On Complex Cartan Spaces

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

In some recent articles ([13, 14]) we have studied the geometry of complex Hamilton spaces.

In brief, the geometry of a complex Hamilton space is the geometry of the dual holomorphic bundle $(T'M)^*$ endowed with a Hermitian metric derived from a Hamiltonian function. In this study the notion of complex nonlinear connection plays a special role. A significant result provides the complex nonlinear connection derived only from the Hamiltonian function.

If in addition a positive Hamiltonian satisfies the condition of homogeneity, then the notion of complex Cartan space is obtained. This is the correspondent of complex Finsler space on the manifold $(T'M)^*$, and coincides with the notion of complex Finsler Hamiltonian introduced by S. Kobayashi ([7, 5]).

In the present paper we make a geometric study of the complex Cartan space and of some its immediate generalizations.

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1 The bundle $(T'M)^*$

Let M be a complex manifold, $dim_C M = n$, and denote by (z^k) the complex coordinates in a local chart. $T'M$ is the holomorphic bundle of $(1,0)-$type vectors and $(T'M)^*$ is its dual bundle. In a local chart on the manifold $(T'M)^*$, a point u^* is characterized by the coordinates $u^* = (z^k, \zeta_k)$, $k = \overline{1,n}$, and the change of local charts determines the following change of coordinates ([14]):

$$z'^k = z'^k(z) \ ; \ \zeta'_k = \frac{\partial z^j}{\partial z'^k}\zeta_j \ ; \ rank(\frac{\partial z^j}{\partial z'^k}) = n \tag{1.1}$$

Now, let us consider the holomorphic bundle $\pi_T^* : T'(T'M)^* \to (T'M)^*$. A local frame in u^* is $\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \zeta_k}\}$ and its changes are imposed by the Jacobi matrix of (1.1).

The vertical subbundle $V(T'M)^* = ker\ \pi_T^*$ is holomorphic too and a local base in the vertical distribution $\mathcal{V}^*$ is $\{\frac{\partial}{\partial \zeta_k}\}$. A complex nonlinear connection (in brief $(c.n.c.)$) on $(T'M)^*$ is a supplementary subbundle of $V(T'M)^*$ in $T'(T'M)^*$, i.e. $T'(T'M)^* =$

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$H(T'M)^* \oplus V(T'M)^*$. If a (*c.n.c.*) is given, by conjugation a decomposition of the whole complexification $T_C(T'M)^*$ is obtained.

In the horizontal distribution $\mathcal{H}^* = \mathcal{H}_{\sqcap^*}(\mathcal{T}'\mathcal{M})^*$ a local basis has the form

$$\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} + \overset{*}{N}_{jk} \frac{\partial}{\partial \zeta_j} \tag{1.2}$$

and this basis is said to be *adapted* if it transforms under the rule:

$$\frac{\delta}{\delta z^i} = \frac{\partial z'^j}{\partial z^i} \frac{\delta}{\delta z'^j}. \tag{1.3}$$

The basis $\{\delta_k = \frac{\delta}{\delta z^k},\ \dot{\partial}^k = \frac{\partial}{\partial \zeta_k}\}$ is an *adapted basis* on $T'_{u^*}(T'M)^*$. The corresponding dual basis $\{dz^k,\ \delta\zeta_k = d\zeta_k - \overset{*}{N}_{kj}\, dz^j\}$ is an adapted basis on $T'^*_{u^*}(T'M)^*$.

Of course, the condition (1.3) involves that the coefficients $\overset{*}{N}_{jk}$ of (*c.n.c.*) obey a certain rule of transformation. Let us note that if $\overset{*}{N}_{jk}$ is a (*c.n.c.*) then $\overset{*}{N}_{kj}$ and $\frac{1}{2}(\overset{*}{N}_{jk} + \overset{*}{N}_{kj})$ are (*c.n.c*) too.

