Gauge Transformations on Holomorphic Bundles

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Abstract

In holomorphic tangent bundle $T'M$ we define a generalization of classical gauge transformation, called complex gauge transformation, and related to it we shall study the invariant geometric objects: d-gauge tensors, nonlinear gauge connection, gauge complex derivatives.

The problem of global invariance concerning a complex Lagrangian is treated in the section related to Einstein-Yang-Mills complex equations. Finally, we shall discuss a few applications regarding infinitesimal complex gauge transformations.

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1 Introduction. The holomorphic bundle $T'M$

In many physical theories related to relativistic quantum, such as spontaneous symmetry-breaking, the Higgs theory, etc., the gauge Lagrangians which are used are complex scalar fields. Consequently, a geometrical approach of such problems requires to create a complex model for gauge geometric fields and their derivatives.

For the real case the geometric methods of fibre bundles with structural group are used for a long period of time in gauge theories ([3]). A generalization for the vectorial bundles of gauge transformation was made in [1], [7].

In a previous paper ([8]) we studied the Lagrange spaces having as base the holomorphic tangent bundle $T'M$, endowed with a nonlinear complex connection. Following such ideas, in the present paper we shall deal with complex gauge transformations on $T'M$.

Briefly, we shall introduce the basic results from [8].

Let us consider M a complex manifold, $\dim_C M = n$, and $(U, (z^i))$ the complex coordinates in a local map. The complexification $T_C M$ of the tangent bundle TM in each $z \in U$ is decomposed in the $(1,0)$-vectors and their conjugates of $(0,1)$-type ([5], [6]),

$$T_C M = T'M \oplus T''M.$$

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The bundle $\pi_T : T'M \longrightarrow M$ is holomorphic and $\dim_C T'M = 2n$. With $V(T'M) = \{\xi \in T'(T'M)' \ \pi_{T*}(\xi) = 0\}$ is denoted the vertical sub-bundle, which is also holomorphic, a local basis in $V(T'M)$ being $\left\{\dfrac{\partial}{\partial \eta^i}\right\}_{i=\overline{1,n}}$.

A nonlinear complex connection, $(nl.c.c.)$ for short, is a distribution N in $T'(T'M)$, $N : u = (z^i, \eta^i) \to H_u(T'M)$ where $H_u(T'M)$ is a supplementary sub-bundle of the vertical bundle in $T'(T'M)$. A local basis in $H_u(T'M)$ is denoted by

$$\left\{\frac{\delta}{\delta z^i} = \frac{\partial}{\partial z^i} - N_i^j \frac{\partial}{\partial \eta^j}\right\}_{i,j=\overline{1,n}},$$

where N_i^j are the $(nl.c.c.)$-coefficients and satisfying the following rules of transformation at the change of local map

$$N'^i_k \frac{\partial z'^k}{\partial z^j} = \frac{\partial z'^i}{\partial z^k} N_j^k - \frac{\partial^2 z'^i}{\partial z^j \partial z^k} \eta^k. \tag{1.1}$$

From (1.1) we can easily verify that the adapted basis $\left\{\dfrac{\delta}{\delta z^i}\right\}$ is changed after the rule

$$\frac{\delta}{\delta z^i} = \frac{\partial z'^j}{\partial z^i} \frac{\delta}{\delta z'^j}. \tag{1.2}$$

Taking the conjugate bundles $\overline{H(T'M)}$ and $\overline{V(T'M)}$ with the corresponding local bases $\left\{\dfrac{\delta}{\delta \overline{z}^i}\right\}$ and $\left\{\dfrac{\partial}{\partial \overline{\eta}^i}\right\}$, we obtain the following decomposition for the complexified bundle:

$$T_C M = H(T'M) \oplus V(T'M) \oplus \overline{H(T'M)} \oplus \overline{V(T'M)}, \tag{1.3}$$

with the corresponding projectors denoted by $h, v, \overline{h}, \overline{v}$.

Furthermore, we shall use the following abbreviate notations: $\{\delta_i, \partial_i, \delta_{\overline{i}}, \partial_{\overline{i}}\}$ for the adapted bases in (1.3) decomposition.

Let us consider D a derivative law on $T_C(T'M)$. For a given $(nl.c.c.)$ N, the derivative D is called *N-linear complex connection,* shortly: $N-(l.c.c.)$, if D preserves the four distributions from (1.3). In the adapted basis $\{\delta_i, \partial_i, \delta_{\overline{i}}, \partial_{\overline{i}}\}$ the $N-(.c.c.)D$ has the following local expression:

