Extrema with Constraints on Points and/or Velocities

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

The main results of this paper refer to three ideas:
- to replace the constraints in an optimum problems by a selector of curves;
- to reformulate and study extremum problems with point constraints and/or velocity constraints;
- to extend the saddle point theory and the Kuhn-Tucker theory to extrema with nonholonomic constraints.

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1 Extremum constrained by a selector of curves

Let D be an open set of $\mathbf{R}^n$. For each point $x \in D$, we denote by Γ_x the set of all C^1 parametrized curves $\alpha : I \to D$ which passs through the point x at a given moment $t \in I$.
1.1. Definition. Any function

$$\hat{\Gamma} : D \to \bigcup_{x \in D} \Gamma_x, \quad \hat{\Gamma}(x) \subset \Gamma_x,$$

is called *selector of curves* on D. The elements of $\hat{\Gamma}(x)$ are called *admissible curves* at the point x.
1.2. Definition. Let $f : D \to \mathbf{R}$ be a function and $\hat{\Gamma}$ be a selector of curves on D. If

$$f(\alpha(t)) \geq f(x_0), \quad \forall t \in [t_0, t_0 + \varepsilon), \quad \forall \alpha \in \hat{\Gamma}(x_0), \quad x_0 = \alpha(t_0),$$

then $x_0 \in D$ is called a *minimum point of f constrained by the selector* $\hat{\Gamma}$.
Examples. Suppose Γ_x is either the set of regular curves at x or the set of C^2 curves having x either as a regular point or as singular point of order 2. Define the selector $\hat{\Gamma}$ by $\hat{\Gamma}(x) = \Gamma_x$.

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1) In ([11]) was proved the following

1.3. Theorem. 1) *$x_0 \in D$ is a minimum point of f iff $x_0 \in D$ is a minimum point of f constrained by the selector $\hat{\Gamma}$.*

2) Let $g^a : D \to \mathbf{R}$, $a = \overline{1,p}$, $p < n$ be C^1 functions. These functions can be used to create equality constraints (equations) or inequality constraints (inequations) on points.

a) The equations $g^a(x) = 0$ introduce the partial selectors

$$\hat{\Gamma}^{\,a}(x) = \begin{cases} \{\alpha \in \Gamma_x | g^a(\operatorname{Im}\alpha) = 0\} & \text{for } x \in D \text{ with } g^a(x) = 0 \\ \emptyset & \text{for } x \in D \text{ with } g^a(x) \neq 0. \end{cases}$$

These produce the general selector

$$\hat{\Gamma}(x) = \bigcap_{a=1}^{p} \hat{\Gamma}^{\,a}(x).$$

Now we can formulate

1.4. Theorem. ([16]) *Suppose g^a are C^1 functions on D satisfying rank $\left[\frac{\partial g^a}{\partial x^j}(x)\right] = p$, at any point x of D, and the subset defined by $g^a(x) = 0$, $a = \overline{1,p}$ is nonvoid. Then $x_0 \in D$ is a minimum point of a function $f : D \to \mathbf{R}$ constrained by the selector $\hat{\Gamma}$ iff $g^a(x_0) = 0$, $a = \overline{1,p}$ and x_0 is a minimum point for f constrained by $g^a(x) = 0$, $a = \overline{1,p}$.*

b) The inequations $g^a(x) \geq 0$ give the partial selectors

$$\hat{\Gamma}^{\,a}(x) = \begin{cases} \{\alpha \in \Gamma_x | g^a(\alpha(t)) \geq 0, t \geq t_0)\} & \text{for } x \in D \text{ with } g^a(x) = 0 \\ \Gamma_x & \text{for } x \in D \text{ with } g^a(x) > 0 \\ \emptyset & \text{for } x \in D \text{ with } g^a(x) < 0. \end{cases}$$

The general selector is

$$\hat{\Gamma}(x) = \bigcap_{a=1}^{p} \hat{\Gamma}^{\,a}(x).$$

1.5. Theorem. ([16]) *Suppose g^a are C^1 functions on D satisfying*

$$\textit{rank} \left[\frac{\partial g^a}{\partial x^j}(x)\right] = p,$$

at any point x of D, and the subset defined by $g^a(x) \geq 0$ is nonvoid. Then x_0 is a minimum point of a continuous function $f : D \to \mathbf{R}$ constrained by the selector Γ iff x_0 is a minimum point for f constrained by $g^a(x) \geq 0$, $a = \overline{1,p}$.

