On Equations and Properties Concerning Some Classes of Chordal Polygons

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

In the paper k-chordal (or k-inscribed) polygons of first and second kind with given index are considered. Existence result is proved for equilateral chordal polygon which side lengths are already known. The convex and nonconvex cases are discussed depending on the orientation of the polygon. Secondly, the number of different radii of circumcircles of equilateral k-inscribed n-gon cannot be greater then

$$\mathbf{s}[n] = \left[\frac{n-1}{2}\right] + \left[\frac{n-3}{2}\right] + \left[\frac{n-5}{2}\right] + \cdots + 2 + 1.$$

A very natural conjecture is formulated on the existence of side lengths of k-chordal n-gons when the minimal number of different circumcircle radii is

$$\boldsymbol{\sigma}[n] := \left[\frac{n-1}{2}\right] + \begin{pmatrix} n \\ 1 \end{pmatrix} \left[\frac{n-3}{2}\right] + \begin{pmatrix} n \\ 2 \end{pmatrix} \left[\frac{n-5}{2}\right] + \cdots + \begin{pmatrix} n \\ \mu \end{pmatrix} \left[\frac{n-2\mu+1}{2}\right],$$

where $n-2\mu = 3(4)$ for n odd (even). Thirdly, the so-called *main equation* (kind of related characteristic algebraic equation for a polygon) is introduced for the class $C_n(a_1, \cdots, a_n)$ of k-chordal *related polygons*. In few illustrative examples we obtain the number and the numerical values of different radii of quadrangle, pentagon, octagon and enneagon, solving the related main equations, when only the side lengths of initial polygons are known. In the final section certain interesting properties of the so-called main equations are discussed, proving that the positive roots of the main equations are the radii of the circumcircles of the chordal n-gons whose sides have the lengths $a_1, \cdots, a_n$. The equilateral pentagon is presented in detail with three different positive solutions of its main equation which is an eigth degree algebraic equation. In the same section the main equation of λn-gons is characterized, when the initial n-gon is λ times continued on the same circumcircle, λ positive integer.

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Key words: Algebraic equations, k-chordal polygon, k-inscribed chordal polygon, main equation, circumcircle, polygon of first kind, polygon of second kind, index of chordal polygon.

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1 Introduction

The subject and the main purposes of this article are very closely connected to the current interests of the first two authors in elementary geometry, more precisely in properties of generalized plane polygons. The main tools and definitions of the frequently mentioned basic geometrical object as k-chordal-, k-tangential-, (k, λ, l)-chordal-, (k, f, l)-chordal polygon, polygon of first/second kind are introduced, treated and discussed in ([6], [7], [8], [9], [10], [11]). There existence results are proved for chordal polygons under necessary and sufficient conditions upon the side lengths, while for (k, f, l)-chordal polygons upon the function f, i.e. upon the lengths $a_1, \cdots, a_n$ and $f(a_1), \cdots, f(a_n)$ being side lengths of a k-chordal and l-chordal polygons respectively in the same time ([6]). The approach in investigations by Góźdź in ([1]) and Pech in ([5]) is more or less different then in previously cited articles, namely, these authors uses complex methods and harmonic/Fourier series methods in getting characteristic inequalities and equations for different polygonal plane structures, such that are convex. The classical works by Kürschák concerns to the isoperimetric questions on the chordal and tangential n-gons to the given circle, where by their original method it was shown that the equilateral case is the extremal ([3], [4]). Some comments and explanations can be found in depending Horváth's essay ([2]). Finally ([12]) is containing many known inequalities relating circumradius (and further characteristics) of planar convex sets, such that could be treated and generalized to our nonconvex, k-inscribed setting and similarly Temesvári's optimization paper could be found interesting in further investigations for the maxima of the power sums of side lengths of some classes of k-chordal polygons, compare ([13]).

In this paper we follow the previous investigations focusing mainly ourselves to the so-called *Main Equation* of the k-chordal n-gon, and to computing all different radii of depending circumcircles, when the polygons side lenghts are already known.

