Notable Curves in Geometrized $J^1(T, M)$ Framework

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

Within the framework of jet spaces endowed with non-linear connection, are characterized the special curves of these spaces (h-paths, v-paths and geodesics, Lorentz-type paths and electromagnetic Lagrangian-action minimizers) which extend the Riemannian classical electromagnetic field model. Remarkable special cases outline the extension and computer-drawn graphic Maple-V plots for paths are provided.

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1 Geometric objects on $J^1(T, M)$

The geometrized framework on osculating first and higher-order osculating spaces was introduced and widely studied by Acad. R.Miron and collaborators ([4], [5]). As a complementary extension of the tangent (first-order osculating) framework in the last decade was developed with significant results the geometric approach on first-order jet spaces ([11], [9], [1], [3]).

In the sequel let $\xi = (E = J^1(T, M), \pi, T \times M)$ be the first order jet bundle of mappings $\varphi : T \to M$, where T and M are $\mathcal{C}^\infty$ real differentiable manifolds ($\dim T = m$, $\dim M = n$). The local jet coordinates on E will be denoted by

$$(t^\alpha, x^i, y^A)_{(\alpha,i,A)\in I_*} \equiv (y^\mu)_{\mu\in I},$$

where the set of indices I splits as follows

$$I = I_h \cup I_v, \quad I_h = I_{h_1} \cup I_{h_2}, \quad I_v = \overline{m+n+1, m+n+mn}$$
$$I_{h_1} = \overline{1, m}, \quad I_{h_2} = \overline{m+1, m+n}, \quad I_* = I_{h_1} \times I_{h_2} \times I_v.$$

and the indices implicitly take values as described below:

$$\alpha, \beta, \ldots \in I_{h_1}; \quad i, j, \ldots \in I_{h_2}; \quad A, B, \ldots \in I_v; \quad \lambda, \mu, \ldots \in I.$$

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As well, when appropriate, we identify $A = m + n + n(i - m - 1) + \alpha$ as $A \equiv \binom{i}{\alpha}$ and denote $y^A \equiv x^{\binom{i}{\alpha}} = \frac{\partial x^i}{\partial t^\alpha}$.

We endow E with a the extended Lagrangian of electrodynamics ([9]) of the form

$$L(t, x, y) = \tilde{g}_{AB}(t, x, y) y^A y^B + U_A(t, x) y^A + \Phi(t, x), \tag{1.1}$$

where $U_A(t, x)$ is a 1-form on E, $\Phi \in F(E)$ and assume the Kronecker decomposition

$$\tilde{g}_{AB} \equiv \tilde{g}_{\binom{i}{\alpha}\binom{j}{\beta}} = h^{\alpha\beta}(t, x) g_{ij}(t, x, y), \tag{1.2}$$

with $h_{\alpha\beta}$ and g_{ij} non-degenerate tensor fields. The derived Euler-Lagrange equations evidentiate a spray, which under certain restrictive conditions provides a *non-linear connection* $N = \{N^A_\mu\}_{\mu \in I_h, A \in I_v}$ on E which leads to the splitting $TE = HE \oplus VE$, where $VE = Ker\ \pi_*$ [11, 5]. As well, N determines the local adapted basis of $\mathcal{X}(E)$

$$\mathcal{B} = \{\delta_\alpha, \delta_i, \delta_A\}_{(\alpha, i, A) \in I_*} \equiv \{\delta_\mu\}_{\mu \in I}, \tag{1.3}$$

with $\partial_\alpha = \frac{\partial}{\partial t^\alpha}, \partial_i = \frac{\partial}{\partial x^i}$ and

$$\delta_\alpha = \partial_\alpha - N^A_\alpha \delta_A, \quad \delta_i = \partial_i - N^A_i \delta_A, \quad \delta_A = \dot{\partial}_A = \frac{\partial}{\partial y^A}. \tag{1.4}$$

The dual basis of $\mathcal{B}$ in (1.3) writes then $\mathcal{B}^* = \{\delta^\alpha, \delta^i, \delta^A\}_{(\alpha, i, A) \in I_*} \equiv \{\delta^\mu\}_{\mu \in I}$, where

$$\delta^\alpha = dt^\alpha, \ \ \delta^i = dx^i, \ \ \delta^A \equiv \delta y^A = dy^A + N^A_\alpha dt^\alpha + N^A_i dx^i. \tag{1.5}$$

The existence of Lagrangian-derived non-linear connections in the general Kronecker case represents still an open problem ([9]). However, in the following cases where $\tilde{g}$ admits a particular Kronecker splitting, the problem is tractable.