3) Let $\omega^a(x) = \sum_{j=1}^{n} \omega^a_j(x) dx^j$, $a = \overline{1,p}$, $p < n$ be C^1 Pfaff forms. These Pfaff forms can be used to create equality constraints (Pfaff equations) or inequality constraints (for example, Pfaff inequalities) on velocities.

a) The Pfaff equations generate the partial selectors

$$\hat{\Gamma}^{\,a}(x) = \{\alpha \in \Gamma_x | \alpha \text{ is an integral curve of the Pfaff equation } \omega^a(x) = 0\},$$

which produce the general selector (associated to the Pfaff system)

$$\hat{\Gamma}(x) = \bigcap_{a=1}^{p} \Gamma^{a}(x).$$

The previous selector is connected to extrema with nonholonomic constraints (see [2], [10], [11]), by the following
1.6. Theorem. *$x_0 \in D$ is a minimum point of the function $f : D \to \mathbf{R}$ constrained by the selector $\hat{\Gamma}$ iff $x_0 \in D$ is a minimum point for f constrained by the Pfaff system $\omega^a(x) = 0$, $a = \overline{1,p}$.*

b) The primitive of each Pfaff form $\omega^a(x)$ defines the partial selectors

$$\hat{\Gamma}^{a}(x_0) = \left\{ \alpha \in \Gamma_{x_0} \middle| \int_{t_0}^{t} (\omega^a(\alpha(u)), \alpha'(u))du \geq 0,\ \forall t \in [t_0, t_0 + \varepsilon) \right\},$$

where $\alpha(t_0) = x_0$. From this point of view, the selector associated to all Pfaff forms is

$$\hat{\Gamma}(x_0) = \bigcap_{a=1}^{p} \hat{\Gamma}^{a}(x_0).$$

1.7. Definition. *$x_0 \in D$ is a minimum point of the function $f : D \to \mathbf{R}$ constrained by $\omega^a \geq 0$, $a = \overline{1,p}$ if x_0 is a minimum point of f constrained by the selector $\hat{\Gamma}$.*

This type of extremum was studied in [11].
Remark. Generally, there are two types of constraints. One given by constraints on points and other as constraints on velocities. A equation constraint on points induces an equation constraint on velocities; this last constrained does not contribute for finding critical points, but contribute in establishing the nature of critical points. The converse is not true. In the case of extrema of type 2), the point constraints select certain semicurves. In the case of extrema of type 3), the point constraints dissapear, any point of D being susceptible to be an extremum point. In the sequel we introduce a type of extremum where the point constraints and velocity constraints are not correlate.

2 Extremum with constraints on points and / or velocities

Let $\omega(x) = \sum_{j=1}^{n} \omega_j(x)dx^j$ be a C^1 Pfaff form with rank $[\omega_j(x)] = 1$. Let $M = S \cup bS$ be a subset of D, where S and bS are disjoint.

