C-Totally Real Submanifolds of $\mathbf{R}^{2n+1}$ Satisfying a Certain Inequality

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Abstract

We establish a sharp inequality between the squared mean curvature and the scalar curvature for a C-totally real submanifold of maximum dimension in a Sasakian space form. In particular we investigate C-totally real submanifolds of $\mathbf{R}^{2n+1}$ satisfying the equality case.

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1 Introduction

Let $\mathbf{C}^n$ denote the complex Euclidean $n-$space with complex structure J defined by

$$J(x_1, x_2, ..., x_{2n}) = (-x_{n+1}, ..., -x_{2n}, x_1, ..., x_n).$$

If $f : M \longrightarrow \mathbf{C}^n$ is an isometric immersion from a Riemannian $n-$manifold M into $\mathbf{C}^n$, then M is called a *Lagrangian submanifold* (or *totally real submanifold* in [5]) if J carries each tangent space of M into its normal space. Lagrangian submanifolds appear naturally in the context of classical mechanics and mathematical physics.

It is well-known, that every curve in $\mathbf{C}^1$ is Lagrangian. For $n \geq 2$, there is a Lagrangian immersion from an $n-$sphere $\mathbf{S}^n$ into $\mathbf{C}^n$ given by Whitney which is a called the *Whitney immersion*. The Whitney immersion is defined as follows :

Let $f : E^{n+1} \longrightarrow \mathbf{C}^n$ be a map from E^{n+1} into the complex Euclidean space $\mathbf{C}^n$ defined by :

$$f(x_0, x_1, ..., x_n) = \frac{1}{1 + x_0^2}(x_1, ..., x_n, x_0x_1, ..., x_0x_n).$$

Denote by $\mathbf{S}^n$ the unit hypersphere of E^{n+1} centered at the origin. The restriction of f to $\mathbf{S}^n$ gives rise to an immersion :

$$w : \mathbf{S}^n \longrightarrow \mathbf{C}^n$$

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which has a unique self-intersection point $f(-1,0,...,0)=f(1,0,...,0)$. With respect to the canonical complex structure J on $\mathbf{C}^n$, $w:\mathbf{S}^n\longrightarrow\mathbf{C}^n$ is a Lagrangian immersion which is the *Whitney immersion*.

Let $\tilde{g}$ denote the metric on $\mathbf{S}^n$ induced from the Euclidien metric on $\mathbf{C}^n$ via w.

We call the Riemannian $n-$manifold $\tilde{\mathbf{S}^n}=(\mathbf{S}^n,\tilde{g})$ the *Whitney n-sphere.*

Let S^n denote the unit hypersphere of $\mathbf{R}^{n+1}$. Consider the spherical coordinates $\{t_1,...,t_n\}$ on S^n defined by

$$x_1=\cos t_1,...,x_i=\cos t_i\prod_{j=1}^{i-1}\sin t_j,...,x_n=\cos t_n\prod_{j=1}^{n-1}\sin t_j, \tag{1.1}$$

$$x_{n+1}=\sin t_n\prod_{j=1}^{n-1}\sin t_j.$$

Recall that the Whitney immersion $w:\mathbf{S}^n\longrightarrow\mathbf{C}^n$ is defined by

$$w(x_0,x_1,...,x_n)=\frac{1}{1+x_0^2}(x_1,...,x_n,x_0x_1,...,x_0x_n). \tag{1.2}$$

for $(x_0,x_1,...,x_n)\in\mathbf{S}^n$ and consider the Whitney $n-$sphere $\tilde{\mathbf{S}^n}=(\mathbf{S}^n,\tilde{g})$ endowed with the Riemannian metric $\tilde{g}$ induced from the Whitney immersion w. (1.1) and (1.2) imply that the components $\tilde{g}_{\alpha\beta}$ of the metric tensor $\tilde{g}$ with respect to the spherical coordinates are given by

$$\tilde{g}_{\alpha\alpha}=\frac{\prod\limits_{j=1}^{\alpha-1}\sin^2 t_j}{1+\cos^2 t_1},\quad \tilde{g}_{\alpha\beta}=0,\quad 1\le\alpha\neq\beta\le n, \tag{1.3}$$

where we put $\prod\limits_{i=1}^{0}\sin^2 t_i=1$.

