupoid $\neq \Gamma_0$ of Γ.

(iii) We let Γ acts on Γ/H on the left as follows.

The mapping $\sigma : \Gamma/H \to \Gamma_0,\ \ xH \to \sigma(xH) = \alpha(x)$, is a differentiable map from the manifold Γ/H into Γ_0, and the corresponding fibre product is $\Gamma \times_{\Gamma_0} \Gamma/H = \{(a,xH) \in \Gamma \times \Gamma\ \times H \mid \sigma(xH) = \beta(a)\ \}$.

The map $\Gamma \times_{\Gamma_0} \Gamma/H \to \Gamma/H,\ (a,xH) \to a \cdot xH = (ax)H$ is an action of Γ on Γ/H on the left. Indeed, we have:

- $\sigma(a \cdot xH) = \sigma((ax)H) = \alpha(ax) = \alpha(a),\ \ (\forall)\ (a,xH) \in \Gamma \times_{\Gamma_0} \Gamma/H$;
- $(a \cdot (b \cdot xH) = a \cdot (bx)H = (a(bx))H = ((ab)x)H = (ab) \cdot xH,\ (\forall)\ (a,b) \in \Gamma_{(2)}$,
- $u \cdot xH = (ux) \cdot H = xH,\ \ (\forall)\ u \in \Gamma_0$,

i.e. the conditions (2.1) - (2.3) hold.$\triangle$

Example 2.2. Given a Lie groupoid Γ and a manifold M, Γ acts freely on $P = M \times \Gamma$ on the right as follows.

We consider the mapping $\rho : P = M \times \Gamma \to \Gamma_0,\ (x,a) \to \rho(x,a) = \beta(a)$. Then $P \times_{\Gamma_0} \Gamma = \{((x,a),b) \in P \times \Gamma \mid \rho(x,a) = \alpha(b)\ \}$ and for all $(x,a) \in P \times_{\Gamma_0} \Gamma$ we have $\alpha(b) = \beta(a)$ and hence $(a,b) \in \Gamma_{(2)}$.

The action of Γ on $M \times \Gamma$ is given by: $(M \times \Gamma) \times_{\Gamma_0} \Gamma \ \to\ M \times \Gamma,\ ((x,a),b) \to (x,ab)$. $\triangle$

3 Principal fibre bundles with structural Lie groupoid

Let Γ be a Lie groupoid acting freely on a manifold P on the right.

On P define an equivalence relation:

$u \sim v \iff (\exists)\ a \in \Gamma$ such that $(u,a) \in P \times_{\Gamma_0} \Gamma$ and $v = R_a u$,

and denote the equivalence classes by [u], $u \in P$ and the set of equivalence classes by P/Γ.

For each $u \in P$, the orbit $u\Gamma = \{R_a u \mid (\forall)\ a \in \Gamma\ \}$ is closed in P.

The quotient space P/Γ admits a structure of manifold and the projection $\pi : P \longrightarrow P/\Gamma,\ u \to \pi(u) = [u]$ is a differentiable map.

Definition 3.1. ([2]) A *(differentiable) principal bundle over* M *with structural Lie groupoid* Γ consists of a manifold P and an action of Γ on P satisfying the following conditions:

(3.1) Γ acts freely on P on the right: $(u,a) \in P \times_{\Gamma_0} \Gamma \to u \cdot a = R_a u \in P$;

(3.2) M is the quotient space of P by the equivalence relation induced by Γ i.e. $M = P/\Gamma$, and the canonical projection $\pi : P \to M$ is differentiable;

(3.3) P is locally trivial, that is, every point $x \in M$ has a neighborhood U such that $\pi^{-1}(U)$ is diffeomorphic with $U \times \Gamma$ in the sense that there is a diffeomorphism $\psi : \pi^{-1}(U) \to U \times \Gamma$ such that $\psi(u) = (\pi(u), \varphi(u))$, where φ is a mapping of $\pi^{-1}(U)$ into Γ satisfying $\varphi(u \cdot a) = \varphi(u) \cdot a$, for all $u \in \pi^{-1}(U)$ and $a \in \Gamma$, with $u \cdot a \in P$. $\triangle$

A principal fibre bundle with structural Lie groupoid Γ will be denoted by (P, π, M, Γ) or simply P. We call P the *total space*, M the *base space*, Γ the *structural groupoid* and π the *projection.*

For $x \in M$, $\pi^{-1}(x)$ is a closed submanifold of P, which is differentiably isomorphic with Γ. It is called the *fibre* over x.