$$\begin{array}{llll}
D_{\delta_k}\delta_j = \overset{1}{L}{}^i_{jk}\ \delta_i; & D_{\partial_k}\delta_j = \overset{1}{C}{}^i_{jk}\ \delta_i; & D_{\delta_{\overline{k}}}\delta_j = \overset{3}{L}{}^i_{j\overline{k}}\ \delta_i; & D_{\partial_{\overline{k}}}\delta_j = \overset{3}{C}{}^i_{j\overline{k}}\ \delta_i; \\
D_{\delta_k}\partial_j = \overset{2}{L}{}^i_{jk}\ \partial_i; & D_{\partial_k}\partial_j = \overset{2}{C}{}^i_{jk}\ \partial_i; & D_{\delta_{\overline{k}}}\partial_j = \overset{4}{L}{}^i_{j\overline{k}}\ \partial_i; & D_{\partial_{\overline{k}}}\partial_j = \overset{4}{C}{}^i_{j\overline{k}}\ \partial_i; \\
D_{\delta_k}\delta_{\overline{j}} = \overset{3}{L}{}^{\overline{i}}_{\overline{j}k}\ \delta_{\overline{i}}; & D_{\partial_k}\delta_j = \overset{3}{C}{}^{\overline{i}}_{\overline{j}k}\ \delta_{\overline{i}}; & D_{\delta_{\overline{k}}}\delta_{\overline{j}} = \overset{1}{L}{}^{\overline{i}}_{\overline{j}\overline{k}}\ \delta_{\overline{i}}; & D_{\partial_{\overline{k}}}\delta_{\overline{j}} = \overset{1}{C}{}^{\overline{i}}_{\overline{j}\overline{k}}\ \delta_{\overline{i}}; \\
D_{\delta_k}\partial_{\overline{j}} = \overset{4}{L}{}^{\overline{i}}_{\overline{j}k}\ \partial_{\overline{i}}; & D_{\partial_k}\partial_{\overline{j}} = \overset{4}{C}{}^{\overline{i}}_{\overline{j}k}\ \partial_{\overline{i}}; & D_{\delta_{\overline{k}}}\partial_{\overline{j}} = \overset{2}{L}{}^{\overline{i}}_{\overline{j}\overline{k}}\ \partial_{\overline{i}}; & D_{\partial_{\overline{k}}}\partial_{\overline{j}} = \overset{2}{C}{}^{\overline{i}}_{\overline{j}\overline{k}}\ \partial_{\overline{i}},
\end{array}$$

with $\overline{D_X Y} = D_{\overline{X}}\overline{Y}$. If

$$\overset{1}{L}{}^{i}_{jk}=\overset{2}{L}{}^{i}_{jk}=L^{i}_{jk};\ \overset{3}{L}{}^{\bar{i}}_{\bar{j}k}=\overset{4}{L}{}^{\bar{i}}_{\bar{j}k}=\overline{L}{}^{\bar{i}}_{\bar{j}k};\ \overset{1}{C}{}^{i}_{jk}=\overset{2}{C}{}^{i}_{jk}=C^{i}_{jk};\ \overset{3}{C}{}^{\bar{i}}_{\bar{j}k}=\overset{4}{C}{}^{\bar{i}}_{\bar{j}k}=C^{\bar{i}}_{\bar{j}k}$$

(and their conjugates coincide too), then the $N-l.c.c.$ D is said to be *normal*, and will be denoted by $M-(l.c.c)$.

The components of the torsion and the curvature tensors of one $M-(l.c.c.)$, D have been calculated in [8].

A $N-(l.c.c.)$ is decomposed in : $D = D' + D''$, where $D'' = \overline{D'}$ and their turn for each of them: $D' = D'^h + D'^v$, $D'' = D''^h + D''^v$, with $D'^h = D_{\delta_k}$, $D'^v = D_{\partial_k}$, $D''^h = D_{\delta_{\bar{k}}}$, $D''^v = D_{\partial_{\bar{k}}}$. A system of functions on $T'M$, $w^{i_1\ldots i_p\bar{i}_1\ldots\bar{i}_r}_{j_1\ldots j_q\bar{j}_1\ldots\bar{j}_s}(z,\eta)$ is called a *d−complex tensor field* of $\begin{pmatrix} p & \overline{r} \\ q & \overline{s}\end{pmatrix}$-type if at change of local maps on M it well be modified after the rule:

$$w'^{i_1\ldots i_p\bar{i}_1\ldots\bar{i}_r}_{j_1\ldots j_q\bar{j}_1\ldots\bar{j}_s}(z',\eta') = \frac{\partial z'^{i_1}}{\partial z^{k_1}}\cdots\frac{\partial z'^{i_p}}{\partial z^{k_p}}\cdot\frac{\partial z^{h_1}}{\partial z'^{j_1}}\cdots\frac{\partial z^{h_q}}{\partial z'^{j_q}}\cdot$$
$$\cdot\frac{\partial \overline{z}'^{\bar{i}_1}}{\partial \overline{z}^{\bar{k}_1}}\cdots\frac{\partial \overline{z}'^{\bar{i}_r}}{\partial \overline{z}^{\bar{k}_r}}\cdot\frac{\partial \overline{z}^{\bar{h}_1}}{\partial \overline{z}'^{\bar{j}_1}}\cdots\frac{\partial \overline{z}^{\bar{h}_r}}{\partial \overline{z}'^{\bar{j}_r}} w^{k_1\ldots k_p\bar{k}_1\ldots\bar{k}_r}_{h_1\ldots h_p\bar{h}_1\ldots\bar{h}_s}(z,\eta).$$

The derivations of the $d-$tensor w will be expressed by "$|$" and by "$||$" for $h-$and respectively $v-$covariant derivative D', and by "$\bar{|}$", "$\overline{||}$" respectively for $\overline{h}-$and $\overline{v}-$covariant derivative D''.

A metric Hermitian structure G on $T_C(T'M)$ is defined by a $d-$complex tensor $g_{i\bar{j}}(z,\eta)$ of $\begin{pmatrix} 0 & \overline{0} \\ 1 & \overline{1}\end{pmatrix}$-type, nondegenerate, so that:

$$G = g_{i\bar{j}}dz^i \otimes d\overline{z}^j + g_{i\bar{j}}\delta\eta^i \otimes \delta\overline{\eta}^j, \tag{1.6}$$

where $(dz^i, \delta\eta^i, d\overline{z}^j, \delta\overline{\eta}^j)$ is the dual adapted basis.