Let N and S denote the points $(1,0,...,0)$ and $(-1,0,...,0)$ in $\mathbf{S}^n$, respectively. From (1.3) we see that $\tilde{\mathbf{S}^n}-\{N,S\}$ is a warped product $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\times_{\rho(t)}\mathbf{S}^{n-1}$ of the open interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and the unit $(n-1)-$sphere with warped product metric given by

$$\tilde{g}=\left(\frac{1}{1+\cos^2 t_1}\right)dt_1^2+\left(\frac{\sin^2 t_1}{1+\cos^2 t_1}\right)g_0,$$

where g_0 is the standard metric on the unit $(n-1)-$sphere $\mathbf{S}^{n-1}$ and $\rho(t)=\frac{\sin t_1}{\sqrt{1+\cos^2 t_1}}$.

Let $\{e_1,...,e_n\}$ be the unit vector fields in the direction of the tangent vector fields $\{\frac{\partial}{\partial t_1},...,\frac{\partial}{\partial t_n}\}$ on $\tilde{\mathbf{S}^n}$ respectively. Then $\{e_1,...,e_n,e_{1*},...,e_{n*}\}$ form an adapted Lagrangian orthonormal frame field. By a direct, long computation, we may prove that the second fundamental form of the Whitney immersion w with respect to this adapted frame field satisfies (see [2])

$$\begin{array}{rcl} h(e_1,e_1) & = & 3\lambda e_{1*}, \quad h(e_2,e_2)=...=h(e_n,e_n)=\lambda e_{1*}, \\ h(e_1,e_j) & = & \lambda e_{j*}, \quad h(e_j,e_k)=0, 2\le j\ne k\le n. \end{array}$$

where

$$\lambda = -\frac{\sin t_1}{\sqrt{1+\cos^2 t_1}}.$$

An orthonormal frame field $e_1,...,e_n,e_{1*},...,e_{n*}$ is called an *adapted frame field* if $e_1,...,e_n$ are orthonormal tangent vector fields and $e_{1*},...,e_{n*}$ are normal vector fields given by

$$e_{1*} = Je_1, ..., e_{n*} = Je_n$$

2 Submanifolds of a Sasakian space form

Let $(\tilde{M}, g)$ be a $(2m+1)$-dimensional Riemannian manifold endowed with an endomorphism φ ($(1,1)-$tensor field) of its tangent bundle $T\tilde{M}$, a vector field ξ and a 1-form η such that

$$\begin{cases} \varphi^2 X = -X + \eta(X)\xi\,,\ \varphi\xi = 0\,,\ \eta\circ\varphi = 0\,,\ \eta(\xi) = 1\,, \\ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)\,,\ \eta(X) = g(X,\xi)\,, \end{cases}$$

for all vector fields $X, Y \in \Gamma(T\tilde{M})$.

If, in addition, $d\eta(X,Y) = g(\varphi X, Y)$, then $\tilde{M}$ is said to have *a contact Riemannian structure* (φ, ξ, η, g). If, moreover, the structure is normal, i.e. if

$$[\varphi X, \varphi Y] + \varphi^2[X,Y] - \varphi[X, \phi Y] - \varphi[\varphi X, Y] = -2d\eta(X,Y)\xi,$$

then the contact Riemannian structure is called a *Sasakian structure* and $\tilde{M}$ is called a *Sasakian manifold*. For more details and background, we refer to the standard references [1], [8].