Remark 3.1. If Γ is a Lie group (i.e. $\Gamma_0 = \{e\}$), then (P, π, M, Γ) is a principal fibre bundle with Γ as structural group. $\triangle$

Example 3.1. (i) Given a Lie groupoid Γ and a manifold B, Γ acts freely on $P = B \times \Gamma$ on the right (see, Example 2.2.) by the action:

$((x,a),b) \in (B \times \Gamma) \times \Gamma \ \to\ (x,ab) \in B \times \Gamma,\ \ (\forall)\ (a,b) \in \Gamma_{(2)}$.

Then $(B \times \Gamma, \pi, B, \Gamma)$, where $\pi : B \times \Gamma \to B$ is the projection on first factor, is a principal fibre bundle with Γ as structural groupoid, called the *trivial principal bundle.*

- (ii) Let (P, π, M, Γ) be a principal fibre bundle with structural groupoid Γ. Given an open submanifold V of M, $\pi^{-1}(V)$ is an open submanifold of P on which Γ acts freely and $(\pi^{-1}(V), \pi, V, \Gamma)$ is a principal fibre bundle over the base V with structural groupoid Γ.

- (iii) Let Γ be a Lie groupoid and H a closed Lie subgroupoid of Γ. H acts differentiably on Γ to the right (see, Example 2.1.(i)) by the action: $(x, a) \in \Gamma \times_{\Gamma_0} H \rightarrow x \cdot a \in \Gamma$.

Then $(\Gamma, \pi, \Gamma/H, H)$ is a principal fibre bundle over the base manifold Γ/H with structural groupoid H.

- (iv) Let $\xi = (P, \pi, M, \Gamma)$ be a principal fibre bundle with Γ as structural groupoid and H a closed Lie subgroupoid of Γ.

Being a Lie subgroupoid of Γ, H acts on P on the right. Let P/H be the quotient space of P by this action of the Lie groupoid H.

We obtain a principal fibre bundle $\xi_H = (P, \pi_H, P/H, H)$ over the base P/H with structural groupoid H, where $\pi_H : P \rightarrow P/H$ is the projection. $\triangle$

Proposition 3.1. *Let $\xi = (P, \pi, M, \Gamma)$ be a principal fibre bundle with structural Lie groupoid Γ. Then for each $u \in \Gamma_0$, the 5-uple*

$\xi_{\Gamma(u)} = (P, \pi_{\Gamma(u)}, P/\Gamma(u), \Gamma(u))$ *is a principal fibre bundle with the isotropy group $\Gamma(u)$ as structural Lie group.*

Proof. The Lie group $\Gamma(u)$ is a Lie subgroupoid of Γ and it acts on P on the right.

We denote by $P/\Gamma(u)$ the quotient space of P by this action of $\Gamma(u)$.

We obtain a principal fibre bundle $\xi_{\Gamma(u)} = (P, \pi_{\Gamma(u)}, P/\Gamma(u), \Gamma(u))$ over the base $P/\Gamma(u)$ with structural group $\Gamma(u)$, where $\pi_{\Gamma(u)} : P \rightarrow P/\Gamma(u)$ is the projection . $\triangle$

By condition (3.3) from Definition 3.1., for a principal fibre bundle (P, π, M, Γ) it is possible to choose an open covering $\{U_i \mid i \in I\}$ of M, each $\pi^{-1}(U_i)$ provided with an diffeomorphism $x \rightarrow (\pi(x), \varphi_i(x))$ of $\pi^{-1}(U_i)$ onto $U_i \times \Gamma$ such that $\varphi_i(x \cdot a) = \varphi_i(x) \cdot a$, $(\forall)$ $(x, a) \in P \times_{\Gamma_0} \Gamma$.

If $x \in \pi^{-1}(U_i \cap U_j)$, and $(x, a) \in P \times_{\Gamma_0} \Gamma$, then we get

$\varphi_j(x \cdot a)(\varphi_i(x \cdot a))^{-1} = \varphi_j(x)(\varphi_i(x))^{-1}$, which shows that $\varphi_j(x)(\varphi_i(x))^{-1}$ depends only on $\pi(x)$ not on x; hence the mapping

$x \in \pi^{-1}(u_i \cap U_j) \rightarrow \varphi_j(x)(\varphi_i(x))^{-1} \in \Gamma$ is constant on each fiber.

We can define a mapping $\psi_{ji} : U_i \cap U_j \rightarrow \Gamma$ by:

$$(3.4) \qquad \psi_{ji}(\pi(x)) = \varphi_j(x)(\varphi_i(x))^{-1}.$$

The familly of mappings ψ_{ji} for all $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$ are called *transitions functions* of the bundle (P, π, M, Γ) corresponding to the open covering $(U_i)_{i \in I}$ of M.