A polygon with vertices $A_1, \cdots, A_n$ (in this order) will be denoted by $\mathbf{A} \equiv A_1 \cdots A_n$ and the lengths of its sides by $a_1, \cdots, a_n$. The interior angle at the vertex A_i will be denoted by α_i or $\angle A_i$. Thus

$$\angle A_i = \angle A_{i-1} A_i A_{i+1}, \qquad i = \overline{1, n},$$

where $A_0 \equiv A_n,\ A_{n+1} \equiv A_1$.

A polygon $\mathbf{A}$ is called chordal if there exists a circle $\mathbf{C_A}$ such that $\forall A_i \in \mathbf{C_A}$. Whenever $\mathbf{A}$ is chordal, then C, ρ and $\mathbf{C_A}$ stand for the centre, radius and the circumcircle of $\mathbf{A}$ respectively. Throughout this paper very important roles are playing by (oriented) angles

$$\beta_i = \angle C A_i A_{i+1}, \tag{1}$$

$$\varphi_i = \angle A_i C A_{i+1}, \qquad i = \overline{1, n}. \tag{2}$$

Also it is important to emphasize that β_i and φ_i are in opposite orientations, compare the following figure.

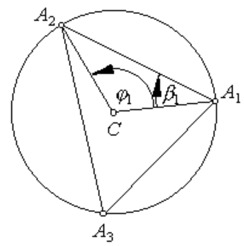

Figure 1. Angles of chordal polygons

Notice 1 *We consider chordal polygons with property that no two of their consecutive vertices are the same.*

Of course, the measure $|\psi|$ of an oriented angle ψ we take depending on the orientation of ψ, in radians. So, by Notice 1. it is

$$0 \le |\beta_i| < \frac{\pi}{2}, \qquad 0 < |\varphi_i| \le \pi,$$

Notice 2 *It will be no confusion there writing β_i, φ_i the measures of oriented angles β_i, φ_i given by* (1),(2).

Notice 3 *In the following we shall suppose that no β_i is zero.*

Let us remark that in the case when some β_i is equal to zero, then we have

$$2\rho = \max\{a_1, \cdots, a_n\}.$$

Accordingly, in the following when we speak about a chordal polygon $\mathbf{A}$ it will be meant (by Notice 1 and Notice 3) that $\mathbf{A}$ has no two the same consecutive vertices and no one of its sides is its diameter.

Definition 1 *Let $\mathbf{A} = A_1 \cdots A_n$ be a chordal polygon. We say that $\mathbf{A}$ is of the* first kind *if inside $\mathbf{C}_\mathbf{A}$ there exists a point O that all oriented angles $\angle A_iOA_{i+1}$ have the same orientation. If such a point O does not exist, i.e. not all $\angle A_iOA_{i+1}$ have the same orientation, we say that $\mathbf{A}$ is of* second kind.

Definition 2 *Let $\mathbf{A}$ be a chordal polygon and let X be a point inside $\mathbf{C}_\mathbf{A}$ such that*

$$\left|\sum_{j=1}^{n} \psi_j\right| = 2\omega(X)\pi,$$

where $\psi_j = \text{measure}\angle A_jXA_{j+1}$ and $\omega(X)$ is a positive integer. Then we say that $\mathbf{A}$ is k-inscribed polygon of the first (second) kind *when*

$$k = \max_{X \in \text{int}(\mathbf{C}_\mathbf{A})} \omega(X).$$

Here $\text{int}(S)$ *stays for the interior of the set S.*

We say that j is the index *of $\mathbf{A}$ if $|\varphi_1 + \cdots + \varphi_n| = 2j\pi,\ j \in \{0, 1, \cdots, k\}$.*

Definition 3 *The polygon* $\mathbf{A}$ *is said to be k-chordal polygon if it is of first kind and if* $j = k$*, where* j *is the index of* $\mathbf{A}$ *and* k *is given by Definition* 2.