2 Gauge complex transformations

Let us consider the holomorphic bundle $\pi_T : T'M \to M$.

Definition 2.1 A *gauge transformation* on complex manifold M is a pair of analytic isomorphisms $\Upsilon = (F^0, F^1)$ on the manifolds $F^0 : M \to M$ and $F^1 : T'M \to T'M$, satisfying the property

$$\pi_T \circ F^1 = F^0 \circ \pi_T. \tag{2.1}$$

We can see that a gauge transformation Υ preserves the geometric structures of the manifold and the whole set determine a group structure with respect to the composition of mappings.

It is useful to obtain a local representation of complex gauge transformation Υ.

Supposing that $\Upsilon = (F^0, F^1)$ applies the point $u = (z^i, \eta^i) \in \pi_T^{-1}(U_\alpha)$ in $\widetilde{u} = (\widetilde{z}^i, \widetilde{\eta}^i) \in \pi_T^{-1}(U_\beta)$ and by adding the condition (2.1), it results

Proposition 2.1 *A gauge complex transformation $\Upsilon : u \to \widetilde{u}$ is locally given by a system of analytic functions:*

$$\widetilde{z}^i = X^i(z), \qquad \widetilde{\eta}^i = Y^i(z,\eta), \tag{2.2}$$

with the regularity condition $\det\left(\frac{\partial X^i}{\partial z^j}\right)\cdot\det\left(\frac{\partial Y^i}{\partial \eta^j}\right)\neq 0$.

At the changes of local coordinates at u and $\widetilde{u}$ it is necessary to require the conditions of global existence of the transformation Υ. If at $\widetilde{u}=(\widetilde{z}^i,\widetilde{\eta}^i)$ we consider the local changes of the coordinates $\widetilde{z}'^i=\widetilde{z}'^i(z)$; $\widetilde{\eta}'^i=\frac{\partial\widetilde{z}'^i}{\partial\widetilde{z}^j}\widetilde{\eta}^j$, then according to (2.2) we have

$$\widetilde{z}'^i = X'^i(z); \qquad \widetilde{\eta}'^i = Y'^i(z,\eta), \tag{2.3}$$

and the inverses $X'^i(z)=\widetilde{z}'^i(X(z))$, $\qquad Y'^i(z,\eta)=\frac{\partial\widetilde{z}'^i}{\partial\widetilde{z}^j}Y^j(z,\eta)$. The global existence of Υ transformation implies:

$$X'^i(z'(z)) = X^i(z) \;\; ; \quad Y'^i(z'(z),\eta'(z,\eta)) = Y^i(z,\eta), \tag{2.4}$$

with the regularity conditions of transformation.

Next, it is convenient to denote the following derivatives by $X^i_j=\frac{\partial X^i}{\partial z^j}$, $Y^i_j=\frac{\partial Y^i}{\partial \eta^j}$, their inverses by $\overset{*}{X}{}^i_j$, $\overset{*}{Y}{}^i_j$, and the conjugates by $\overline{X^i_j}$, etc.

From (2.4) we infer

Proposition 2.2 *At the change of local maps, the following derivatives changed by the rules*

$$X'^i_k=\frac{\partial\widetilde{z}'^i}{\partial\widetilde{z}^j}X^j_k,\quad Y'^i_k=\frac{\partial\widetilde{z}'^i}{\partial\widetilde{z}^j}Y^j_k,\quad X^i_k=\frac{\partial z^i}{\partial z'^j}X'^j_k,\quad Y^i_k=\frac{\partial z^i}{\partial z'^j}Y'^j_k.$$

Definition 2.2 *A d−gauge complex tensor* of $\begin{pmatrix} p & \overline{r} \\ q & \overline{s}\end{pmatrix}$ type is a system of functions $w^{i_1\ldots i_p\overline{i}_1\ldots\overline{i}_r}_{j_1\ldots j_q\overline{j}_1\ldots\overline{j}_s}(z,\eta)$, which satisfies in addition to (1.5) the following rules of change

$$\begin{aligned}\widetilde{w}^{i_1\ldots i_p\overline{i}_1\ldots\overline{i}_r}_{j_1\ldots j_q\overline{j}_1\ldots\overline{j}_s}(\widetilde{z},\widetilde{\eta}) &= X^{i_1}_{h_1}\cdots X^{i_p}_{h_p}\cdot\overset{*}{X}{}^{k_1}_{j_1}\cdot\ldots\cdot\overset{*}{X}{}^{k_q}_{j_q}\cdot\overline{X}^{\overline{i}_1}_{\overline{h}_1}\cdot\ldots\cdot \\ &\cdot\;\overline{X}^{\overline{i}_r}_{\overline{h}_r}\cdot\overline{\overset{*}{X}}{}^{\overline{k}_1}_{\overline{j}_1}\cdot\ldots\cdot\overline{\overset{*}{X}}{}^{\overline{k}_s}_{\overline{j}_s}\cdot\cdot w^{h_1\ldots h_p\overline{h}_1\ldots\overline{h}_r}_{k_1\ldots k_q\overline{k}_1\ldots\overline{k}_s}(z,\eta).\end{aligned} \tag{2.5}$$

Denote by J the natural almost tangent structure on $T'M$, which is defined by $J\left(\frac{\partial}{\partial z^i}\right)=\frac{\partial}{\partial\eta^i}$, $J\left(\frac{\partial}{\partial\eta^i}\right)=0$, and is globally defined. The kernel of J is just the vertical distribution, and if N is a $(nl.c.c.)$ then $J(\frac{\delta}{\delta z^i})=\frac{\partial}{\partial\eta^i}$. This means that the image of the horizontal distribution is the vertical one.