We note as particular case *the ARL (almost Riemann Lagrange) jet case*, where the tensor field $h_{\alpha\beta}(t)$ is a metric tensor field on T; then the Lagrangian (1.1) produces the canonical nonlinear connection $N = \{N^{\binom{i}{\alpha}}_\beta, N^{\binom{i}{\alpha}}_j\}$ of coefficients

$$N^{\binom{i}{\alpha}}_\beta = -\begin{vmatrix} \gamma \\ \alpha\beta \end{vmatrix} y^{\binom{i}{\gamma}}, \quad N^{\binom{i}{\alpha}}_j = \begin{vmatrix} i \\ jk \end{vmatrix} y^{\binom{k}{\alpha}} + \tfrac{1}{4} g^{ik}(2\partial_\alpha g_{jk} + h_{\alpha\beta} U_{\binom{k}{\beta})j}), \tag{1.6}$$

where $U_{\binom{k}{\beta})j} = \delta_{[j} U_{\binom{k]}{\beta}}$ means the h_2-curl of U; generally, we denote $\tau_{[i\dots j]} = \tau_{i\dots j} - \tau_{j\dots i}$ and $\tau_{\{i\dots j\}} = \tau_{i\dots j} + \tau_{j\dots i}$. Also we have

$$\tilde{g}_{AB} = \tfrac{1}{2} \dot{\partial}^2_{AB} L. \tag{1.7}$$

More particular, in the ARLS (almost Riemann Lagrange separated) jet case, g_{ij} is a metric tensor field on M, and both the nondegenerate metric tensors h, g and the potentials U_A determine the nonlinear connection N of coefficients

$$N^{\binom{i}{\alpha}}_{\beta} = -\left|{}^{\gamma}_{\alpha\beta}\right| y^{\binom{i}{\gamma}}, \quad N^{\binom{i}{\alpha}}_{j} = \left|{}^{i}_{jk}\right| y^{\binom{k}{\alpha}} + \tfrac{1}{4} g^{ik} \cdot h_{\alpha\beta} U_{\binom{k}{\beta} j}. \tag{1.8}$$

If E is endowed with a non-linear connection $N = \{N^A_\alpha, N^A_i\}$, any linear connection $\nabla = \{L^\lambda_{\mu\nu}\}_{\lambda,\mu,\nu\in I}$ on E has its components relative to the adapted basis (1.3) provided by the relations $\delta^\lambda(\nabla_{\delta_\nu}\delta_\mu) = L^\lambda_{\mu\nu}, \ \ \forall\lambda,\mu,\nu \in I$. According to the three sets of indices I_{h_1}, I_{h_2}, I_v, these components group in $3^3 = 27$ distinct subsets.

The subsets of nontrivial coefficients of ∇ can be strongly reduced for the connections $\Gamma(N)$ (called "N-connections"), whose covariant derivative preserves the sections $\mathcal{S}(HE)$ and $\mathcal{S}(VE)$; these obey the conditions

$$L^\lambda_{\mu\nu} = 0, \ \ \forall\, (\lambda,\mu) \in (I_h \times I_v) \cup (I_v \times I_h). \tag{1.9}$$

Further, one may consider the *special N-connections* $\Gamma_*(N)$, whose covariant derivatives preserve the distributions $Span(\delta_\alpha)_{\alpha\in I_{h_1}}$ and $Span(\delta_i)_{i\in I_{h_2}}$; they satisfy the supplementary relations

$$L^\lambda_{\mu\nu} = 0, \ \ \forall\, (\lambda,\mu) \in (I_{h_1} \times I_{h_2}) \cup (I_{h_2} \times I_{h_1}). \tag{1.10}$$

More particular, the so-called "Γ-linear h-normal connections" $\Gamma_n(N)$ [9] have just four essential sets of components

$$\{L^\alpha_{\beta\gamma}, L^i_{j\gamma}, L^i_{jk}, L^i_{jA}\} \equiv \nabla, \tag{1.11}$$

which provide the other 5 derived sets by means of

$$\begin{aligned}
&L^A_{B\gamma} \equiv L^{\binom{i}{\alpha}}_{\binom{j}{\beta}\gamma} = \delta^\beta_\alpha L^i_{j\gamma} - \delta^i_j \left|{}^{\beta}_{\alpha\gamma}\right|, \ \ L^A_{Bk} \equiv L^{\binom{i}{\alpha}}_{\binom{j}{\beta}k} = \delta^\beta_\alpha \left|{}^{i}_{jk}\right|,\\
&L^A_{BC} \equiv L^{\binom{i}{\alpha}}_{\binom{j}{\beta}C} = \delta^\beta_\alpha L^i_{jC}, \ \ L^\alpha_{\beta j} = 0, \quad L^\alpha_{\beta C} = 0.
\end{aligned}$$