Proposition 1.1 *If* $\overset{*}{N}_{jk}$ *is a* (*c.n.c.*) *then* $\frac{\partial \overset{*}{N}_{jk}}{\partial \zeta_m}\zeta_m$ *determines a* (*c.n.c.*), *called the spray connection of* $\overset{*}{N}_{jk}$.

In our approach a special meaning have those geometrical objects, called $d-$*complex tensors,* which are transformed only by means of the matrices $(\partial z^i/\partial z'^j)$ or $(\partial \bar{z}^i/\partial \bar{z}'^j)$ for the bar indices, and with their inverses, in a similar way as on the base manifold M.

A linear connection $D : \chi_C(T'M)^* \times \chi_C(T'M)^* \to \chi_C(T'M)^*$ is said to be a $\overset{*}{N}$ *−complex linear connection* (shortly $\overset{*}{N}$ −(*c.l.c.*)) if for a given (*c.n.c.*) it preserves the four distributions of $T_C(T'M)^*$ and its coefficients coincide two by two ([14]). Note that for a $d-$(*c.l.c.*) D we have $\overline{D_X Y} = D_{\overline{X}}\overline{Y}$, and so it is well defined in respect to the adapted base if the following local expression is given:

$$D_{\delta_k}\delta_j = H^i_{jk}\delta_i\ ;\ D_{\delta_k}\dot{\partial}^i = -H^i_{jk}\dot{\partial}^j\ ;\ D_{\delta_k}\delta_{\bar{j}} = H^{\bar{i}}_{\bar{j}k}\delta_{\bar{i}}\ ;\ D_{\delta_k}\dot{\partial}^{\bar{i}} = -H^{\bar{i}}_{\bar{j}k}\dot{\partial}^{\bar{j}}$$

$$D_{\dot{\partial}^k}\delta_j = C^{ik}_j\delta_i\ ;\ D_{\dot{\partial}^k}\dot{\partial}^i = -C^{ik}_j\dot{\partial}^j\ ;\ D_{\dot{\partial}^k}\delta_{\bar{j}} = C^{\bar{i}k}_{\bar{j}}\delta_{\bar{i}}\ ;\ D_{\dot{\partial}^k}\dot{\partial}^{\bar{i}} = -C^{\bar{i}k}_{\bar{j}}\dot{\partial}^{\bar{j}} \tag{1.4}$$

Therefore, a $\overset{*}{N}$ −(*c.l.c.*) is characterized only by the set of coefficients $(H^i_{jk}\,;H^{\bar{i}}_{\bar{j}k};\ C^{ik}_j;C^{\bar{i}k}_{\bar{j}})$, and their conjugates. The covariant derivatives of a $d-$complex tensor in respect to a $\overset{*}{N}$ −(*c.l.c.*) D will be denoted by "$_{|k}$", "$\,|_k$" or "$_{|\bar{k}}$", "$\,|_{\bar{k}}$". The local expressions of curvatures and torsions of a $\overset{*}{N}$ −(*c.l.c.*) are calculated in [14].

2 Complex Hamilton space

Let $\overset{*}{N}$ be a fixed (*c.n.c.*) and $g_{i\bar{j}}(z,\zeta)$ a Hermitian metric on $(T'M)^*$, i.e. $g_{i\bar{j}}$ is a $d-$complex tensor, $\overline{g_{i\bar{j}}} = g_{j\bar{i}}$ and $det(g_{i\bar{j}}) \neq 0$. By $(g^{\bar{i}j})$ we denote the inverse matrix of $(g_{i\bar{j}})$. The following metric structure on $T_C(T'M)^*$, (2.1) $$G = g_{i\bar{j}}dz^i \otimes d\bar{z}^j + g^{\bar{j}i}\delta\zeta_i \otimes \delta\bar{\zeta}_j$$

is called the $\overset{*}{N} -lift$ of the metric structure $g_{i\bar{j}}$.