Definition 2.3 A nonlinear complex gauge connection (shortly $(nl.c.g.c)$) is a $(nl.c.c)$ N which is preserved by the tangent map of the transformation, i.e., $\Upsilon_*: T'_u(T'M)\to T'_{\widetilde{u}}(T'M)$ preserves the distributions.

If $\left\{ \frac{\delta}{\delta\widetilde{z}^i} = \frac{\partial}{\partial\widetilde{z}^i} - \widetilde{N}_i^j \frac{\partial}{\partial\eta^j};\ \frac{\partial}{\partial\widetilde{\eta}^i} \right\}$ is the adapted basis in $T'_{\widetilde{u}}(T'M)$ then for Υ_* to preserve the distributions it is necessary that $J(\frac{\delta}{\delta\widetilde{z}^i}) = \frac{\partial}{\partial\widetilde{\eta}^i}$. For this reason N is a $(nl.c.g.c.)$ if

$$\text{(2.6)} \qquad \frac{\delta}{\delta z^i} = X_i^j \frac{\delta}{\delta\widetilde{z}^j} \quad \text{and} \quad \frac{\partial}{\partial\eta^i} = X_i^j \frac{\partial}{\partial\widetilde{\eta}^j},$$

this means that the elements of the adapted bases are $d-$gauge complex tensors.
Proposition 2.3. *The coefficients $N_i^j(z,\eta)$ of one (nl.c.g.c.) satisfy in addition to the condition (1.1) the next transformation law* $\widetilde{N}_k^j X_j^k = X_k^i N_j^k - \frac{\partial Y^i}{\partial z^k}$. The proof results from the first relation (2.6)

Considering the conjugate $\widetilde{N}_i^j$ one of the $(nl.c.g.c.)$ and the corresponding adapted bases on $T''(T'M)$, we obtain that the extension Υ_* of the tangent gauge transformation to the whole complexification $T_C(T'M)$ preserves the four distributions.

3 Complex gauge derivatives

For a given $(nl.c.g.c.)N$, let us consider the components $D^h,\ D^v,\ D^{\widetilde{h}},\ D^{\widetilde{v}}$ of one $N-(l.c.c.)$ D on $T_C(T'M)$ with the local coefficients written in formula (1.4), and which at the change of local maps on M are transformed by the rules

$$\text{(3.1)} \quad \overset{\alpha}{L}{}'^{i}_{jk} \frac{\partial z'^j}{\partial z^h}\frac{\partial z'^k}{\partial z^m} = \overset{\alpha}{L}{}^{l}_{hm} \frac{\partial z'^i}{\partial z^l} - \frac{\partial^2 z'^i}{\partial z^h \partial z^m} \qquad \overset{\alpha}{C}{}'^{i}_{jk} \frac{\partial z'^j}{\partial z^h}\frac{\partial z'^k}{\partial z^m} = \overset{\alpha}{C}{}^{l}_{hm} \frac{\partial z'^i}{\partial z^l}\ ;\ \alpha = 1,2.$$

For the conjugate indices the rules are obtained from (3.1) deriving with respect to the conjugate basis.

The covariant derivatives $D^h,\ D^v,\ D^{\widetilde{h}},\ D^{\widetilde{v}}$ act on one complex $d-$tensor field w of $\begin{pmatrix} p & \overline{r} \\ q & \overline{s} \end{pmatrix}-$type, determining the following $d-$tensor fields

$$w^{i_1\dots i_p \overline{i}_1\dots\overline{i}_r}_{j_1\dots j_q \overline{j}_1\dots\overline{j}_s\,|m}\ ;\ w^{i_1\dots i_p \overline{i}_1\dots\overline{i}_r}_{j_1\dots j_q \overline{j}_1\dots\overline{j}_s\,\|m},\quad w^{i_1\dots i_p \overline{i}_1\dots\overline{i}_r}_{j_1\dots j_q \overline{j}_1\dots\overline{j}_s\,\overline{|}m},\quad w^{i_1\dots i_p \overline{i}_1\dots\overline{i}_r}_{j_1\dots j_q \overline{j}_1\dots\overline{j}_s\,\overline{\|}m}.$$

Proposition 3.1 *The covariant derivatives $D^h,\ D^v,\ D^{\widetilde{h}},\ D^{\widetilde{v}}$ of one complex $d-$gauge tensor field are $d-$complex gauge tensor fields if and only if the local coefficients of D derivative satisfy in addition to (3.1) the following rules*

$$\text{(3.2)} \quad \overset{\alpha}{\widetilde{L}}{}^{i}_{jk} = \overset{*}{X}{}^h_j \overset{*}{X}{}^m_k\ X^i_l\ \overset{\alpha}{L}{}^{l}_{hm} - \overset{*}{X}{}^h_j \overset{*}{X}{}^m_k \frac{\delta X^i_m}{\delta z^h} \qquad \overset{\alpha}{\widetilde{C}}{}^{i}_{jk} = \overset{*}{X}{}^h_j \overset{*}{X}{}^m_k\ X^i_l\ \overset{\alpha}{C}{}^{l}_{hm}, \quad \alpha = 1,2.$$

and analogous conditions for the conjugates indices, the $\overset{*}{X}{}^i_j$ is replaced by its conjugate $\overset{*}{\overline{X}}{}^{\overline{i}}_{\overline{j}}$.