It is easy to see that $\mathbf{A}$ is k-chordal **iff**

$$|\beta_1 + \cdots + \beta_n| = (n - 2k)\frac{\pi}{2},$$

where $\beta_i > 0$ $i = \overline{1, n}$ or $\beta_i < 0$, $i = \overline{1, n}$. For example, if $\forall \beta_i > 0$, then $\varphi_i < 0$, $i = \overline{1, n}$, and it is valid

$$\varphi_1 + \cdots + \varphi_n = -2k\pi,$$

or in other words $2\beta_1 + \cdots + 2\beta_n = n\pi - 2k\pi$, since $\varphi_i = -\pi + 2\beta_i$. Thus, if $\beta_i > 0$, $i = \overline{1, n}$, then

$$\beta_1 + \cdots + \beta_n = (n - 2k)\frac{\pi}{2}.$$

The sign of the sum $\beta_1 + \cdots + \beta_n$ depends of the orientation of the polygon, we discuss this in brief. Let $\mathbf{A}$ be a chordal k-inscribed polygon and let $\mathbf{B} = B_1 \cdots B_n$ be a polygon with vertices $B_j = A_{n-1+j}$, $j = \overline{1, n}$. Then $\mathbf{A} \equiv \mathbf{B}$ but their orientations are opposite (orientation of $\mathbf{A}$ is positive (negative) depending on the circumscription of $\mathbf{C}_{\mathbf{A}}$ to $\mathbf{A}$ "counter-clockwise" ("clockwise")).

If $\mathbf{A}$ is k-chordal, then β_j, $j = \overline{1, n}$ are negative if $\mathbf{A}$ is positively oriented and vice versa. But in the case when $\mathbf{A}$ is a chordal polygon of second kind, then there are β_j's of opposite signes.

Notice 4 *In the following we shall assume that polygons are negatively oriented. Then* $\varphi_1 + \cdots + \varphi_n \leq 0$ *but* $\beta_1 + \cdots + \beta_n \geq 0$.

So, for example, the case Fig. 2.(a) gives $\beta_1 + \cdots + \beta_5 < 0$, and the case Fig. 2.$(b)$ one concludes $\beta_1 + \cdots + \beta_5 > 0$.

Lemma 1 *If* $\mathbf{A} = A_1 \cdots A_n$ *is a k-inscribed chordal polygon whose index is* j*, then*

$$\beta_1 + \cdots + \beta_n = (n - 2(\nu + j))\frac{\pi}{2}, \tag{3}$$

where $\nu = \sharp\{m |\ \beta_m < 0\}$ *and*

$$|\beta_1| + \cdots + |\beta_n| = (n - 2(\nu + j))\frac{\pi}{2} + 2\tau, \tag{4}$$

where $\tau = -\sum_{m:\ \beta_m < 0} \beta_m$.

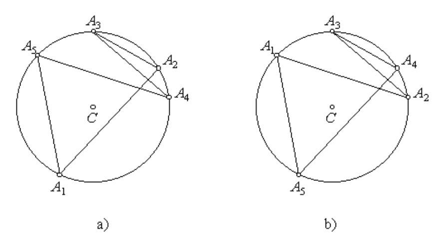

Figure 2. Chordal pentagons of opposite orientations

Proof. Since $\varphi_j = -\pi + 2\beta_j$ if $\beta_j > 0$, and $\varphi_j = \pi + 2\beta_j$ for $\beta_j < 0$, the equality $\varphi_1 + \cdots + \varphi_n = -2j\pi$ can be written as

$$2\beta_1 + \cdots + 2\beta_n + \nu\pi - (n-\nu)\pi = -2j\pi,$$

from which follows (3). Now, by (3) we get (4) easily.

At this point we introduce certain symbols such that we will use frequently in the sequel.

1. $[a]$ denotes the largest integer contained in a. Obviously, if $\mathbf{A} = A_1 \cdots A_n$ is k-inscribed chordal polygon, then

$$k \le \left[\frac{n-1}{2}\right], \tag{5}$$

because it has to be $n - 2k > 0$. Of course, there are extremal cases, when the equality holds in (5), e.g. when $\mathbf{A}$ is equilateral, i.e. $a_1 = \cdots = a_n$.