A $\overset{*}{N}$ $-(c.l.c.)$ D is metrical, that is $DG = 0$, iff $g_{i\bar{j}|k} = g_{i\bar{j}}\ |_k = g_{i\bar{j}|\bar{k}} = g_{i\bar{j}}\ |_{\bar{k}} = 0$.

A remarkable example of metrical $\overset{*}{N}$ $-(c.l.c.)$ on $(T'M)^*$ is given by

Theorem 2.1 *([14]).The following* $\overset{*}{N}$ *$-(c.l.c.)$, denoted by* $\overset{c}{D}$, *is metrical :*

(2.2) $$\begin{aligned} \overset{c}{H^i_{jk}} &= \frac{1}{2}g^{\bar{h}i}(\frac{\delta g_{j\bar{h}}}{\delta z^k} + \frac{\delta g_{k\bar{h}}}{\delta z^j}) \quad ; \quad \overset{c}{C^{ik}_j} = -\frac{1}{2}g_{j\bar{h}}(\frac{\partial g^{\bar{h}i}}{\partial \zeta_k} + \frac{\partial g^{\bar{h}k}}{\partial \zeta_i}) \\ \overset{c}{H^{\bar{i}}_{\bar{j}k}} &= \frac{1}{2}g^{\bar{i}h}(\frac{\delta g_{h\bar{j}}}{\delta z^k} - \frac{\delta g_{k\bar{j}}}{\delta z^h}) \quad ; \quad \overset{c}{C^{\bar{i}k}_{\bar{j}}} = -\frac{1}{2}g_{h\bar{j}}(\frac{\partial g^{\bar{i}h}}{\partial \zeta_k} - \frac{\partial g^{\bar{i}k}}{\partial \zeta_h}) \end{aligned}$$

and has the following zero torsions $hT(hX, hY) = vT(vX, vY) = 0$.

The notion of Hermitian metric has a special signification if it is derived from a complex Hamiltonian. A *complex Hamiltonian* is given by a $C^\infty-$differentiable function $H : (T'M)^* \to R$ with the property that the following $d-$complex tensor is nondegenerate

(2.3) $$g^{\bar{j}i}(z, \zeta) = \frac{\partial^2 H}{\partial \zeta_i \partial \bar{\zeta}_j} \ \ , \ rank(g^{\bar{j}i}) = n.$$

The pair (M, H) is said to be a *complex Hamilton space.*

In [15] we made an extension of the well-known Legendre transformation to the complexified of $(T'M)^*$. As a product, a special result gives a very simple form of a $(c.n.c.)$

Theorem 2.2 *The following functions*

(2.4) $$\overset{*c}{N}_{ji} = -g_{j\bar{h}}\frac{\partial^2 H}{\partial z^i \partial \bar{\zeta}_h}$$

are the coefficients of a $(c.n.c.)$ on $(T'M)^*$, depending only on the complex Hamiltonian function H.

A straight computation of the bracket $[\delta_j, \delta_k] = \Omega_{ijk}\dot{\partial}^i$ yields to $\Omega_{ijk} = \delta_j(\overset{*c}{N}_{ik})$ $-\delta_k(\overset{*c}{N}_{ij}) = 0$ and consequently, the $\overset{*c}{N}_{ji}$ $(c.n.c.)$ plays a special role.

In respect to the adapted basis of the $(c.n.c.)$ given by (2.4), we consider the connection $\overset{c}{D}$ from (2.2). So, the set $\overset{c}{\Gamma H} = (\overset{*c}{N}_{jk}, \overset{c}{H^i_{jk}}, \overset{c}{H^{\bar{i}}_{\bar{j}k}}, \overset{c}{C^{ik}_j}, \overset{c}{C^{\bar{i}k}_{\bar{j}}})$ will be called the *canonical* $(c.l.c.)$ of the complex Hamilton space (M, H).

In the next lines we shall describe another method to obtain a $\overset{*}{N}$ $-(c.l.c.)$ which generalizes to the dual case the idea of vertical connections ([1]) from the theory of complex Finsler spaces.