The pair (ω, M) determines the following selector of curves:

$$\hat{\Gamma}(x_0) = \begin{cases} \Gamma_{x_0} \text{ if } x_0 \in S \\ \left\{ \alpha \in \Gamma_{x_0} \middle| \int_{t_0}^{t} (\omega(\alpha(u)), \alpha'(u))du \geq 0,\ \forall t \in [t_0, t_0 + \varepsilon) \right\} \text{ if } x_0 \in bS \\ \emptyset \text{ if } x_0 \in D \setminus M. \end{cases}$$

Similarly, for each $a = \overline{1,p}$, we consider the pair (ω^a, M_a), where $\omega^a(x) = \sum_{j=1}^{n} \omega^a_j(x)dx^j$ are C^1 Pfaff forms, and $M_a = S_a \cup bS_a$. The pair (ω^a, M_a) determines the selector $\hat{\Gamma}^{\,a}$. If we denote $\omega = \{\omega^a | a = \overline{1,p}\}$, $S = \bigcap_{i=1}^{p} S_a$, $bS = \bigcup_{a=1}^{p} bS_a$, $M = S \cup bS$, then the pair (ω, M) induces the selector

$$\hat{\Gamma}(x_0) = \bigcap_{a=1}^{p} \hat{\Gamma}^{\,a}(x_0), \quad \forall x_0 \in D.$$

Let $f : D \to \mathbf{R}$ be a function. If $x_0 \in M$ is a minimum point of f constrained by the selector $\hat{\Gamma}$, then we say that x_0 is a minimum of f constrained by the pair (ω, M). Specifying $M_a = S_a \cup bS_a$, we obtain all the examples in §1.

Example 1: $S_a = D$, $bS_a = \emptyset$.

Example 2a): $S_a = \emptyset$, $bS_a = \{x \in D | g^a(x) = 0,\ a = \overline{1,p}\}$, $\omega^a = dg^a$.

Example 2b): $S_a = \{x \in D | g^a(x) > 0\}$, $bS_a = \{x \in D | g^a(x) = 0\}$, $\omega^a = dg^a$.

Example 3a): $S_a = \emptyset$, $bS_a = D$.

$$\hat{\Gamma}^{\,a}(x_0) = \left\{ \alpha \in \Gamma_{x_0} |\int_{t_0}^{t} (\omega^a(\alpha(u)), \alpha'(u))du = 0,\ \forall t \in [t_0, t_0 + \varepsilon) \right\}.$$

Example 3b): $S_a = \emptyset$, $bS_a = D$.

Remark. In the Theorems 2.1 and 2.2, the set Γ_x is the family of all C^1 parametrized curves passing by x and regular at x. In Theorem 2.2, the set Γ_x can be also the family of all C^2 parametrized curves passing by x and with singularities of order 2 at x.

2.1. Theorem. *Let $f : D \to \mathbf{R}$ be a C^1 function, and (ω, M) be the pair described above. If $x_0 \in M$ is a minimum point of f constrained by the pair (ω, M), then there exist $\lambda_a \geq 0$, $a = \overline{1,p}$ such that $df(x_0) = \sum_{a=1}^{p} \lambda_a \omega^a(x_0)$. Moreover, if $\lambda_a \neq 0$, then $x_0 \in bS_a$.*

Proof. Let $B(x_0) = \{a | x_0 \in bS_a\}$. Let $v \neq 0$ be a vector of $\mathbf{R}^n$ such that

$$(\omega^a(x_0), v) \geq 0, \quad \forall a \in B(x_0).$$

We denote

$$J(x_0) = \{a \in B(x_0) | (\omega^a(x_0), v) = 0\}.$$

Let α be an integral curve of the Pfaff system $\omega^a(x) = 0$, $a \in J(x_0)$, satisfying $\alpha(t_0) = x_0$, $\alpha'(t_0) = v$. The existence of this curve is ensured by the hypothesis on the rank of the Pfaff form (In case $J(x_0) = \emptyset$, the curve α is arbitrary, with $\alpha(t_0) = x_0$, $\alpha'(t_0) = v$). If $a \in J(x_0)$, then it follows $\int_{t_0}^{t} (\omega^a(\alpha(u)), \alpha'(u))du = 0$, $\forall t \in [t_0, t_0 + \varepsilon)$; if $a \in B(x_0) \setminus J(x_0)$, then $\int_{t_0}^{t} (\omega^i(\alpha(u)), \alpha'(u))du \geq 0$, $\forall t \in [t_0, t_0 + \varepsilon)$. Since x_0 is a minimum point of f constrained by (ω, M), i.e., $f(\alpha(t)) \geq f(\alpha(t_0))$, $\forall t \in [t_0, t_0 + \varepsilon)$, we have $(\operatorname{grad} f(x_0), v) \geq 0$. By the Farkas Lemma we find $df(x_0) = \sum_{a \in B(x_0)} \lambda_a \omega^a(x_0)$ with $\lambda_a \geq 0$. For $a \notin B(x_0)$, we take $\lambda_a = 0$.