2. $P(a_1, \cdots, a_n; \beta_1, \cdots, \beta_n)$. Let $\mathbf{A} = A_1 \cdots A_n$ be a chordal n-gon. Then this n-gon will also be written as

$$P(a_1, \cdots, a_n; \beta_1, \cdots, \beta_n). \tag{6}$$

Sometimes instead of (6) we write

$$P(a_1, \cdots, a_n; \beta_1, \cdots, \beta_n; \rho). \tag{7}$$

In the equilateral case ($a_j = a$) it stays $P(a; \beta_1, \cdots, \beta_n)$ or $P(a; \beta_1, \cdots, \beta_n; \rho)$.

3. $P(a_1, \cdots, a_n; i_1, \cdots, i_\nu)$. If $\beta_{i_j},\ j = \overline{1,\nu}$ are negative, then this situation we note as

$$P(a_1, \cdots, a_n; i_1, \cdots, i_\nu), \tag{8}$$

or appropriately $P(a; i_1, \cdots, i_\nu)$.

4. $\mathbf{S}_p^n$. Let p be an integer such that $0 \le p \le n$. If $p = 0$, then

$$\mathbf{S}_0^n := \cos\beta_1 \cdots \cos\beta_n.$$

If $p > 0$, then $\mathbf{S}_p^n$ is the sum of $\binom{n}{p}$ products of the form

$$\sin\beta_{i_1} \cdots \sin\beta_{i_p} \cos\beta_{i_{p+1}} \cdots \cos\beta_{i_n},$$

where $(i_1, \cdots, i_n)$ is a permutation of the set $\{1, \cdots, n\}$, e.g.

$$\mathbf{S}_2^3 = \sin\beta_1 \sin\beta_2 \cos\beta_3 + \sin\beta_1 \cos\beta_2 \sin\beta_3 + \cos\beta_1 \sin\beta_2 \sin\beta_3.$$

2 Existence of k-inscribed chordal polygon

Let $a_1, \cdots, a_n$ be given lengths such that satisfies the constraint

$$\sum_{j=1}^{n} a_j > 2\max\{a_1, \cdots, a_n\}. \tag{9}$$

Then there exists (at least) n-gon $P(a_1, \cdots, a_n; \beta_1, \cdots, \beta_n)$, where $\forall \beta_j > 0$ and $\beta_1 + \cdots + \beta_n = (n-2)\pi/2$. Namely, then there exists a positive real ρ (which is in fact certain length), such that satisfies

$$\sum_{j=1}^{n} \arccos \frac{a_j}{2\rho} = (n-2)\frac{\pi}{2}. \tag{10}$$

Intuitively it is easy to see this result. For example, observe the situation on Fig. 3.

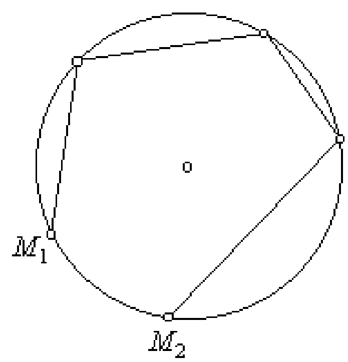

Figure 3. Tightening of circumscribed circle one gets closed chordal polygon

Taking a sequence of circles with respect to decreasing radii we achieve the case $M_1 \equiv M_2$; see the strong proof of (10) in ([7, proof of Theorem 2]).

Let us remark that (10) exists **iff** there are lengths $r, R;\ r \le R$, such that

$$\sum_{j=1}^{n} \arccos \frac{a_j}{2r} \le (n-2)\frac{\pi}{2} \le \sum_{j=1}^{n} \arccos \frac{a_j}{2R}.$$

Generally speaking, if $\sum_{j=1}^{n} \beta_j = \kappa\pi/2$, for κ integer, and if $\beta_j = (-1)^{\epsilon_j}|\beta_j|$, where $\epsilon_j \in \{0, 1\}$, then there exists $P(a_1, \cdots, a_n; \beta_1, \cdots, \beta_n)$ **iff** there are lengths (positive reals) $r, R; r \le R$ such that

$$\sum_{j=1}^{n} (-1)^{\epsilon_j} \arccos \frac{a_j}{2r} \le \kappa\frac{\pi}{2} \le \sum_{j=1}^{n} (-1)^{\epsilon_j} \arccos \frac{a_j}{2R}.$$