Let $\nabla : \chi(T'M)^* \times V(T'M)^* \to V(T'M)^*$ be a linear connection on the vertical bundle, locally given by its coefficients Γ^j_{ik} and C^{jk}_i, where

$$\nabla_{\frac{\partial}{\partial z^k}} \dot{\partial}^j = -\Gamma^j_{ik} \dot{\partial}^i \quad ; \quad \nabla_{\dot{\partial}^k} \dot{\partial}^j = C^{jk}_i \dot{\partial}^i \ .$$

By d^k_i is denoted the $d-$complex tensor $d^k_i = \delta^k_i - C^{jk}_i \zeta_j$. As in [4] we prove that $\Gamma^0_{ik} = \Gamma^j_{ik}\zeta_j$ are transformed by the rule:

$$\Gamma'^0_{jk} = \frac{\partial z^p}{\partial z'^j} \frac{\partial z^q}{\partial z'^k} \Gamma^0_{pq} + d'^p_j \zeta_q \frac{\partial^2 z^q}{\partial z'^p \partial z'^k} \tag{2.5}$$

Therefore, if there exists the inverse $(d^k_i)^{-1} = b^k_i$ then $\overset{*}{N}_{ik} = b^k_i \Gamma^0_{jk}$ satisfies the rule of change of a (*c.n.c.*) on $(T'M)^*$. If there exist b^k_i, by analogy with [1], we say that ∇ is a good vertical connection on $(T'M)^*$.

Based on (2.5), it follows

Proposition 2.1 *Any good vertical connection determines a* (*c.n.c.*) *on* $(T'M)^*$.

Moreover, a good vertical connection determines a $\overset{*}{N}$ $-$(*c.l.c.*) of (1,0)-type as follows. The coefficients C^{jk}_i of a good vertical connection satisfy the same rule of transformation as C^{jk}_i of one $\overset{*}{N}$ $-$(*c.l.c.*)D and H^j_{ik} is directly obtained from the calculation of $D_{\delta_k} \dot{\partial}^j = \nabla_{(\frac{\partial}{\partial z^k} + \overset{*}{N}_{hk} \dot{\partial}^h)} \dot{\partial}^j$. So we have that $H^j_{ik} = \Gamma^j_{ik} + \overset{*}{N}_{hk} C^{jh}_i$ are the horizontal coefficients of a $\overset{*}{N}$ $-$(*c.l.c.*) on $(T'M)^*$. The coefficients $C^{\bar{j}k}_{\bar{i}}, H^{\bar{j}}_{\bar{i}k}$ can be zero (since they are $d-$tensors) and then the obtained $\overset{*}{N}$ $-$(*c.l.c.*)D is of $(1,0)-$type.

Let us consider the whole vertical complexified bundle $V(T'M)^* \oplus \overline{V(T'M)^*}$ and let $\mathcal{G} = \}^{\overleftarrow{\dagger}} \lceil \zeta_\rangle \otimes \lceil \overleftarrow{\zeta}_\rceil$ be a Hermitian vertical metric. We assume that ∇ is a metric linear connection of $(1.0)-$type, i.e. $(\nabla_X \mathcal{G})(\mathcal{U}, \mathcal{V}) = \mathcal{X}\mathcal{G}(\mathcal{U}, \mathcal{V}) - \mathcal{G}(\nabla_{\mathcal{X}} \mathcal{U}, \mathcal{V}) - \mathcal{G}(\mathcal{U}, \nabla_{\mathcal{X}} \mathcal{V}) = {\prime}$ and $C^{\bar{j}h}_{\bar{i}} = \Gamma^{\bar{j}}_{\bar{i}h} = 0$. Then by choosing $U = \dot{\partial}^j$, $V = \dot{\partial}^{\bar{k}}$ and $X = \frac{\partial}{\partial z^h}$ or $\frac{\partial}{\partial \bar{z}^h}$ it results that:

$$\begin{aligned} \Gamma^j_{ih} &= -g_{i\bar{k}} \frac{\partial g^{\bar{k}j}}{\partial z^h} \quad ; \quad C^{jh}_i = -g_{i\bar{k}} \frac{\partial g^{\bar{k}j}}{\partial \zeta_h} \\ \overset{*}{N}_{ik} &= -b^j_i g_{j\bar{h}} \frac{\partial g^{\bar{h}l}}{\partial z^k} \zeta_l \quad ; \quad H^j_{ik} = -g_{i\bar{m}} \frac{\delta g^{\bar{m}j}}{\delta z^k}. \end{aligned} \tag{2.6}$$