The hypothesis regarding the rank of the Pfaff forms can be replaced with a regularity condition of Kuhn-Tucker type.

2.2. Definition. We say that (ω, M) satisfies the *Kuhn-Tucker regularity condition* at $x_0 \in bS$ if for each vector $v \neq 0$ with $\langle \omega^a(x_0), v\rangle \geq 0$, $\forall a \in B(x_0) = \{a | x_0 \in bS_a\}$, there exists a parametrized curve $\alpha \in \hat{\Gamma}_{x_0}(\alpha(t_0) = x_0)$ such that $\alpha'(t_0) = v$.

2.3. Theorem. *Let $f : D \to \mathbf{R}$ be a C^1 function and (ω, M) be the pair described above. If (ω, M) satisfies the Kuhn-Tucker regularity condition at $x_0 \in M$ and x_0 is a minimum point of f, then there exist $\lambda_a \geq 0$ such that*

$$df(x_0) = \sum_{a=1}^{p} \lambda_a \omega^a(x_0).$$

Moreover, if $\lambda_a \neq 0$, then $x_0 \in bS_a$.

Proof. Let $v \neq 0$ be a vector of $\mathbf{R}^n$ with $(\omega^a(x_0), v) \geq 0$, $\forall a \in B(x_0)$. By the Kuhn-Tucker regularity condition there exists a parametrized curve $(\alpha \in \hat{\Gamma}_{x_0}$ $(\alpha(t_0) = x_0)$ with $\alpha'(t_0) = v$. Consequently, $\displaystyle\int_{t_0}^{t} (\omega^i(\alpha(u)), \alpha'(u))du \geq 0$, $\forall t \in [t_0, t_0 + \varepsilon)$. In rest, the same arguments as in the proof of the Theorem 2.1.

2.4. Theorem . *Let Γ_x be the set of all C^2 parametrized curves passing by the point x and being regular at x. Suppose the Pfaff forms ω and the function f be of class C^1 on D. Let (ω, M) be the pair described above and $x_0 \in M$.*

Suppose: i) there exist $\lambda_a \geq 0$, $a = \overline{1,p}$, such that

$$df(x_0) = \sum_{a=1}^{p} \lambda_a \omega^a(x_0),$$

and, if $\lambda_a \neq 0$, then $x_0 \in bS_a$;

ii) the restriction of the quadratic form

$$d^2 f(x_0) - \frac{1}{2} \sum_{a=1}^{p} \lambda_a \sum_{j,k=1}^{n} \left(\frac{\partial \omega_j^a}{\partial x^k} + \frac{\partial \omega_k^a}{\partial x^j} \right)(x_0) dx^j dx^k$$

to the subspace

$$\sum_{j=1}^{n} \omega_j^a(x_0) dx^j = 0,\ a \in J^1 = \{a \in B(x_0) | \lambda_a > 0\}$$

is positive definite.

Then x_0 is a minimum point of f constrained by the pair (ω, M).