We shall further consider the case when $h_{\alpha\beta}(t)$ and $g_{ij}(t,x,y)$ in the Lagrangian L in (1.1) are non-degenerate, and we endow E with a semi-Riemannian metric

$$G = \underbrace{h_{\alpha\beta}(t)dt^\alpha \otimes dt^\beta}_{h} + \underbrace{g_{ij}(t,x,y)dx^i \otimes dx^j}_{g} + \underbrace{\tilde{g}_{AB}(t,x,y)\delta y^A \otimes \delta y^B}_{\tilde{g}}, \tag{1.12}$$

where $\tilde{g}_{AB} \equiv \tilde{g}_{\binom{i}{\alpha}\binom{j}{\beta}} = h^{\alpha\beta}(t) g_{ij}(t,x,y)$. In this case the so-called *the Cartan linear connection*, which is an $h-$normal connection, is metrical and satisfies the conditions ([11], [9])

$$L^i_{j\gamma} = \frac{g_{ik}}{2}\partial_\gamma g_{jk}, \quad L^i_{[jk]} = 0, \quad L^i_{[j\binom{k]}{\alpha}} = 0.$$

Its four essential sets of coefficients (1.11) are given by

$$\begin{aligned}
&L^\alpha_{\beta\gamma} = \left|{}^{\alpha}_{\beta\gamma}\right|, \ \ L^i_{j\gamma} = \tfrac{1}{2} g^{ik}\delta_\gamma g_{kj}, \ \ L^i_{jk} = \left|{}^{i}_{jk}\right|,\\
&L^i_{jA} \equiv L^i_{j\binom{k}{\gamma}} = \tfrac{1}{2} g^{il}(\delta_{\binom{\{k}{\gamma}} g_{jl\}} - \delta_{\binom{l}{\gamma}} g_{jk}).
\end{aligned} \tag{1.13}$$

The adapted components of the torsion $\mathcal{T}$ and of the curvature $\mathcal{R}$ of ∇ are defined by the relations

$$\delta^\lambda(\mathcal{T}(\delta_\nu,\delta_\mu)) = T^\lambda_{\mu\nu},\ \ \delta^\lambda(\mathcal{R}(\delta_\nu,\delta_\mu)\delta_\rho) = R_\rho{}^\lambda{}_{\mu\nu},\ \forall\ \lambda,\mu,\nu,\rho\in I.$$

Then the Cartan essential torsion coefficients are ([9]; for ARL case [11, Theorem 4.4])

$$\{T_\gamma{}^{\binom{i}{\alpha}}{}_{\binom{j}{\beta}}, T_k{}^{\binom{i}{\alpha}}{}_{\binom{j}{\beta}}, T_{\binom{j}{\beta}}{}^{\binom{i}{\alpha}}{}_{\binom{k}{\gamma}}, T_\beta{}^i{}_j, T^i_{jA}, T_\beta{}^A{}_\gamma, T_\beta{}^A{}_j, T_i{}^A{}_j\}.$$

The nontrivial non-holonomy coefficients $\omega^\lambda_{\mu\nu}$ are described by the relations

$$\begin{aligned} &[\delta_\mu,\delta_\nu] = \omega^A_{\mu\nu}\delta_A \equiv T^A_{\mu\nu}\delta_A, &&\forall\mu,\nu\in I_h,\\ &[\delta_\mu,\delta_B] = \omega^A_{\mu B}\delta_A \equiv \partial_B N^A_\mu\delta_A, &&\forall\mu\in I_h,\end{aligned}$$

and are explicitly provided for the ARL case in [9, Theorem 2.3]. Ultimately, the five essential and three derived nontrivial sets of curvature N-tensor fields are respectively

$$\{R_\beta{}^\alpha{}_{\gamma\delta},\ R_j{}^i{}_{km},\ R_j{}^i{}_{\gamma\lambda},\ R_j{}^i{}_{\lambda A},\ R_j{}^i{}_{CD}\},\quad \{R_{\binom{j}{\beta}}{}^{\binom{i}{\alpha}}{}_{\gamma\delta}, R_{\binom{j}{\beta}}{}^{\binom{i}{\alpha}}{}_{\lambda k}, R_{\binom{j}{\beta}}{}^{\binom{i}{\alpha}}{}_{\mu A}\},$$

for $\lambda\in I_h,\ \mu\in I$.