Thus, we have:

Theorem 2.3 *A good vertical connection on a complex Hamilton space* (M, H) *determines a* $\overset{*}{N}$ $-$(*c.l.c.*) *of* $(1,0)-$*type,* $\overset{CH}{\Gamma H} = (\overset{*}{N}_{ik}, H^j_{ik}, 0, C^{jh}_i, 0)$ *given by (2.6), and called the Chern-Hamilton connection.*

3 Complex Cartan spaces

In the geometry of complex Finsler spaces there already exists a large reference ([1, 2, 3, 6, 11, 17]), the geometric support of such geometry being the holomorphic bundle $T'M$.

Concerning the Lagrangian-Hamiltonian duality from the classical mechanics we have considered necessary to make a study of complex Hamilton spaces based on the manifold $(T'M)^*$. The correspondent of complex Finsler spaces in $(T'M)^*$ are the complex Cartan spaces, defined as follows:

Definition 3.1 *A complex Cartan space is a complex Hamilton space* (M, H) *for which the function* $H : (T'M)^* - \{0\} \to R_+$ *satisfies the homogeneity condition:*

$$H(z, \lambda\zeta) = |\, \lambda \,|^2 \, H(z, \zeta) \quad , \quad \forall \lambda \in \mathbf{C}. \tag{3.1}$$

We see that this notion coincides with that of the complex Finsler Hamiltonian initially introduced by S.Kobayashi ([7]), but here we prefer to use the notion of complex Cartan space by analogy with the real known terminology ([8, 9, 10]).

Accordingly, the Hamilton metric $g^{\bar{j}i}(z, \zeta) = \partial^2 H / \partial\zeta_i \partial\bar{\zeta}_j$ is $0-$homogeneous and, applying the complex version of the Euler Theorem, a Cartan space is characterized by

Proposition 3.1 *In a complex Cartan space the following terms are true:*

$$\frac{\partial H}{\partial \zeta_i}\zeta_i = H \quad ; \quad \frac{\partial H}{\partial \bar{\zeta}_i}\bar{\zeta}_i = H \tag{3.2}$$

$$g^{\bar{j}i}\zeta_i = \frac{\partial H}{\partial \bar{\zeta}_j} \quad ; \quad g^{\bar{j}i}\bar{\zeta}_j = \frac{\partial H}{\partial \zeta_i} \quad ; \quad g^{\bar{j}i}\zeta_i\bar{\zeta}_j = H \tag{3.3}$$

$$\frac{\partial g^{\bar{j}i}}{\partial \zeta_k}\zeta_i = \frac{\partial g^{\bar{j}k}}{\partial \zeta_i}\zeta_i = 0 \quad ; \quad \frac{\partial g^{\bar{j}i}}{\partial \bar{\zeta}_k}\bar{\zeta}_j = \frac{\partial g^{\bar{j}k}}{\partial \bar{\zeta}_i}\bar{\zeta}_i = 0 \tag{3.4}$$

$$\frac{\partial g^{\bar{j}i}}{\partial \zeta_k}\bar{\zeta}_j = g^{ik} \quad ; \quad \frac{\partial^2 H}{\partial z^k \partial \zeta_i}\zeta_i = \frac{\partial H}{\partial z^k} \quad ; \quad \frac{\partial^2 H}{\partial z^k \partial \bar{\zeta}_i}\bar{\zeta}_i = \frac{\partial H}{\partial z^k} \tag{3.5}$$

$$g^{ij}\zeta_j = 0 \quad ; \quad g^{ij}\zeta_i\zeta_j = 0 \quad ; \quad \frac{\partial g^{ij}}{\partial \zeta_k}\zeta_j = -g^{ik}. \tag{3.6}$$