Proof. For instance, starting with the first relation (1.4) $D_{\delta_k}\delta_j = \overset{1}{L}{}^i_{jk}\ \delta_i$, and considering (2.6) and their conjugates it results directly the relations (3.2).

Definition 3.1 A $N-$linear complex connection D satisfying the conditions from *Prop.3.1* is named a $N-$*linear complex gauge connection* $(N-(l.c.g.c.))$.

The corresponding gauge complex derivatives will be specified by $\mathcal{D}^h$, $\mathcal{D}^v$, $\mathcal{D}^{\widetilde{h}}$, $\mathcal{D}^{\widetilde{v}}$, replacing the short, long and conjugate bar of derivation of a $d-$complex gauge tensor.

Definition 3.2. A $N-(l.c.g.c.), D$ is said to be *metrical* if

$$\mathcal{D}^h g_{i\bar{j}} = 0 \ ; \ \mathcal{D}^v g_{i\bar{j}} = 0, \quad \mathcal{D}^{\overline{h}} g_{i\bar{j}} = 0, \quad \mathcal{D}^{\overline{v}} g_{i\bar{j}} = 0, \tag{3.3}$$

where $g_{i\bar{j}}$ defines the Hermitian metric structure (1.6).

If we use [8] we obtain a $N-(l.c.c.)$, denoted by $\overset{c}{D}$ and called *canonical connection*, which is metrical and $M-$connection, too:

$$\begin{aligned} \overset{c}{L}{}^i_{jk} &= \tfrac{1}{2} g^{\bar{l}i} \left(\frac{\delta g_{j\bar{l}}}{\delta z^k} + \frac{\delta g_{k\bar{l}}}{\delta z^j} \right), \quad \overset{c}{C}{}^i_{jk} = \frac{1}{2} g^{\bar{l}i} \left(\frac{\partial g_{j\bar{l}}}{\partial \eta^k} + \frac{\partial g_{k\bar{l}}}{\partial \eta^j} \right), \\ \overset{c}{L}{}^{\bar{i}}_{\bar{j}k} &= \frac{1}{2} g^{\bar{i}l} \left(\frac{\delta g_{l\bar{j}}}{\delta z^k} - \frac{\delta g_{k\bar{j}}}{\delta z^l} \right), \quad \overset{c}{C}{}^{\bar{i}}_{\bar{j}k} = \frac{1}{2} g^{\bar{i}l} \left(\frac{\partial g_{l\bar{j}}}{\partial \eta^k} - \frac{\partial g_{k\bar{j}}}{\partial \eta^l} \right), \end{aligned} \tag{3.4}$$

and the conjugates.

Reiterating the calculus in Proposition 3.1 for the canonical connection $\overset{c}{D}$, using the conditions (2.6) of $(nl.c.g.c.)$, we infer

Theorem 3.2. *The canonical connection* $\overset{c}{D}$ *is a* $N-(l.c.g.c.)$ *with the* T^i_{jk} *and* S^i_{jk} *vanishing torsions.*

The connection (3.4) will be called the *Miron canonical complex connection.*

4 Complex Einstein-Yang-Mills equations

Let us consider N a $(nl.c.g.c.)$ and $L_0 : T'M \to R$ a complex Lagrangian, i.e.,

$$g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \overline{\eta}^J} \tag{4.1}$$

is a nondegenerated $d-$tensor.

From (4.1) we remark that $g_{i\bar{j}}$ is a gauge complex tensor. The $(nl.c.g.c.), N$ can be determined from L_0 as is shown in [8].

In calculus L_0 depends on the point $u = (z, \eta)$ via the wave functions $\Phi^A, A = \overline{1,p}$, which are gauge fields, and on their derivatives which are supposed to be with respect to the adapted basis

$$L_0(z, \eta) = L \left(\Phi^A, \frac{\delta \Phi^A}{\delta z^i}, \frac{\delta \Phi^A}{\delta \overline{z}^i}, \frac{\partial \Phi^A}{\partial \eta^i}, \frac{\partial \Phi^A}{\partial \overline{\eta}^i} \right). \tag{4.2}$$

The Euler equation gives the extremes of action of the L_0 Lagrangian. But this action depends on the local maps. To remove this drawback it is useful to consider the following modified Lagrangian

$$\mathcal{L}(z, \eta) = L_0(z, \eta) \cdot |g|^2, \tag{4.3}$$

where $|g| = |\det(g_{i\overline{j}})|$.

Then the action $I = \int \mathcal{L}(z,\eta)d\omega$ does not depend on the local coordinates, and if Φ is one of the generic field Φ^A, then the direct calculus gives the following

Proposition 4.1. *The extremum of the action, $\delta I(\Phi) = 0$, determines the Euler-Lagrange complex equation*

(4.4)

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \frac{\partial}{\partial z^i}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \Phi}{\partial z^i}\right)}\right) - \frac{\partial}{\partial \overline{z}^i}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \Phi}{\partial \overline{z}^i}\right)}\right) - \frac{\partial}{\partial \eta^i}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \Phi}{\partial \eta^i}\right)}\right) - \frac{\partial}{\partial \overline{\eta}^i}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \Phi}{\partial \overline{\eta}^i}\right)}\right) = 0.$$