Proof. Let $\alpha : I \to D$ be a C^2 curve with $\alpha(t_0) = x_0$, regular at the point x_0 and $\alpha \in \hat{\Gamma}(x_0)$. It follows

$$\int_{t_0}^{t} (\omega^a(\alpha(u)), \alpha'(u))du \geq 0,\ \forall a \in [t_0, t_0 + \varepsilon),\ \forall j \in B(x_0) = \{a = | x_0 \in bS_a\}.$$

Case 1. If there exists $a_0 \in J'$ with $(\omega^a(x_0), \alpha'(t_0)) > 0$, then $df(x_0)(\alpha'(t_0)) = \displaystyle\sum_{a=1}^{p} \lambda_a (\omega^a(x_0), \alpha'(t_0)) > 0$. Using Taylor expansion

$$f(x) - f(x_0) = df(x_0)(x - x_0) + \mathcal{O}(\|x - x_0\|)$$

and

$$\alpha(t) - \alpha(t_0) = \alpha'(t_0)(t - t_0) + \beta(t) \cdot (t - t_0),$$

with $\lim_{t \to t_0} \beta(t) = 0$, we find

$$f(\alpha(t)) - f(\alpha(t_0)) = (t - t_0) df(x_0)(\alpha'(t_0)) + (t - t_0) df(x_0)(\beta(t)) +$$

$$+\mathcal{O}(\|\alpha(t) - \alpha(t_0)\|) = (t - t_0) df(x_0)(\alpha'(t_0)) + \mathcal{O}(t - t_0) \geq 0, \quad \forall t \in [t_0, t_0 + \varepsilon).$$

Case 2. Suppose

$$(\omega^a(x_0),\ \alpha'(t_0)) = 0, \quad \forall a \in J'.$$

The function

$$\varphi(t) = f(\alpha(t)) - \sum_{a=1}^{p} \lambda_a \int_{t_0}^{t} (\omega^a(\alpha(\tau)),\ \alpha'(\tau)) d\tau.$$

has the derivative

$$\varphi'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\alpha(t)) \frac{dx^i}{dt} - \sum_{a=1}^{p} \lambda_a \sum_{i=1}^{n} \omega_i^a(\alpha(t)) \frac{dx^i}{dt},$$

and hence

$$\varphi'(t_0) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x^i}(x_0) - \sum_{a=1}^{p} \lambda_a \omega_i^a(x_0) \right) \frac{dx^i}{dt} =$$

$$= (df(x_0) - \sum_{a=1}^{p} \lambda_a \omega^a(x_0))(\alpha'(t_0)) = 0.$$

Also

$$\varphi''(t) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x^i \partial x^j}(\alpha(t)) \frac{dx^i}{dt} \frac{dx^j}{dt} + \sum_{i=1}^{n} \frac{\partial f}{d x^i}(\alpha(t)) \frac{d^2 x^i}{dt^2} -$$

$$-\frac{1}{2} \sum_{a=1}^{p} \lambda_a \sum_{i,j=1}^{n} \left(\frac{\partial \omega_i^a}{\partial x^j} + \frac{\partial \omega_j^a}{\partial x^i} \right) (\alpha(t)) \frac{dx^i}{dt} \frac{dx^j}{dt} + \sum_{a=1}^{p} \lambda_a \sum_{i=1}^{n} \omega_i^a(\alpha(t)) \frac{d^2 x^i}{dt^2}.$$

Then

$$\varphi''(t_0) = d^2 f(x_0) - \frac{1}{2} \sum_{a=1}^{p} \lambda_a \sum_{i,j=1}^{n} \left(\frac{\partial \omega_i^a}{\partial x^j} + \frac{\partial \omega_j^a}{\partial x^i} \right) (x_0) \frac{dx^i}{dt}(t_0) \frac{dx^j}{dt}(t_0) +$$

$$+ \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x^i}(x_0) - \sum_{a=1}^{p} \lambda_a \omega_i^a(x_0) \right) \frac{d^2 x^i}{dt^2}(t_0) =$$

$$= d^2 f(x_0) - \frac{1}{2} \sum_{a=1}^{p} \lambda_a \sum_{i,j=1}^{n} \left(\frac{\partial \omega_i^a}{\partial x^j} + \frac{\partial \omega_j^a}{\partial x^i} \right) (x_0) \frac{dx^i}{dt}(t_0) \frac{dx^j}{dt}(t_0).$$