In view of (3.4) we note that the coefficients C_i^{jh} from (2.6) obey the condition $C_i^{jk}\zeta_j = 0$ and then $b_i^k = d_i^k = \delta_i^k$; therefore the vertical connection is good. Consequently, in a complex Cartan space, from (2.6) it results the following (*c.n.c.*)

$$\overset{*K}{N}_{ji} = -g_{j\bar{h}}\frac{\partial g^{\bar{h}l}}{\partial z^i}\zeta_l \tag{3.7}$$

and taking into account (3.3), we remark that it coincides with $\overset{*c}{N}_{ji}$.

Now we can consider the following (*c.l.c.*): the canonical metrical connection $\overset{c}{\Gamma H} = (\overset{*c}{N}_{jk}, \overset{c}{H}{}^i_{jk}, \overset{c}{H}{}^{\bar{i}}_{\bar{j}k}, \overset{c}{C}{}^{ik}_j, \overset{c}{C}{}^{\bar{i}k}_{\bar{j}})$ from (2.2), and in the same time the Chern-Cartan metrical connection $\overset{K}{\Gamma H} = (\overset{*K}{N}_{ji}, \overset{K}{H}{}^i_{jk}, 0, \overset{K}{C}{}^{ik}_j, 0)$ with the coefficients given by (2.6). Like in the complex Finsler case ([13]), we can consider the transformations group of metrical connections and then express the $d-$ tensors which ties this pair of connections (possible with others that may be considered: Rund, Berwald type complex connections).

We emphasize only the fact that, although the Chern-Cartan connection being of $(1,0)-$type is simpler, the canonical connection is $h-$ and $v-$ symmetrical and therefore easy to use in calculations. For the complex Finsler space this aspect was clearly proved by us in a paper that will appear.

Now let us summarize some direct properties of the canonical metrical connection.

Proposition 3.2 *The following assertions are true:*

1. $\Gamma\overset{c}{H}$ *depends only on the Hamilton function* $H(z,\zeta)$
2. *We have:* $\overset{K}{H}{}^{i}_{jk}=\dot{\partial}^{i}(\overset{*c}{N}_{jk})$
3. $\overset{c}{C}{}_{i}^{jk}=\overset{K}{C}{}_{i}^{jk}$; $\overset{c}{C}{}_{\bar{i}}^{\bar{j}k}=\overset{K}{C}{}_{\bar{i}}^{\bar{j}k}=0$
4. $\overset{c}{C}{}_{i}^{0k}=\overset{c}{C}{}_{i}^{jk}\,\zeta_j=0$; $\overset{c}{C}{}^{\bar{i}jk}=-\frac{\partial g^{\bar{i}j}}{\partial\zeta_k}$; $\overset{c}{C}{}^{\bar{0}jk}=\overset{c}{C}{}^{\bar{i}0k}=\overset{c}{C}{}^{\bar{i}j0}=0$
5. $\Gamma\overset{c}{H}$ *has only the following nonzero torsions*