In this framework, the Liouville field $\mathcal{C} = y^A\delta_A$ on (E,N,∇) produces the *deflection tensor fields*

$$d^A_\mu = \delta^A\nabla_{\delta_\mu}\mathcal{C},\quad \mu\in I, A\in I_v,$$

which lead further to the associated to N and ∇ *electromagnetic 2-form* $F = F_{A\mu}\delta y^A\wedge\delta y^\mu$, of nontrivial components

$$\text{(1.14)}\qquad \begin{cases} F_{A\beta}\equiv F_{\binom{i}{\alpha}\beta} = \frac{1}{2}\left(h^{\alpha\gamma}g_{ik}y^{\binom{k}{[\gamma}}\right)_{|\beta]}\\ F_{AB}\equiv F_{\binom{i}{\alpha}\binom{j}{\beta}} = \frac{1}{2}\tilde{g}_{\binom{[i}{\alpha}C}y^C{}_{|\binom{j]}{\beta}}\\ F_{Aj}\equiv F_{\binom{i}{\alpha}j} = \frac{1}{2}\,d_{\binom{[i}{\alpha}j]} = \frac{1}{2}\,y_{\binom{[i}{\alpha}|j]} = \frac{1}{2}\left(y^{\binom{k}{\gamma}}h^{\alpha\gamma}g_{k[i}\right)_{|j]},\end{cases}$$

where $|\alpha$, $|i$ and $|A$ are the covariant derivations given by ∇_{δ_μ}, for $\mu\in I_{h_1}, I_{h_2}$ and I_v respectively. Considering the raising/lowering of the indices performed by the metric tensor field G, F provides *the electromagnetic force*

$$\text{(1.15)}\qquad \tilde{F} = F_A{}^\mu\delta_\mu\otimes\delta^A$$

of nontrivial essential components,

$$F^\alpha_A = h^{\alpha\beta}F_{A\beta},\ F^i_A = g^{ij}F_{Aj},\ F^C_A = g^{CD}F_{AD}.$$

We note that in the particular ARLS case, the Cartan connection has just two basic nontrivial coefficients

$$\{L^{\alpha}_{\beta\gamma} = \left|{}^{\alpha}_{\beta\gamma}\right|,\ L^{i}_{jk} = \left|{}^{i}_{jk}\right|\},$$

and its non-trivial torsion $N-$fields are $\{T^{\binom{m}{\gamma}}_{\alpha\beta}, T^{\binom{m}{\gamma}}_{ij}, T^{\binom{m}{\gamma}}_{\alpha j}\}$ ([9]).

Moreover, for $m = 1,\ n = 4$ and $h_{11} = 1$, one finds as particular case, the pseudo-Riemannian weak gravitational model endowed with the metric $g_{ij}(x) = \eta_{ij} + \varepsilon_{ij}(x)$, where the weakness of the gravitational field g_{ij} is expressed by its decomposition into the flat Minkowski metric $n_{ij} = diag(-1, 1, 1, 1)$ and a small perturbation $\varepsilon_{ij}(x)$, a symmetric tensor field with $|\varepsilon_{ij}(x)| << 1$.

2 Paths and Lorentz curves on $J^1(T, M)$

We consider in the following on (E, N, ∇) smooth curves $c : J = [a, b] \subset \mathbb{R} \to E$, having their images inside a chart $\tilde{U} \subset E$, locally given by

$$c(s) = (t^{\alpha}(s), x^{i}(s), y^{A}(s)) \equiv (y^{\mu}(s)), \forall t \in J.$$

Definitions. a) The field $\mathcal{V}^{\mu} = \frac{\delta y^{\mu}}{\mathrm{d}s}$ defined on c is called N*-velocity field* of the curve c. Its components are explicitely given by

$$\{\mathcal{V}^{\mu}\}_{\mu\in I} \equiv \left(\dot{t}^{\alpha}, \dot{x}^{i}, \frac{\delta y^{a}}{\mathrm{d}s} = \dot{y}^{A} + N^{A}_{\beta}\dot{t}^{\beta} + N^{A}_{j}\dot{x}^{j}\right)_{(\alpha,i,A)\in I_*}$$

where we denote by dot the s-derivation. We denote by $\mathcal{F} = \mathcal{F}^{\mu}\delta_{\mu}$ the N*-force field* on c, which provides the motion of the test-body along c and whose components are explicitely given by

$$\mathcal{F}^{\mu} = \frac{\nabla\mathcal{V}^{\mu}}{ds} \overset{not}{=} \frac{\delta\mathcal{V}^{\mu}}{ds} + L^{\mu}_{\nu\rho}\mathcal{V}^{\nu}\mathcal{V}^{\rho}.$$

b) We call c *stationary curve* with respect to ∇ iff $\mathcal{F} = 0$ along the curve.

c) The curve c is called

- $h-$*curve*, if $\pi_v(\mathcal{V}) = 0$, and
- $v-$*curve*, if $\pi_h(\mathcal{V}) = 0$,

where by π_h and π_v we denoted respectively the $h-$ and $v-$projectors of the canonic splitting induced by N.

d) An $h - /v-$curve which satisfies also the extra condition $\mathcal{F} = 0$, is called $h - /v-$*path*, respectively.