The complex $(E-L)$ equation can be rewritten in an adapted basis $\{\delta_i, \partial_i, \delta_{\overline{i}}, \partial_{\overline{i}}\}$ in the equivalent form

$$\begin{aligned} |g|^2 \Bigg\{ \frac{\partial L}{\partial \Phi} - \frac{\delta}{\delta z^i}\left(\frac{\partial L}{\partial\left(\frac{\delta \Phi}{\delta z^i}\right)}\right) - \frac{\delta}{\delta \overline{z}^i}\left(\frac{\partial L}{\partial\left(\frac{\delta \Phi}{\delta \overline{z}^i}\right)}\right) - \frac{\partial}{\partial \eta^i}\left(\frac{\partial L}{\partial\left(\frac{\partial \Phi}{\partial \eta^i}\right)}\right)_{|c_1} - \\ - \frac{\partial}{\partial \overline{\eta}^i}\left(\frac{\partial L}{\partial\left(\frac{\partial \Phi}{\partial \overline{\eta}^i}\right)}\right)_{|c_2} + \frac{\partial N_j^i}{\partial \eta^i}\frac{\partial L}{\partial\left(\frac{\delta \Phi}{\delta z^j}\right)} + \frac{\partial \overline{N}_j^i}{\partial \overline{\eta}^i}\frac{\partial L}{\partial\left(\frac{\delta \Phi}{\delta \overline{z}^j}\right)} \Bigg\} = \\ = \frac{\delta |g|^2}{\delta z^i}\cdot\frac{\partial L}{\partial\left(\frac{\delta \Phi}{\delta z^i}\right)} + \frac{\delta |g|^2}{\delta \overline{z}^i}\cdot\frac{\partial L}{\partial\left(\frac{\delta \Phi}{\delta \overline{z}^i}\right)} + \frac{\partial |g|^2}{\partial \eta^i}\cdot\frac{\partial L}{\partial\left(\frac{\partial \Phi}{\partial \eta^i}\right)} + \frac{\partial |g|^2}{\partial \overline{\eta}^i}\cdot\frac{\partial L}{\partial\left(\frac{\partial \Phi}{\partial \overline{\eta}^i}\right)}, \end{aligned} \tag{4.5}$$

with $c_1 = \dfrac{\delta \Phi}{\delta z^i}$; $c_2 = \dfrac{\delta \Phi}{\delta \overline{z}^i}$ taking constant values.

The formula (4.5) shows that the $(E.L)$ equation is invariant to the local change of coordinates.

Next, we shall use the notations

$$\overset{h}{\Phi}{}^i_A = \frac{\partial L}{\partial\left(\frac{\delta \Phi}{\delta z^i}\right)}, \quad \overset{\overline{h}}{\Phi}{}^{\overline{i}}_A = \frac{\partial L}{\partial\left(\frac{\delta \Phi}{\delta \overline{z}^i}\right)}, \quad \overset{v}{\Phi}{}^i_A = \frac{\partial L}{\partial\left(\frac{\partial \Phi}{\partial \eta^i}\right)}, \quad \overset{\overline{v}}{\Phi}{}^{\overline{i}}_A = \frac{\partial L}{\partial\left(\frac{\partial \Phi}{\partial \overline{\eta}^i}\right)},$$

in order to abbreviate the written form of (4.5).

Also, for a given $N-(l.c.g.c.)D$ (for instance the (3.4) canonical connection), let us consider the $h-, v-, \overline{h}-, \overline{v}-$derivations of one vector $X = (X^i)$,

$$\begin{aligned} X^i_{|k} &= \frac{\delta X^i}{\delta z^k} + L^i_{jk}X^j, \qquad & X^i_{\|k} &= \frac{\partial X^i}{\partial \eta^k} + C^i_{jk}X^j, \\ X^i_{|\overline{k}} &= \frac{\delta X^i}{\delta \overline{z}^k} + L^i_{j\overline{k}}X^j, \qquad & X^i_{\|\overline{k}} &= \frac{\partial X^i}{\partial \overline{\eta}^k} + C^i_{j\overline{k}}X^j, \end{aligned} \tag{4.6}$$

and their conjugates. So, the formula (4.5) is written in the equivalent form

$$\frac{\partial L}{\partial \Phi^A} - \overset{h}{\Phi}{}^i_{A|i} - \overset{\overline{h}}{\Phi}{}^{\overline{i}}_{A|\overline{i}} - \overset{v}{\Phi}{}^i_{A\|i} - \overset{\overline{v}}{\Phi}{}^{\overline{i}}_{A\|\overline{i}} = E_A, \tag{4.7}$$

where

$$E_A = \frac{1}{|g|^2}\left\{\frac{\delta|g|^2}{\delta z^i}\overset{h}{\Phi}{}^i_A + \frac{\delta|g|^2}{\delta\overline{z}^i}\overset{\overline{h}}{\Phi}{}^{\overline{i}}_A + \frac{\partial|g|^2}{\partial\eta^i}\overset{v}{\Phi}{}^i_A + \frac{\partial|g|^2}{\partial\overline{\eta}^i}\overset{\overline{v}}{\Phi}{}^{\overline{i}}_A\right\} - \tag{4.8}$$

$$- \left(L^i_{jk} + \frac{\partial N^i_j}{\partial\eta^i}\right)\overset{h}{\Phi}{}^i_A - \left(L^{\overline{i}}_{\overline{j}\overline{i}} + \frac{\partial\overline{N}^{\overline{i}}_{\overline{j}}}{\partial\overline{\eta}^{\overline{i}}}\right)\overset{\overline{h}}{\Phi}{}^{\overline{i}}_A - C^i_{ji}\overset{v}{\Phi}{}^i_A - C^{\overline{i}}_{\overline{j}\overline{i}}\overset{\overline{v}}{\Phi}{}^{\overline{i}}_A$$

are gauge scalars if D is a $N-(l.c.g.c.)$.