They enjoy the following properties:

$$(3.5) \quad \psi_{ki}(x) = \psi_{kj}(x) \cdot \psi_{ji}(x), \quad (\forall) \quad x \in U_i \cap U_j \cap U_k;$$

$$(3.6) \quad \psi_{ii}(x) = u, \quad \text{where } u \in \Gamma_0 \quad \text{for all } x \in U_i;$$

$$(3.7) \quad \psi_{ij}(x) \cdot \psi_{ji}(x) = u, \quad \text{where } u \in \Gamma_0 \quad \text{for all } x \in U_i \cap U_j.$$

Conversely, given an open covering $(U_i)_{i \in I}$ of M and a set of differentiable mappings ψ_{ji} of $U_i \cap U_j \neq \emptyset$ into a Lie groupoid Γ which satisfy the relation (3.5), we shall construct a differentiable principal fibre bundle with transitions functions $(\psi_{ji})_{i,j \in I}$.

Theorem 3.1. *Let M be a manifold, $(U_i)_{i \in I}$ an open covering of M and Γ a Lie groupoid. For every familly of differentiable mappings $(\psi_{ji})_{i,j \in I}$, which satisfy* *the relation (3.5), where* $\psi_{ji} : U_i \cap U_j \to \Gamma$, *there exists a differentiable principal fibre bundle* (P, π, M, Γ) *with transition functions* $(\psi_{ji})_{i,j\in I}$.

Proof. For each i, consider a space X_i which is the direct product $U_i \times \Gamma$ and let $X = \cup_{i\in I} X_i$ be the topological sum of X_i, $i \in I$. By taking open sets in each X_i as open sets in X, X becomes a differentiable manifold.

We introduce an equivalence relation ρ in X as follows: $(x, a) \in U_i \times \Gamma \subset X$ is equivalent to $(x, b) \in U_j \times \Gamma \subset X$ iff $b = \psi_{ji}(x) \cdot a$ in Γ.

The condition (3.5) and its consequences (3.6) and (3.7) imply that this is an veritable equivalence relation.

Now we define P as the quotient space of X by ρ, i.e. $P = X/\rho$.

We claim that P is the required principal fibre bundle (P, π, M, Γ).

In order to prove this, we first note that Γ acts on P on the right by the following rule:

if $[(x, a)] \in P$ is the equivalence class of $(x, a) \in U_i \times \Gamma$, then we define $[(x, a)] \cdot b$ to the equivalence class of $(x, ab) \in U_i \times \Gamma$, i.e. $[(x, a)] \cdot b = [(x, ab)]$. Γ acts freely on P, as we easily see.

That P admits a structure of differentiable manifold and that Γ acts differentiably on P will be seen latter on.

At any rate, the quotient of P by the equivalence relation by Γ (two points of P are equivalent iff one is mapped into another by some element of Γ) gives a projection π of P onto M.

Observe then that the projection of X onto P is one-to-one on the set X_i so that $\pi^{-1}(U_i)$ is the set of ρ - classes of (x, a), $x \in U_i$ and $a \in \Gamma$ which is hence in one-to-one correspondence with product $U_i \times \Gamma$.

We now introduce a structure of differentiable manifold in P by requiring that $\pi^{-1}(U_i)$ is an open submanifold of P which is diffeomorphic with $U_i \times \Gamma$. This is possible since every $[(x, a)] \in P$ is contained in $\pi^{-1}(U_i)$ for some i and since the identification $(x, a) \in U_i \times \Gamma$ with $(x, \psi_{ji}(x) \cdot a) \in U_j \times \Gamma$ is made by means of differentiable mappings ψ_{ji}.

With this differentiable structure in P, it is no difficult to verify all the conditions so that P becomes a differentiable principal fibre bundle over the base M with Γ as structural groupoid.

Moreover, the transition functions of (P, π, M, Γ) corresponding to the covering $(U_i)_{i\in I}$ are precisely the given $(\psi_{ji})_{i,j\in I}$. $\triangle$

Definition 3.2. Let (P, π, M, Γ) and (P', π', M', Γ') two principal fibre bundles with structural Lie groupoids Γ, Γ', respectively. A *homomorphism* of (P, π, M, Γ) into (P', π', M', Γ') consists of a differentiable mapping $f : P \to P'$ and a Lie groupoid morphism $g : \Gamma \to \Gamma'$ such that $f(x \cdot a) = f(x) \cdot g(a)$, for all $x \in P$ and $a \in \Gamma$ with $(x, a) \in P \times_{\Gamma_0} \Gamma$.