Analytically, these curves are described by

Theorem 1. *Let $c : J \subset \mathbb{R} \to E$ be a curve. Then the curves defined above are characterized as follows:*

a) c is an $h-$curve iff

$$\mathcal{V}^{A} = 0 \Leftrightarrow \tfrac{\delta y^{A}}{ds} = 0 \Leftrightarrow \dot{y}^{A} + N^{A}_{\alpha}\dot{t}^{\alpha} + N^{A}_{j}\dot{x}^{j} = 0,\ \forall A \in I_v. \tag{2.16}$$

b) c is a v−curve iff

$$\mathcal{V}^\mu = 0,\ \forall \mu \in I_h \ \Leftrightarrow\ \tfrac{\delta y^\mu}{ds} = 0, \forall \mu \in I_h \ \Leftrightarrow\ c(s) = (t_0, x_0, y(s)), s \in J. \tag{2.17}$$

c) c is an h−path ("stationary h−curve or "horizontal geodesic") iff besides (2.16) it satisfies

$$\frac{d\mathcal{V}^\mu}{ds} + L^\mu_{\nu\rho}\mathcal{V}^\nu\mathcal{V}^\rho = 0,\ \forall \mu \in I_h. \tag{2.18}$$

d) c is a v−path ("stationary v−curve or "vertical geodesic") iff besides (2.17) it satisfies

$$\frac{\delta\mathcal{V}^A}{ds} + L^A_{BC}\mathcal{V}^B\mathcal{V}^C = 0,\ \forall A \in I_v. \tag{2.19}$$

We note that the implicit sum in the right term of (2.18)/(2.19) involves just horizontal/vertical index types. The proof is computational.

Consider the triple (E, N, G), where the metric G in the one in (1.12), N is a fixed nonlinear connection, and ∇ is the Cartan connection attached to G of basic coefficients (1.13). Then qne can derive the electromagnetic tensor fields in (1.14) and (1.15) and we have

Definition. A curve c is called *Lorentz curve* on (E, N, G) iff

$$G_{\nu\rho}\frac{\nabla\mathcal{V}^\rho}{ds} = F_{A\nu}\mathcal{V}^A \quad \Leftrightarrow \quad \frac{\nabla\mathcal{V}^\mu}{ds} = F_A{}^\mu\mathcal{V}^A. \tag{2.20}$$

Theorem 2. ([1, 3]) *The Lorentz equations (2.20) have the equivalent form*

$$\ddot{t}^\alpha + L^\alpha_{\beta C}\dot{t}^\beta\mathcal{V}^C + L^\alpha_{jC}\dot{x}^j\mathcal{V}^C + L^\alpha_{\beta\gamma}\dot{t}^\beta\dot{t}^\gamma + L^\alpha_{j\gamma}\dot{x}^j\dot{t}^\gamma + L^\alpha_{\beta k}\dot{t}^\beta\dot{x}^k + L^\alpha_{jk}\dot{x}^j\dot{x}^k = F^\alpha_B\mathcal{V}^B \tag{2.21}$$

$$\ddot{x}^i + L^i_{\beta C}\dot{t}^\beta\mathcal{V}^C + L^i_{jC}\dot{x}^j\mathcal{V}^C + L^i_{\beta\gamma}\dot{t}^\beta\dot{t}^\gamma + L^i_{j\gamma}\dot{x}^j\dot{t}^\gamma + L^i_{\beta k}\dot{t}^\beta\dot{x}^k + L^i_{jk}\dot{x}^j\dot{x}^k = F^i_B\mathcal{V}^B \tag{2.22}$$

$$\dot{\mathcal{V}}^A + N^A_\alpha\dot{t}^\alpha + N^A_i\dot{x}^i + L^A_{C\beta}\mathcal{V}^C\dot{t}^\beta + L^A_{Cj}\mathcal{V}^C\dot{x}^j + L^A_{BC}\mathcal{V}^B\mathcal{V}^C = F^A_B\mathcal{V}^B, \tag{2.23}$$

where $\mathcal{V}^A = \dot{y}^A + N^A_\beta\dot{t}^\beta + N^A_i\dot{x}^i,\ A \in I_v$.