Let us note that the invariance of the $(E-L)$ equation is assured if the Lagrangian L is gauge invariant.

We shall analyze the gauge invariance of Lagrangian L in the particular case of gauge infinitesimal transformations.

Let us consider G a group of transformations, $\dim G = m$, that acts on $T'M$,

$$\widetilde{u} = \Upsilon(u,a), \qquad u = \Upsilon(u,0), \tag{4.9}$$

with $u = (z^i, \eta^i)$ and $a = (a^1, a^2, \dots a^m) \in G$.

At an infinitesimal holomorphic of the group G, we have

$$\widetilde{z}^i = z^i + \overset{h}{\xi}{}^i_\lambda\,\varepsilon^\lambda, \quad \widetilde{\eta}^i = \eta^i + \overset{v}{\xi}{}^i_\lambda\,\varepsilon^\lambda. \tag{4.10}$$

Accordingly, the gauge fields $\Phi^A(u)$ will be transformed by the rules

$$\widetilde{\Phi}^A = \Phi^A + (X_\lambda\Phi^A)\varepsilon^\lambda, \tag{4.11}$$

where $X_\lambda = \overset{h}{\xi}{}^i_\lambda\frac{\partial}{\partial z^i} + \overset{v}{\xi}{}^i_\lambda\frac{\partial}{\partial\eta^i}$ are the generators of the group.

As in classical theory, let us consider a $p-$dimensional complex representation of the generators $X_\lambda \to [X_\lambda]^A_B$. Then the infinitesimal transformation of the gauge fields becomes

$$\widetilde{\Phi}^A = \Phi^A + ([X_\lambda]^A_B\Phi^B)\varepsilon^\lambda. \tag{4.12}$$

The gauge infinitesimal invariance condition of the Lagrangian L is that of vanishing its variation,

$$\frac{\partial L}{\partial\Phi^A}\delta\Phi^A + \overset{h}{\Phi}{}^i_A\,\delta\left(\frac{\delta\Phi^A}{\delta z^i}\right) + \overset{\overline{h}}{\Phi}{}^{\overline{i}}_A\,\delta\left(\frac{\delta\Phi^A}{\delta\overline{z}^i}\right) + \overset{v}{\Phi}{}^i_A\,\delta\left(\frac{\partial\Phi^A}{\partial\eta^i}\right) + \overset{\overline{v}}{\Phi}{}^{\overline{i}}_A\,\delta\left(\frac{\partial\Phi^A}{\partial\overline{\eta}^i}\right) = 0.$$

We propose now to discuss only the global invariance, meaning that ε^λ are constants, and hence the results

$$\delta\left(\frac{\delta\Phi^A}{\delta z^i}\right) = \frac{\delta}{\delta z^i}(\delta\Phi^A) = \varepsilon^\lambda[X_\lambda]^A_B\frac{\delta\Phi^B}{\delta z^i}, \quad \delta\left(\frac{\partial\Phi^A}{\partial\eta^i}\right) = \varepsilon^\lambda[X_\lambda]^A_B\frac{\partial\Phi^B}{\partial\eta^i},$$

$$\delta\left(\frac{\delta\Phi^A}{\delta\overline{z}^i}\right) = \frac{\delta}{\delta\overline{z}^i}(\delta\Phi^A) = \varepsilon^\lambda[X_\lambda]^A_B\frac{\delta\Phi^B}{\delta\overline{z}^i}, \quad \delta\left(\frac{\partial\Phi^A}{\partial\overline{\eta}^i}\right) = \varepsilon^\lambda[X_\lambda]^A_B\frac{\partial\Phi^B}{\partial\overline{\eta}^i}.$$

Replacing these variations in the invariance condition of the Lagrangian, it results

$$\left\{\frac{\partial L}{\partial \Phi^A}\Phi^B + \Phi^i_A\frac{\delta \Phi^B}{\delta z^i} + \Phi^{\bar{i}}_A\frac{\delta \Phi^B}{\delta \overline{z}^i} + \Phi^i_A\frac{\partial \Phi^B}{\partial \eta^i} + \Phi^{\bar{i}}_A\frac{\partial \Phi^B}{\partial \overline{\eta}^i}\right\}[X_\lambda]^A_B = 0. \tag{4.13}$$

Now, combining the formulae (4.7) and (4.13) we obtain the global invariance law of complex Lagrangian L at the gauge infinitesimal transformation (4.11)

$$\begin{aligned}&\Big\{E_A\Phi^B + \overset{h}{\Phi^i_{A|i}}\Phi^B + \overset{\overline{h}}{\Phi^{\bar{i}}_{A|\bar{i}}}\Phi^B + \overset{v}{\Phi^i_{A\|i}}\Phi^B + \overset{\overline{v}}{\Phi^{\bar{i}}_{A\|\bar{i}}}\Phi^B +\\ &+ \overset{h}{\Phi^i_A}\frac{\delta \Phi^A}{\delta z^i} + \overset{\overline{h}}{\Phi^{\bar{i}}_A}\frac{\delta \Phi^A}{\delta \overline{z}^i} + \overset{v}{\Phi^i_A}\frac{\partial \Phi^A}{\partial \eta^i} + \overset{\overline{v}}{\Phi^{\bar{i}}_A}\frac{\partial \Phi^A}{\partial \overline{\eta}^i}\Big\}[X_\lambda]^A_B = 0.\end{aligned} \tag{4.14}$$