Finally,

$$\varphi(t) - \varphi(t_0) = \frac{1}{2} \varphi''(t_0)(t - t_0)^2 + \mathcal{O}((t - t_0)^2),$$

whence $\varphi(t) \geq \varphi(t_0)$, $\forall t \in (t_0 - \varepsilon, t_0 + \varepsilon)$. But $\varphi(t_0) = f(x_0)$ and, for $t \geq t_0$, $f(\alpha(t)) \geq \varphi(t)$ so that there follows $f(\alpha(t)) \geq f(x_0)$ for $t \in [t_0, t_0 + \varepsilon)$.

3 Extremum constrained by inequalities associated to a family of primitives of a Pfaff form

Let D be an open set in $\mathbf{R}^n$ and $\omega(x) = \sum_{j=1}^{n} \omega_j(x)dx^j$ be a C^0 Pfaff form. Let Γ be the family of all C^1 parametrized curves $\alpha : I \to D$. Each curve α generates a family $\{g_\alpha\}$ of functions,

$$g_\alpha : I \to \mathbf{R}, \quad g'_\alpha(t) = (\omega(\alpha(t)), \alpha'(t)),$$

called the *primitives of* ω *along* α. On the other hand, each curve α defines an equivalence class $\{\beta = \alpha \circ \varphi | \varphi : J \to I\}$ is C^1 diffeomorphism.

3.1. Definition. Let g: be a function which associates to each parametrized curve α a function g_α from the family $\{g_\alpha\}$. If $g_\beta = g_\alpha \circ \varphi$, for any equivalent curves α and β, then the function g is called system of ω-primitives.

For a Pfaff form ω and its associated system of primitives we can associate the set $M = S \cup bS$, where

$$bS = \{x_0 \in D | \exists \alpha \in \Gamma_{x_0},\ \alpha(t_0) = x_0,\ g_\alpha(t_0) = 0\}$$
$$S = \{x_0 \in D \setminus bS | \exists \alpha \in \Gamma_{x_0},\ \alpha(t_0) = x_0,\ g_\alpha(t_0) > 0\}.$$

The pair (ω, M) induces a selector $\hat{\Gamma}$ of curves.

Similarly, for each $a = \overline{1,p}$, we consider the pair (ω^a, M_a), where $\omega^a(x) = \sum_{j=1}^{n} \omega^a_j(x)dx^j$ are C^0 Pfaff forms, and $M_a = S_a \cup bS_a$ are defined using the system of ω^a-primitives g^a. The pair (ω, M) induces a selector of curves via the system of primitives $g = (g^a)$.

Let $f : D \to \mathbf{R}$ be a C^0 function. Using the previous ingredients we define the Lagrange function

$$L_\alpha(t, \lambda) = f(\alpha(t)) - \sum_{a=1}^{p} \lambda_a g^a_\alpha(t),\ \forall t \in I,\ \forall \lambda_a \geq 0.$$

This function is defined along each curve $\alpha : I \to D$, using the restriction of function f to α and the primitives of the Pfaff forms ω^a along α. In this way obtain a family of Lagrange functions, which will satisfy conditions of saddle point type.