Example 1 *Put* $a_j = 2 + .1j,\ j = \overline{1,5}$ *and* $\sum_{j=1}^{5} \beta_j = \pi/2$, *where* $\beta_1 < 0$, *then there is a pentagon* $P(2.1, \cdots, 2.5; \beta_1, \cdots, \beta_5)$, *see Figure 4. below. In this case may be taken* $2r = 2.806,\ 2R = 2.807$, *namely then the radius* ρ *of the circumcircle* $\mathbf{C_A}$ *satisfies* $2.806 < 2\rho < 2.807$. *But for* $\beta_5 < 0$ *no pentagon is there, since it is*

$$\arccos\frac{2.1}{2\rho} + \arccos\frac{2.2}{2\rho} + \arccos\frac{2.3}{2\rho} + \arccos\frac{2.4}{2\rho} - \arccos\frac{2.5}{2\rho} > \frac{\pi}{2}, \qquad 2\rho \geq a_5.$$

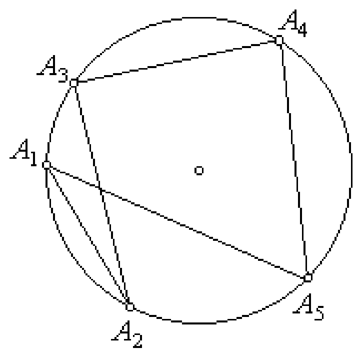

Figure 4. $P(2.1, \cdots, 2.5; \beta_1, \cdots, \beta_5)$ with $\sum_{j=1}^{5} \beta_j = \frac{\pi}{2};\ \beta_1 < 0$.

Theorem 1 *If* $a_j = a,\ j = \overline{1,n}$, *then for each angle*

$$\beta(k) = (n-2k)\frac{\pi}{2n}, \qquad k = \overline{1, [(n-1)/2]}$$

there is a k*-chordal equilateral* n*-gon* $P(a; \beta(k))$.

Proof. It is easy to see that for $2\rho_k = a/\cos(n-2k)\frac{\pi}{2n}$ it is

$$n \arccos\frac{a}{2\rho_k} = (n-2k)\frac{\pi}{2}.$$

As an ilustration of this result we give the case of equilateral 2-chordal pentagon $P(1; \beta(2))$ presented on Figure 5.:

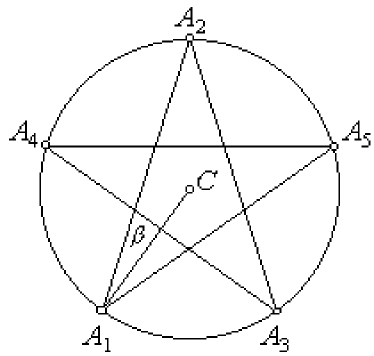

Figure 5. 2 - chordal pentagon $P(1, \beta(2))$

3 Number of k-inscribed chordal polygons

Let $\mathbf{s}[n]$ be defined by

$$\mathbf{s}[n] := \left[\frac{n-1}{2}\right] + \left[\frac{n-3}{2}\right] + \left[\frac{n-5}{2}\right] + \cdots + 2 + 1. \tag{11}$$

Theorem 2 *If $a_1 = \cdots = a_n = a$, then the number of k-inscribed chordal n-gons wich have not equal radii cannot overgrow $\mathbf{s}[n]$.*

Proof. From $(n-2\nu)\beta(k,\nu) = (n-2(k+\nu))\pi/2$ it follows that

$$\beta(k,\nu) = \left(1 - \frac{2k}{n-2\nu}\right)\frac{\pi}{2}, \tag{12}$$

where

$$k \in \left\{1, 2, \cdots, \left[\frac{n-1}{2}\right]\right\}, \quad \nu \in \left\{0, 1, \cdots, \left[\frac{n-3}{2}\right]\right\}, \quad k + \nu = \left[\frac{n-1}{2}\right].$$

Then by the Theorem 1. using the notation $\nu = \sharp\{m |\, \beta_m < 0\}$ introduced in Lemma 1. we deduce that there are

$$\left[\frac{n-2\nu-1}{2}\right] \quad \text{polygons} \quad P(a; \beta(k,\nu)), \qquad k \in \left\{1, 2, \cdots, \left[\frac{n-2\nu-1}{2}\right]\right\}.$$

Let us remark that here k refers to the term *k-inscribed* polygon in Definition 2. Now obvious transformations lead to the assertion of the theorem.