Remarks. a) The *Lorentz h-paths* satisfy the extra conditions $\mathcal{V}^A = 0,\ A \in I_v$ and since the right side of (2.21)-(2.23) is identically vanishing, they *coincide with the usual h-paths of* (E, N, G).

b) The *Lorentz v-paths* have fixed base-point, i.e.,

$$\mathcal{V}^\mu = 0,\ \mu \in I_h \quad \Leftrightarrow \quad (t, x) = (t_0, x_0) \in T \times M,$$

and hence the associated Lorentz equations rewrite

$$F^\alpha_B\mathcal{V}^B = 0, \quad F^i_B\mathcal{V}^B = 0, \quad F^A_B\mathcal{V}^B = \dot{\mathcal{V}}^A + L^A_{BC}\mathcal{V}^B\mathcal{V}^C.$$

c) In the ARLS case *with the nonlinear connection (1.6) induced by the Lagrangian*, the electromagnetic tensors simplify to

$$(2.24)\qquad F_A^\alpha \equiv F^\alpha_{\binom{i}{\beta}\gamma}=0,\ \ F^i_A = g^{ij}\tilde{F}_{Aj}=-\frac{1}{4}g^{ij}U_{Aj},\ \ F^B_A=0,$$

and the nonvanishing Cartan connection essential coefficients reduce to

$$L^\alpha_{\beta\gamma}=\left|{}^{\alpha}_{\beta\gamma}\right|,\quad L^i_{jk}=\left|{}^{i}_{jk}\right|,\ \ L^A_{B\gamma}\equiv L^{\binom{i}{\alpha}}_{\binom{j}{\beta}\gamma}=-\delta^i_j\left|{}^{\beta}_{\alpha\gamma}\right|,\ \ L^A_{Bk}\equiv L^{\binom{i}{\alpha}}_{\binom{j}{\beta}k}=-\delta^\beta_\alpha\left|{}^{i}_{jk}\right|.$$

Then the Lorentz equations (2.21)-(2.23) get the typical shape

$$\ddot{t}^\alpha+\left|{}^{\alpha}_{\beta\gamma}\right|\dot{t}^\beta\dot{t}^\gamma=0,\quad \ddot{x}^i+\left|{}^{i}_{jk}\right|\dot{x}^j\dot{x}^k=-\frac{1}{4}g^{ij}U_{Aj}\mathcal{V}^A,\quad \dot{\mathcal{V}}^A=0.$$

Note that in this case (g dependent on x only), the Berwald connection [11] has the same coefficients as the Cartan connection, and hence the associated Lorentz curves, h- and v-paths are described by the same equations. The Lorentz h-paths obey the extra equations

$$\dot{y}^A+N^A_\beta\dot{t}^\beta+N^A_j\dot{x}^j=0,\ A\in I_v,$$

which write explicitely

$$\dot{y}^{\binom{i}{\alpha}}-\left|{}^{\gamma}_{\alpha\beta}\right|y^{\binom{i}{\gamma}}\dot{t}^\beta+\left(\left|{}^{i}_{jk}\right|y^{\binom{k}{\alpha}}+\frac{1}{4}g^{ik}h_{\alpha\beta}U_{\binom{k}{\beta}j}\right)\dot{x}^j=0.$$

As well, the Lorentz v-paths for the Cartan connection satisfy the extra condition $-\mathcal{V}^A=2\dot{\mathcal{V}}^A$, having as solutions flat rays within the fibers of E - in accordance with the particular case $J^1(\mathbb{R},M)\equiv TM$ studied in [6].

d) In the *ARLSU* case (ARLS uniparametric case, for $m=1$ and $s=t^1=t$, [2]), for $h_{11}=1$, we recapture the known results derived in [4, 6] for the tangent space case. For this, after shifting the indices left by one unit ($I_{h_2}=\overline{1,n},\ I_v=\overline{n+1,2n}$), $y^A\equiv y^{\binom{i}{1}}\overset{not}{=}y^i$, set locally $h_{11}=1$ and we can use *the Finsler-Lagrange tangent space notations* from [5].