A plane section σ in $T_p\tilde{M}$ of a Sasakian manifold $\tilde{M}$ is called a *φ-section* if it is spanned by X and φX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature $\bar{K}(\sigma)$ w.r.t. a φ-section σ is called a φ-sectional curvature. If a Sasakian manifold $\tilde{M}$ has constant φ-sectional curvature c, then it is called a *Sasakian space form* and is denoted by $\tilde{M}(c)$.

The curvature tensor $\tilde{R}$ of a Sasakian space form $\tilde{M}(c)$ is given by [1]:

$$\begin{array}{rcl} \widetilde{R}(X,Y)Z & = & \frac{c+3}{4}(g(Y,Z)X - g(X,Z)Y) + \\ & + & \frac{c-1}{4}(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \\ & + & g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z), \end{array}$$

for any tangent vector fields X, Y, Z to $\tilde{M}(c)$.

An n-dimensional submanifold M of a Sasakian space form $\tilde{M}(c)$ is called a $C-$*totally real submanifold* if ξ is a normal vector field on M. A direct consequence of this definition is that $\varphi(TM) \subset T^{\perp}M$, i.e. that M is an anti-invariant submanifold of $\tilde{M}(c)$, (hence their name of "contact"-totally real submanifolds); see e.g. [6].

As examples of Sasakian space forms we mention $\mathbf{R}^{2m+1}$ and $\mathbf{S}^{2m+1}$, with standard Sasakian structures.

If M is a Riemannian $n-$manifold isometrically immersed in a Euclidian $m-$space E^m, one may consider extrinsic invariants as well as intrinsic invariants on M .

Let M be an n-dimensional Riemannian manifold. Denote by $K(\pi)$ the *sectional curvature of the plane section* $\pi \subset T_pM$, $p \in M$. For any orthonormal basis $\{e_1, ..., e_n\}$ of the tangent space T_pM, the *scalar curvature* τ at p is defined by

$$\tau = \sum_{1 \le i < j \le n} K(e_i \wedge e_j) \ .$$

Let $p \in M$ and $\{e_1, ..., e_n\}$ an orthonormal basis of the tangent space T_pM. We denote by H the *mean curvature vector*, that is

$$H(p) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i)$$

Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r), \quad \|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

3 Main results

Theorem 1. *If M^n is a C-totally real submanifold of a Sasakian space form $\tilde{M}^{2n+1}(c)$, then the mean curvature H and the scalar curvature τ of M satisfy*

$$\|H\|^2 \ge \frac{2(n+2)}{n^2(n-1)}\tau - \left(\frac{n+2}{n}\right)\left(\frac{c+3}{4}\right). \tag{3.1}$$

Moreover the equality sign holds if and only if, with to respect an adapted frame field $e_1, ..., e_n, e_{1*}, ..., e_{n*}, e_{2n+1} = \xi$ with e_{1*} parallel to H , the second fundamental form of M^n in $\tilde{M}^{2n+1}(c)$ takes the following form:

$$h(e_1, e_1) = 3\lambda e_{1*}, \quad h(e_2, e_2) = ... = h(e_n, e_n) = \lambda e_{1*},$$

$$h(e_1, e_j) = \lambda e_{j*} \quad h(e_j, e_k) = 0, \quad 2 \le j \ne k \le n,$$

with $\lambda \in C^\infty(M)$.

Proof. Let M^n be a C-totally real submanifold of a Sasakian space form $\tilde{M}^{2n+1}(c)$, and $e_1, ..., e_n, e_{1*}, ..., e_{n*}, e_{2n+1} = \xi$ a local adapted frame field on M^n.

Put $h_{jk}^i = g\left(h(e_j, e_k), e_{i*}\right)$.