3.2. Definition. Let $x_0 \in D$, and $\lambda^0 = (\lambda^0_a)$ with $\lambda^0_a \geq 0$, $a = \overline{1,p}$. The point (x_0, λ^0) is called *saddle point* for the family of all Lagrange functions L_α if

a) $L_\alpha(t_0, \lambda^0) \leq L_\alpha(t, \lambda^0)$, $\forall \alpha \in \Gamma_{x_0}$, $\alpha(t_0) = x_0$, $\forall t \in [t_0, t_0 + \varepsilon)$;

b) there exists $\alpha \in \Gamma_{x_0}$ with $\alpha(t_0) = x_0$ such that

$$L_\alpha(t_0, \lambda^0) \geq L(t_0, \lambda),\ \forall \lambda = (\lambda_a) \geq 0.$$

3.3. Lemma. *The condition b) in the Definition 3.2 is equivalent to:*

b′) there exists $\alpha \in \Gamma_{x_0}$, $\alpha(t_0) = x_0$, *such that* $g^a_\alpha(t_0) \geq 0$, $a = \overline{1,p}$ *and* $\lambda^0_a g^a_\alpha(t_0) = 0$ *(no summ).*

This condition afirms: if $x_0 \in M$ *and if* $\lambda^0_a > 0$, *then* $x_0 \in bS_a$.

Proof. Suppose b) is true: $\exists \alpha \in \Gamma_{x_0}$ with $\alpha(t_0) = x_0$, and $L_\alpha(t_0, \lambda^0) \geq L(t_0, \lambda)$, $\forall \lambda = (\lambda_a) \geq 0$. It follows $(*)$ $\sum_{a=1}^{p} (\lambda_a - \lambda_a^0) g_\alpha^a(t_0) \geq 0$, $\forall \lambda_a \geq 0$. Suppose, $\exists a_0 \in \overline{1,p}$ with $g_\alpha^{a_0}(t_0) < 0$; taking $\lambda_{a_0} > \lambda_{a_0}^0$ and $\lambda_a = \lambda_a^0$ for $a \neq a_0$ we obtain a contradiction with $(*)$. Taking $\lambda_a = 0$, $\forall a = \overline{1,p}$, in $(*)$, we obtain $\sum_{a=1}^{p} \lambda_a^0 g_\alpha^a(t_0) \leq 0$, i.e., $\lambda_a^0 g_\alpha^a(t_0) = 0$ for each $a = \overline{1,p}$.

The converse is automatically true.

3.4. Theorem. *Let $x_0 \in D$. If there exists $\lambda^0 = (\lambda_a^0) \geq 0$, $a = \overline{1,p}$ such that (x_0, λ^0) is a saddle point for the family for all Lagrange functions L_α, then x_0 is a minimum point of f constrained by (ω, M).*

Proof. The condition b′) from Lemma 4.3 shows that $x_0 \in M$. Suppose $\lambda_a^0 = 0$, $\forall a = \overline{1,p}$. From the condition a) it follows: for each $\alpha \in \Gamma_{x_0}$, with $\alpha(t_0) = x_0$, $f(\alpha(t)) \geq f(x_0)$, $\forall t \in [t_0, t_0 + \varepsilon)$. Hence, x_0 is a minimum point of f constrained by (ω, M); the point x_0 is a free minimum point if Γ_{x_0} is the set of all C^1 parametrized curves, regular at x_0 or the set of all C^2 parametrized curves having x_0 as a regular point or as a singular point of order 2 (Theorem 1.3.)

Suppose $J' = \{a | \lambda_a^0 > 0\}$ is nonvoid. According the condition b′), it follows $x_0 \in bS_a$, $\forall a \in J'$. Let $\alpha \in \hat{\Gamma}_{x_0}$, with $\alpha(t_0) = x_0$. Since $\int_{t_0}^{t} (\omega^a(\alpha(u)), \alpha'(u)) du \geq 0$, $\forall t \in [t_0, t_0 + \varepsilon)$, $\forall a \in J'$, we find $g_\alpha^a(t) - g_\alpha^a(t_0) \geq 0$, $\forall t \in [t_0, t_0 + \eta)$, $\forall a \in J'$. From the condition a) we get

$$f(\alpha(t)) - f(x_0) \geq \sum_{a \in J'} \lambda_a^0 (g_\alpha^a(t) - g_\alpha^a(t_0)).$$

Consequently x_0 is also a minimum point of f constrained by the pair (ω, M).

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