Using the complex currents

$$\begin{aligned}&\overset{h}{J^i_A} = -\overset{h}{\Phi^i_A}[X_\lambda]^A_B\Phi^B, \quad \overset{v}{J^i_A} = -\overset{v}{\Phi^i_A}[X_\lambda]^A_B\Phi^B,\\ &\overset{\overline{h}}{J^{\bar{i}}_A} = -\overset{\overline{h}}{\Phi^{\bar{i}}_A}[X_\lambda]^A_B\Phi^B, \quad \overset{\overline{v}}{J^{\bar{i}}_A} = -\overset{\overline{v}}{\Phi^{\bar{i}}_A}[X_\lambda]^A_B\Phi^B,\end{aligned} \tag{4.15}$$

the global conservative law (4.13) is written in the form

$$\overset{h}{J^i_{A|i}} + \overset{\overline{h}}{J^{\bar{i}}_{A|\bar{i}}} + \overset{v}{J^i_{A\|i}} + \overset{\overline{v}}{J^{\bar{i}}_{A\|\bar{i}}} = E_A[X_\lambda]^A_B\Phi^B. \tag{4.16}$$

5 Applications

Let us consider a metric Hermitian $d-$tensor $\gamma_{i\bar{j}}(z)$ on M, and $\Phi : T'M \to \mathbf{C}$ one scalar field $(A = 1)$. For a nonlinear complex connection $N^i_j(z,\eta)$ we can consider the one determined by the Christoffel symbols Γ^k_{ij} of the metric $\gamma_{i\bar{j}}$, i.e., $\overset{c}{N^i_j} = \Gamma^i_{jk}\eta^k$ (see [8]).

In the adapted basis $\{\delta_i, \partial_i, \delta_{\bar{i}}, \partial_{\bar{i}}\}$ we shall consider a generalization of the classical exact symmetry Lagrangian ([4]) for the complex gauge field $\Phi : T'M \to \mathbf{C}$,

$$L = \gamma^{\bar{i}j}(z)\frac{\delta\overline{\Phi}}{\delta\overline{z}^i}\frac{\delta\Phi}{\delta z^j} - m^2\overline{\Phi}\cdot\Phi - \frac{1}{4}f\cdot(\overline{\Phi}\cdot\Phi)^2, \tag{5.1}$$

where $m^2 > 0$ is the mass and $f > 0$ is the coupling constant.

The Lagrangian L is invariant to the change of local maps on $T'M$ and is also gauge invariant with respect to the phase transformation of U_1 group,

$$\Phi \to \widetilde{\Phi}(z,\eta) = e^{-ig\varepsilon}\Phi(z,\eta) \ \ ; \ \ \overline{\Phi} \to \widetilde{\overline{\Phi}}(z,\eta) = e^{ig\varepsilon}\Phi(z,\eta), \tag{5.2}$$

determined by the infinitesimal variations

$$\delta z^i = -ig\varepsilon, \quad \delta\eta^i = ig\varepsilon. \tag{5.3}$$

The U_1 group has one parameter $\varepsilon^1 = \varepsilon$; the variations of gauge field are $\delta\Phi = -ig\varepsilon\Phi$, and the generator of the group is $X = -ig\left(\frac{\partial}{\partial z^i} - \frac{\partial}{\partial \eta^i}\right)$.

In classical theories, in order to study the gauge invariance, there are considered two gauge fields: $\Phi^1 = \Phi$ and $\Phi^2 = \overline{\Phi}$. This doubles the dimension of the representation, in comparison to the present approach, where we consider only one generator.

The energy of the system, $E = \gamma^{\bar{i}j}(z)\dfrac{\delta\overline{\Phi}}{\delta\overline{z}^i}\dfrac{\delta\Phi}{\delta z^j} + m^2\overline{\Phi}\cdot\Phi + \dfrac{1}{4}f\cdot(\overline{\Phi}\cdot\Phi)^2$ is minimal if $\dfrac{\partial E}{\partial\Phi} = \dfrac{\partial E}{\partial\overline{\Phi}} = 0$; it results that $\Phi = \overline{\Phi} = 0$, and hence the vacuum state is nondegenerate, preserving the exact symmetry.

Another generalization of classical case is that of the spontaneously broken symmetry, where it is considered the complex Lagrangian

$$L_1 = \gamma^{\bar{i}j}(z)\frac{\delta\overline{\Phi}}{\delta\overline{z}^i}\frac{\delta\Phi}{\delta z^j} + m^2\overline{\Phi}\cdot\Phi - \frac{1}{4}f\cdot(\overline{\Phi}\cdot\Phi)^2, \tag{5.4}$$

which is also gauge invariant at (5.2) infinitesimal transformation.

The energy of the system, $E_1 = \gamma^{\bar{i}j}(z)\dfrac{\delta\overline{\Phi}}{\delta\overline{z}^i}\dfrac{\delta\Phi}{\delta z^j} - m^2\overline{\Phi}\cdot\Phi + \dfrac{1}{4}f\cdot(\overline{\Phi}\cdot\Phi)^2$ is minimal if $|\Phi| = \sqrt{2}\dfrac{m}{\sqrt{f}}$ and hence, the vacuum states are degenerated, i.e., we have in the holomorphic bundle a spontaneously broken symmetry.

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