If we consider the Lagrangian (1.1) of the particular form

$$(2.25)\qquad L(x,y)=mc\,\gamma_{ij}(x)y^iy^j+\frac{2e}{m}U_i(x)y^i+\Phi(x),$$

with γ_{ij} pseudo-Riemannian metric, $U=U_idx^i$ 1-form on M and $\Phi\in F(M)$, then the fundamental tensor derived from L via (1.7) is

$$\tilde{g}_{\binom{i}{1}\binom{j}{1}}(t,x,y)=g_{ij}(x)=mc\,\gamma_{ij}(x),$$

the non-linear connection induced by L has the components

$$N^A_1=0, N_j^{\binom{i}{1}}=\left|{}^{i}_{jk}\right|y^k+g^{ik}U_{\binom{k}{1}j},\ i=\overline{1,n},\ A=\overline{n+1,2n},$$

where $U_{\binom{k}{1}}=\frac{e}{m}A_k$. In this case, the Cartan (1.13) and Berwald canonic connections have just null and Christoffel (re-indexed) components. For ∇ Cartan connection, the

Lorentz equations (2.22) confine to the known ones of Lagrange spaces ([5], [4, p. 171])

$$\ddot{x}^i + 2G^i(x,y) = 0, \;\; y^i = \frac{dx^i}{ds}, \; i = \overline{1,m} \tag{2.26}$$

of the Lagrangian spray derived from the Lagrangian L in (2.25) for Φ constant,

$$G^i = \frac{1}{2}\gamma^i_{jk}y^j y^k + \frac{e}{2m^2c}\gamma^{ij}A_{[j;k]}y^k,$$

where "$;k$" expresses the canonic covariant derivative on (M, γ_{ij}).

We note that in the absence of the electromagnetic force F_{μ_A}, the equations (2.20) rewritten in the form (2.26) become the equations of stationary curves of the connection ∇. Hence, in the absence of the covector potential U, the equations (2.20) become *the equations of geodesics* of the manifold M and the equations of $h - paths$ become the Lorentz equations.

3 Electromagnetic Lagrangian extremals

In the ARLS case the extremals of the energy action

$$E(L) = \int_T L(t,x,y)\, \mathrm{d}vol_T \tag{3.27}$$

of the Lagrangian L in (1.1) are shown to satisfy the PDE system ([8])

$$h^{\alpha\beta}(\partial_\beta y^{\binom{i}{\alpha}} + 2G^{\binom{i}{\alpha}}_\beta) = 0, \; i = \overline{1,n}. \tag{3.28}$$

In (3.28), an essential role plays *the spray* $G^{\binom{i}{\alpha}}_\beta = {}^1G^{\binom{i}{\alpha}}_\beta + {}^2G^{\binom{i}{\alpha}}_\beta$ associated to L, where

$$\begin{cases} {}^1G^{\binom{i}{\alpha}}_\beta = -\frac{1}{2}\left|{}^{\gamma}_{\alpha\beta}\right| y^{\binom{i}{\gamma}} \\ {}^2G^{\binom{i}{\alpha}}_\beta = \frac{1}{2}\left|{}^{i}_{jk}\right| y^{\binom{j}{\alpha}}y^{\binom{k}{\beta}} + \frac{1}{4m}\tilde{g}^{\binom{i}{\alpha}\binom{l}{\beta}}(U_{\binom{l}{\varepsilon}s}y^{\binom{s}{\varepsilon}} + \partial_\varepsilon U_{\binom{l}{\varepsilon}} + U_{\binom{l}{\gamma}}\left|{}^{\varepsilon}_{\gamma\varepsilon}\right| - \partial_l\Phi), \end{cases}$$

which provides the canonic L-induced nonlinear connection N in (1.8) via

$$N^{\binom{i}{\alpha}}_\beta = 2\frac{\partial({}^1G^{\binom{i}{\alpha}}_\beta)}{\partial y^{\binom{j}{\gamma}}}y^{\binom{j}{\gamma}}, \qquad N^{\binom{i}{\alpha}}_j = 2\frac{\partial({}^2G^{\binom{i}{\delta}}_\varepsilon h^{\delta\varepsilon})}{\partial y^{\binom{j}{\gamma}}}h_{\alpha\gamma}.$$

We note that in the ARLSU case for $m = 1$ and $h_{11} = 1$, using the conventions above, the extremals of the Lagrangian action are characterized by the equations

$$\ddot{x}^i + \left|{}^{i}_{jk}\right| \dot{x}^j\dot{x}^k = \frac{1}{4}(F_j{}^i y^j + g^{ij}\partial_j\Phi),$$

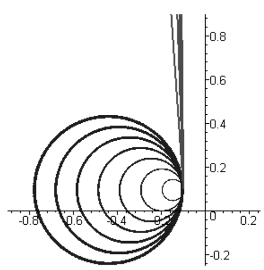
and for constant Φ these coincide with the extended Lorentz paths produced by the Liouville tensor field.