$$\begin{array}{rclcl}
vT(\dot{\partial}^k,\delta_j) & = & [\overset{c}{H}{}^{i}_{jk}-\dot{\partial}^k(\overset{*c}{N}_{ij})]\dot{\partial}^i & ; & hT(\dot{\partial}^k,\delta_j)=\overset{c}{C}{}_{j}^{ik}\,\delta_i\\
vT(\dot{\partial}^{\bar{k}},\delta_j) & = & -\dot{\partial}^{\bar{k}}(\overset{*c}{N}_{ij})\dot{\partial}^i & ; & hT(\delta_{\bar{k}},\delta_j)=\overset{c}{H}{}^{i}_{j\bar{k}}\,\delta_i\\
vT(\delta_{\bar{k}},\delta_j) & = & -\delta_{\bar{k}}(\overset{*c}{N}_{ij})\dot{\partial}^i & ; & \bar{h}T(\delta_{\bar{k}},\delta_j)=-\overset{c}{H}{}^{\bar{i}}_{\bar{k}j}\,\delta_i\\
\bar{v}T(\delta_{\bar{k}},\delta_j) & = & -\delta_j(\overset{*c}{\bar{N}}_{ik})\dot{\partial}^{\bar{i}} & ; & \bar{h}T(\delta_{\bar{k}},\dot{\partial}^j)=-\dot{\partial}^j(\overset{*c}{\bar{N}}_{ik})\dot{\partial}^{\bar{i}}
\end{array}$$

6. $\theta=dz^k\wedge\delta\zeta_k+d\bar{z}^k\wedge\delta\bar{\zeta}_k$ *is a symplectic form on* $(T'M)^*$.

It seems that the class of complex Cartan spaces is poor enough (as well as that of complex Finsler spaces). For the moment we have two classical examples: one provided from a Hermitian metric on the base manifold M and, the Kobayashi Finsler Hamiltonian metric ([7, 5]). The homogeneity condition (3.1) with $\lambda\in\mathbf{C}$ is more restrictive. If we consider (3.1) only for all $\lambda\in\mathbf{R}$ (which is not an uninteresting case for geometry, taking in account that the parameter on a curve is real, unlike for the complex function theory) the class of examples is wider. If $\alpha^2=a^{\bar{j}i}(z)\zeta_i\bar{\zeta}_j$ and $\beta=2Re\{A^i(z)\zeta_i\}$, where $a^{\bar{j}i}(z)$ is a Hermitian metric on M and $A^i(z)$ is a vector field, then in analogy to the real case we can discuss on $\mathbf{R}-$complex Randers-Cartan spaces, Kropina-Cartan spaces or, more general, on $\mathbf{R}-$complex $(\alpha,\beta)-$Cartan spaces.

A complex Hamilton space (M,H) is said to be an *almost Cartan-Hamilton* $(a.C-H)$ *space* if the metric tensor $g^{\bar{j}i}(z,\zeta)=\partial^2H/\partial\zeta_i\partial\zeta_{\bar{j}}$ is $0-$

Let us note that in an $(a.C-H)$ space we have $\overset{c}{C}{}_{i}^{0k}=0$. Hence $b_i^k=d_i^k=\delta_i^k$, and then in an $(a.C-H)$ a $(c.n.c)$ is $\overset{*c}{N}_{ji}$ too.

Theorem 3.1 *A complex Hamilton space* (M,H) *is an* $(a.C-H)$ *space if and only if the Hamilton function has the form:*

$$H(z,\zeta)=g^{\bar{j}i}(z,\zeta)\zeta_i\bar{\zeta}_j+2Re\{A^i(z)\zeta_i\}+B(z)$$

where $A^i(z)$ *is a vector and* $B(z)$ *is a real valued function.*

The proof is based on the fact that $\dot{\partial}^i\dot{\partial}^{\bar{j}}(H-E)=0$ and by $H(z,\zeta)=\overline{H(z,\zeta)}$, where $E=g^{\bar{j}i}(z,\zeta)\zeta_i\bar{\zeta}_j$ is the complex energy.

A complex Hamilton space is said to be of *local Minkowski type* if at any point u^* there exists a local chart where $g^{\bar{j}i}$ depend only on the variable ζ.

Particularly, the complex Cartan space of local Minkowski type is obtained.

In a complex local Minkowski space there exists a local chart in which the coefficients of one (*c.n.c.*) obtained from a good vertical connection are zero, and therefore $\delta_i=\partial/\partial z^i$. For such a choice of local atlas one obtains simplified forms of torsions and curvatures of (*c.l.c.*).

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