Then, by

$$A_{\varphi X}Y = -\varphi h(X, Y) = A_{\varphi Y}X \quad \forall X, Y \in \Gamma(TM), \tag{3.2}$$

we have

$$h_{jk}^i = h_{ik}^j = h_{ij}^k, \quad i, j, k = 1, ..., n.$$

From the definition of the mean curvature function we have

$$n^2\|H\|^2=\sum_i\left(\sum_j\left(h^i_{jj}\right)^2+2\sum_{j<k}h^i_{jj}h^i_{kk}\right).$$

From the equation of Gauss we have

$$2\tau=n(n-1)\left(\frac{c+3}{4}\right)+n^2\|H\|^2-\|h\|^2=n(n-1)\left(\frac{c+3}{4}\right)+n^2\|H\|^2-\sum_{i,j,k=1}^{n}\left(h^i_{jk}\right)^2.$$

Thus, by applying precedent relations, we obtain

$$\tau=\frac{n(n-1)}{2}\left(\frac{c+3}{4}\right)+\sum_i\sum_{j<k}h^i_{jj}h^i_{kk}-\sum_{i\neq j}\left(h^i_{jj}\right)^2-3\sum_{i<j<k}\left(h^i_{jk}\right)^2.$$

Let $m=\dfrac{n+2}{n-1}$. Then, we get

$$\begin{aligned}
n^2\|H\|^2 &- m\left(2\tau-n(n-1)\left(\frac{c+3}{4}\right)\right)=\sum_i\left(h^i_{ii}\right)^2+(1+2m)\sum_{i\neq j}\left(h^i_{jj}\right)^2+\\
&+ 6m\sum_{i<j<k}\left(h^i_{jk}\right)^2-2(m-1)\sum_i\sum_{j<k}h^i_{jj}h^i_{kk}=\\
&= \sum_i\left(h^i_{ii}\right)^2+6m\sum_{i<j<k}\left(h^i_{jk}\right)^2+(m-1)\sum_i\sum_{j<k}\left(h^i_{jj}-h^i_{kk}\right)^2+\\
&+ (1+2m-(n-2)(m-1))\sum_{j\neq i}\left(h^i_{jj}\right)^2-2(m-1)\sum_{j\neq i}h^i_{jj}h^i_{jj}=\\
&= 6m\sum_{i<j<k}\left(h^i_{jk}\right)^2+(m-1)\sum_{i\neq j,k}\sum_{j<k}\left(h^i_{jj}-h^i_{kk}\right)^2+\\
&+ \frac{1}{n-1}\sum_{j\neq i}\left(h^i_{ii}-(n-1)(m-1)h^i_{jj}\right)^2\geq 0
\end{aligned}$$

which implies inequality (3.1). We see that the equality sign of (3.1) holds if and only if $h^i_{ii}=3h^i_{jj},h^i_{jk}=0$, for distinct i,j,k. In particular, if choose $e_1,...,e_n$ in such way that φe_1 is parallel to the mean curvature vector H, we also have $h^j_{kk}=0$ for $j>1,k=1,...,n$.

□

Theorem 2. *Let $i:M^n\longrightarrow\mathbf{R}^{2n+1}$ be a C-totally real isometric immersion satisfying the equality case*

$$\|H\|^2=\frac{2(n+2)}{n^2(n-1)}\tau \tag{3.3}$$

Then either M is a totally geodesic submanifold and hence M is locally isometric to the real space $\mathbf{R}^n$ or the set U of non-totally geodesic points in M is a dense subset of M, U is an open portion of a $\tilde{\mathbf{S}^n}$ Withney sphere with $a>1$ and, up to rigid motions of $\mathbf{R}^{2n+1}$, the immersion i is given by $\widetilde{w}$, where $\widetilde{w}:\mathbf{S}^n\longrightarrow\mathbf{R}^{2n+1}$ is the immersion lifted from the Whitney immersion.

Proof. It follows from Theorem 1 that the function $\phi = \left(\frac{n}{n-2}\right)^2 \|H\|^2 = \lambda^2$ is a well-defined function on M. If the function ϕ vanishes identically, then M is a totally geodesic submanifold of $\mathbf{R}^{2n+1}$. So, for simplicity, we may assume from now on that M is non-totally geodesic, i.e. $\phi \neq 0$. Thus, $U = \{p \in M \mid \phi(p) \neq 0\}$ is a non-empty open subset of M.