4 Numerical simulation

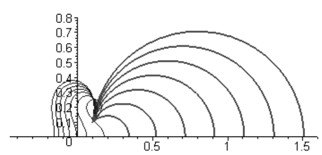
In the ARLS uniparametric case detailed above, consider $n = 2$, M endowed with the Lagrangian L in (1.1) with $g = mc\gamma_{ij}$, $\Phi = 0$ and the potential $\bar{U}$ given by $\bar{U} = \varepsilon(x^1 dx^2 - x^2 dx^1)$, $\varepsilon \in \mathbb{R}$. Then, denoting by $a = \varepsilon e(m^2 c)^{-1}$ the control parameter of electromagnetid field strength, the appropriately rescaled Lorentz-type equations (2.26) read

$$\ddot{x}^i + \left|{i \atop jk}\right| \dot{x}^j \dot{x}^k = (-1)^{i+1} a(g^{i1}\dot{x}^2 + g^{i2}\dot{x}^1),\ i = \overline{1,2}. \tag{4.29}$$

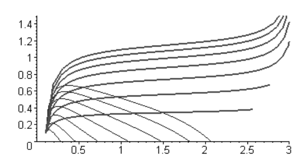
We exemplify further the influence of the electromagnetic force F derived from $\tilde{U}$ via (2.24) on h-paths for three cases: $\mathbb{R}^2$, H^2 and S^2. Using Maple V programming were obtained computer-drawn images representing the Lorentz-type sheaves of curves (the left-bended lines in the drawings) which are obtained for fixed non-zero values of a ($a = -512$ for Euclidean case, $a = -1024$ for the Poincare half-plane, $a = 2$ for the sphere respectively).



$\mathbb{R}^2$



H^2



S^2

We note that, when the influence of the generalized electric potentials $U_i(x)$ disappears (i.e., for $a = 0$ regarded as a limit case), one obtains the sheaves of geodesics of the manifold M (marked with thick lines). Hence the geodesics - the solutions for $a = 0$ of the system (4.29) deform to Lorentz curves, under the controlled by a influence of the generalized electromagnetic tensor field.

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References

[1] V. Balan, *Lorentz-type equations in first-order jet spaces endowed with nonlinear connection*, Proceedings of The First French-Romanian Colloquium of Numerical Physics, October 30-31, 2000, Bucharest, Romania, Geometry Balkan Press 2002, pp. 105-114.

[2] V.Balan, *Synge-Beil and Riemann-Jacobi jet structures with applications to physics*, Jour. of Math. and Math. Sci, Hindawi Publ. Corp., 27 (2003), 1693-1702.

[3] V. Balan, K. Trencevski, *Stationary Lorentz curves in geometrized jet spaces. Numerical simulation and analytic solutions*, Algebras, Groups and Geometries, I.B.R., Palm Harbor, to appear.

[4] R. Miron, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Acad. Publishers, 1994.

[5] R. Miron, M.Anastasiei, *Vector Bundles. Lagrange Spaces. Applications to Relativity*, Geometry Balkan Press, 1996.

[6] R. Miron, R.Roşca, M.Anastasiei, K.Buchner, *New aspects in Lagrangian relativity*, Found. of Phys. Lett., v. 5 (1992), n.2, 141-171.

[7] R. Miron, M. Tatoiu-Radivoiovici, *A Lagrangian theory of electromagnetism*, Rep. Math. Phys., 27 (1989), 49-84.

[8] M.Neagu, *The geometry of autonomous metrical multi-time Lagrange space of electrodynamics*, International Journal of Mathematics and Mathematical Sciences, Hindawi Publishing Corporation, (2001), http://xxx.lanl.gov/abs/math.DG/0010091, (2000).

[9] M. Neagu, *Generalized metrical multi-time Lagrangian geometry of physical fields*, http://xxx.lanl.gov/abs/math.DG/0011003, 2000.

[10] M. Neagu, C. Udrişte, *From PDE Systems and Metrics to Geometric Multi-Time Field Theories*, Sem. Mec. 79, West Univ. of Timisoara, 2001. http://xxx.lanl.gov/abs/math.DG/0009071, 2000.

[11] M. Neagu, C. Udrişte, *The geometry of metrical multi-time Lagrange spaces*, http://xxx.lanl.gov/abs/math.DG/0009071, 2000.

[12] D.J. Saunders, *The Geometry of Jet Bundles*, Cambridge University Press, 1989.

[13] C.Udrişte, *Geometric Dynamics*, Kluwer Acad. Publishers, 2000.

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