We will denote it by $(f, g) : (P, \pi, M, \Gamma) \to (P', \pi', M', \Gamma')$ or $(f, g) : P \to P'$. $\triangle$

Every homomorphism $(f, g) : P \to P'$ maps each fibre of P into a fibre of P' and hence induces a mapping of $f_0 : M \to M'$.

A homomorphism $(f, g) : (P, \pi, M, \Gamma) \to (P', \pi', M', \Gamma')$ is an *isomorphism* of principal fibre bundles with structural groupoids, if $f : P \to P'$ is a diffeomorphism and $g : \Gamma \to \Gamma'$ is an isomorphism of Lie groupoids.

Applying the Proposition 1.1.(i) and Propositionn 3.1. we have the following proposition.

Proposition 3.2. *Let* $\xi = (P, \pi, M, \Gamma)$ *be a principal fibre bundle with a transitive Lie groupoid* Γ *as structural groupoid. Then the principal fibre bundles* $\xi_{\Gamma(u)} = (P, \pi_{\Gamma(u)}, P/\Gamma(u), \Gamma(u))$, $(\forall)$ $u \in \Gamma_0$ *are isomorphic.* $\triangle$

4 Fibre bundles with type fibre and structural groupoid

Let $\xi = (P, \pi, M, \Gamma)$ be a principal fibre bundle and F a manifold on which Γ acts on the left by $(a, x) \in \Gamma \times_{\Gamma_0} F \rightarrow a \cdot x \in F$.

We shall construct a fibre bundle $\xi_F = (E, \pi_E, M, F, \Gamma)$ associated to ξ with fibre F and groupoid Γ, where $E = P \times_\Gamma F$.

On the product $P \times F$, we let Γ acts on the right as follows.

An element $a \in \Gamma$ maps $(x, y) \in P \times F$ into $(x \cdot a, a^{-1} \cdot y) \in P \times F$, for all $(x, a) \in P \times_{\Gamma_0} \Gamma$ and $(a, y) \in \Gamma \times_{\Gamma_0} F$.

The quotient space of $P \times F$ by this groupoid action is denoted by $E = P \times_\Gamma F$.

In E will be introduced a differentiable structure such that E becomes a differentiable manifold. The mapping $P \times F \to M$ which maps (x, y) into $\pi(x)$ induces a mapping π_E, called the *projection* of E onto M. For each $z \in M$, the set $\pi_E^{-1}(z)$ is called the *fibre* of E over z.

Every point $z \in M$ has a neighboorhood U such that $\pi^1(U)$ is diffeomorphic to $U \times \Gamma$. Identifying $\pi^{-1}(U)$ with $U \times \Gamma$, we see that the action of Γ on $\pi^{-1}(U) \times F$ on the right is given by:

$(x, a, y) \rightarrow (x, a \cdot b, b^{-1} \cdot y)$, for $(x, a, y) \in U \times \Gamma \times F$ and $b \in \Gamma$ such that $(a, b) \in \Gamma_{(2)}$ and $(b, y) \in \Gamma \times_{\Gamma_0} F$.

It follows that the diffeomorphism $\pi^{-1}(U) \cong U \times \Gamma$ induces a diffeomorphism $\pi_E^{-1}(U) \cong U \times F$.

Now, we can therefore introduce a differentiable structure in E by requirement that $\pi_E^{-1}(U)$ is an open submanifold of E wich is diffeomorphic with $U \times F$ under the diffeomorphism $\pi_E^{-1}(U) \cong U \times F$.

The projection π_E is then a differentiable map of E onto M.

We call $(E, \pi_E, M, F, \Gamma) = \xi_F$, the *fibre bundle over the base* M *with (type) fibre* F *and (structural) groupoid* Γ *associated to principal fibre bundle* $\xi = (P, \pi, M, \Gamma)$.

Example 4.1. Let $\xi = (P, \pi, M, \Gamma)$ be a principal fibre bundle and H a closed subgroupoid of Γ.

(i) In a natural way, Γ acts on the quotient space Γ/H on the left (see, Example 2.1.(iii)). Then the fibre bundle over M , with fibre Γ/H and groupoid Γ which is associated to principal fibre bundle $\xi = (P, \pi, M, \Gamma)$ is $\xi_{\Gamma/H} = (E = P \times_\Gamma \Gamma/H, \pi_E, M, \Gamma/H, \Gamma)$.

(ii) Since Γ acts on Γ on the left by translations it follows that the fibre bundle over M with fibre Γ and groupoid Γ associated to ξ is identified with ξ .$\triangle$

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