Let $\omega^1, ..., \omega^n$ denote the dual 1-forms of $e_1, ..., e_n$ and denoted by (ω_B^A), $A, B = 1, ..., n, 1*, ...n*, 2n+1$, the connection forms on M defined by

$$\widetilde{\nabla} e_i = \sum_{j=1}^n \omega_i^j e_j + \sum_{j=1}^n \omega_i^{j*} e_{j*}, \quad \widetilde{\nabla} e_{i*} = \sum_{j=1}^n \omega_{i*}^j e_j + \sum_{j=1}^n \omega_{i*}^{j*} e_{j*},\ i = 1, ..., n,$$

where $\omega_i^j = -\omega_j^i$, $\omega_{i*}^{j*} = -\omega_{j*}^{i*}$

For a C-totally real submanifold M^n of a $\mathbf{R}^{2n+1}$, (3.2) yields

$$\omega_j^{i*} = \omega_i^{j*}, \omega_i^j = \omega_{i*}^{j*}, \quad \omega_j^{i*} = \sum_{k=1}^n h_{jk}^i \omega^k.$$

We find

$$\omega_1^{1*} = 3\lambda\omega^1,\ \omega_i^{1*} = \lambda\omega^i, \quad \omega_i^{i*} = \lambda\omega^1, \quad \omega_j^{i*} = 0,\ 2 \leq i \neq j \leq n. \tag{3.4}$$

By applying the equation of Codazzi, we obtain

$$e_1\lambda = \lambda\omega_1^2(e_2) = ... = \lambda\omega_1^n(e_n), \quad e_2\lambda = ... = e_n\lambda = 0, \tag{3.5}$$

$$\omega_1^j(e_k) = 0, \quad 1 < j \neq k \leq n. \tag{3.6}$$

By precedent formulas yield

$$\omega_1^j = e_1(\ln\lambda)\omega^j, \quad j = 2, ..., n \tag{3.7}$$

From Cartan's structure equations and (3.7) we get $d\omega^1 = 0$ and $\nabla_{e_1} e_1 = 0$.

Therefore, we have the following

Lemma 3. *On U, the integral curves of φH (or, equivalently, of e_1) are geodesics of M.*

Let $\mathcal{D}$ denote the distribution spanned by φH and $\mathcal{D}^\perp$ denote the orthogonal complementary distribution of $\mathcal{D}$ on U. Then $\mathcal{D}$ and $\mathcal{D}^\perp$ are spanned by $\{\varphi H\}$ and $\{e_2, ..., e_n\}$, respectively.

By using (3.6) we obtain the following.

Lemma 4. *On U, the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are both integrable.*

Proof. For any $j, k > 1$, (3.6) implies

$$\langle [e_j, e_k], e_1 \rangle = \omega_k^1(e_j) - \omega_j^1(e_k) = 0$$

Thus, the distribution $\mathcal{D}^\perp$ is completely integrable. The integrability of $\mathcal{D}$ is obvious, since $\mathcal{D}$ is a 1-dimensional distribution.

Now, we give the following.

Lemma 5. *On U, there exist local coordinate systems $\{x_1, ..., x_n\}$ satisfying the following conditions :*

(a) $\mathcal{D}$ is spanned by $\{\frac{\partial}{\partial x}\}$ and $\mathcal{D}^\perp$ is spanned by $\{\frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\}$,

(b) $e_1 = \frac{\partial}{\partial x}$, $\omega^1 = dx$,

(c) the metric tensor g takes the form : $g = dx^2 + \sum_{j,k=2}^{n} g_{jk}(x, x_2, ..., x_n)dx_j dx_k$,

where $x = x_1$.

Proof. It is well-know, that there exists a local coordinate systems $\{y_1, ..., y_n\}$ such that $e_1 = \frac{\partial}{\partial y_1}$. Since $\mathcal{D}^\perp$ is completely integrable, there also exists a local coordinate systems $\{z_1, ..., z_n\}$ such that $\frac{\partial}{\partial z_2}, .., \frac{\partial}{\partial z_n}$ span $\mathcal{D}^\perp$. Put $x = x_1 = y_1, x_2 = z_2, ..., x_n = z_n$, then $\{x_1, ..., x_n\}$ is a desired coordinate system.

(3.5) and Lemma 5 imply that λ depends only on $x = x_1$, i.e. $\lambda = \lambda(x)$. Let λ' and λ'' denote the first and second derivates of λ with respect to x.

Lemma 6. *On U, the function λ satisfies the following second order ordinary differential equation:*

$$\frac{d^2\lambda}{dx^2} + 2\lambda^3 = 0 \tag{3.8}$$

Proof. By taking the exterior differentiation of (3.7) and using (3.4), (3.7) and Cartan's structure equations, we find

$$(\ln \lambda)'' + (\ln \lambda)'^2 = -2\lambda^2$$

which is equivalent to (3.8).

Lemma 7. *The solution of the second order ordinary differential equation (3.8) are given by*

$$\lambda(x) = -\frac{\sin(t(x) + b)}{a\sqrt{1 + \cos^2(t(x) + b)}}, \tag{3.9}$$

where $t(x)$ is the inverse function of $x(t)$ defined by

$$x = \int_0^t \frac{a du}{\sqrt{1 + \cos^2(u + b)}} \tag{3.10}$$

and a and b are constants with $a > 0$ and $0 \leq b < 2\pi$.

Proof. (3.10) implies that $x(t)$ is a strictly increasing differentiable function of t.

Thus, $x = x(t)$ has an inverse function, denoted by $t = t(x)$. From (3.10) we get

$$\frac{dt}{dx} = \frac{1}{a}\sqrt{1 + \cos^2(t(x) + b)}, \tag{3.11}$$

Thus by (3.9), (3.11), and chain rule, we find

$$\begin{aligned}\frac{d\lambda}{dx} &= -\frac{2\cos(t(x)+b)}{a^2(1+\cos^2(t(x)+b))} \\ \frac{d^2\lambda}{dx^2} &= -\frac{2\sin^3(t(x)+b)}{a^3(1+\cos^2(t(x)+b))^{\frac{3}{2}}}\end{aligned} \tag{3.12}$$

(3.9) and (3.12) imply that, for any a and b are constants with $a > 0$ and $0 \leq b < 2\pi$, the function λ given by (3.9) is a solution of the differential equation (3.8).

Let $f = f(x, \lambda, \lambda^{'}) = -2\lambda^3$. Then $f, \dfrac{\partial f}{\partial \lambda}, \dfrac{\partial^2 f}{\partial \lambda^2}$ are continous functions on the 3-space $\mathbf{R}^3$. Thus, by Existence and Uniquenss Theorem of second ordinary differential equation, the differential equation (3.8) together with the given initial conditions : $\lambda(x_0) = \lambda_0, \quad \lambda^{'}(x_0) = \lambda_0'$, has a unique solution.

Since for any two arbitrary constants λ_0, λ_0' we may find real number a and b with $a > 0$ and $0 \leq b < 2\pi$ which satisfy the following two conditions :

$$\frac{\sin(t(x_0)+b)}{a\sqrt{1+\cos^2(t(x_0)+b)}} = \lambda_0, \quad \frac{2\cos(t(x_0)+b)}{a^2(1+\cos^2(t(x_0)+b)} = \lambda_0',$$

therefore every solution of the differential equation (3.8) takes the form given by (3.10). The rigidy theorem of C-totally real immersions in $\mathbf{R}^{2n+1}$ achieves the proof. $\square$

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