As an example we give the heptagon with parameters $a = 1,\ k = 1,\ \nu = 2$. If $i_1 = 1,\ i_2 = 5$ (compare (8)), then the heptagon $P(1; i_1 = 1, i_2 = 5)$ is presented on Figure 6 a). On Figure 6 b) the heptagon $P(1; i_1 = 1, i_2 = 2)$ is shown. Although these heptagons are not equal, they have equal radii.

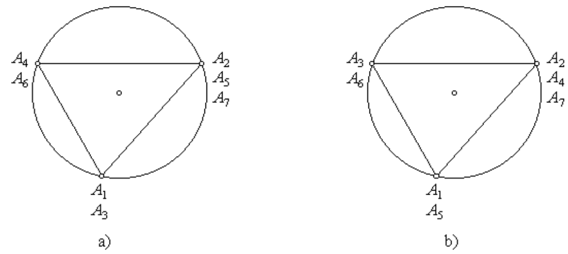

Figure 6. a) Heptagon $P(1; i_1 = 1, i_2 = 5)$; b) Heptagon $P(1; i_1 = 1, i_2 = 2)$

Remark 1 *If n is even then there is one more equilateral n-gon $\mathbf{A} = A_1 \cdots A_n$, where $A_1 \equiv A_3 \equiv \cdots \equiv A_{n-1}$, $A_2 \equiv A_4 \equiv \cdots \equiv A_n$. But in the Notice 3 terminology speaking it is not included in $\mathbf{s}[n]$.*

In the following considerations it is very important to see that the number of different radii of corresponding circumcircles of the chordal polygons is less then $\mathbf{s}[n]$ if $n > 7$. So, for example when $n = 9$ then the 3-chordal enneagon and the 1-inscribed chordal enneagon with three negative angles have equal radii since $\beta(3,0) = \pi/6 = \beta(1,3)$.

Generally speaking if $n - 2i$ and $n - 2j$ are different entries of the sequences

$$\begin{array}{ll} n, n-2, n-4, \cdots, 3 & \textit{where } n \textit{ is odd} \\ n, n-2, n-4, \cdots, 4 & \textit{where } n \textit{ is even} \end{array}$$

and $\mathrm{GCD}(n-2i, n-2j) \geq 3(4)$ *for n odd (even) respectively, then the number of different radii is less then $\mathbf{s}[n]$.*

In the continuation we introduce the symbol

$$(13)\quad \boldsymbol{\sigma}[n] := \left[\frac{n-1}{2}\right] + \binom{n}{1}\left[\frac{n-3}{2}\right] + \binom{n}{2}\left[\frac{n-5}{2}\right] + \cdots + \binom{n}{\mu}\left[\frac{n-2\mu+1}{2}\right],$$

where

$$(14)\qquad n - 2\mu = \begin{cases} 3 & n \text{ in odd} \\ 4 & n \text{ in even.} \end{cases}$$

Having in mind Theorem 2 and the dicussion about Figure 3, we give the following hypothesis.

Conjecture. There are the lengths $a_1, \cdots, a_n$ such that the number of different radii of the circumcircles is at least $\boldsymbol{\sigma}[n]$.

Here we point out that we have proved the assertion of the Conjecture for $n = 3, 4, 5, 6, 7$. Something about this will be exposed in following examples.

Intuitively, this conjecture is very reasonable having in mind the following fact (in connection to Figure 3): if $\epsilon_1, \cdots, \epsilon_n$ are different positive numbers, $a_j = a + \epsilon_j$, $j = \overline{1, n}$ and

$$\mathbf{p} = na + \sum_{j=1}^{n} \epsilon_j,$$

then

$$\mathbf{p} - (a + \epsilon_j) \neq \mathbf{p} - (a + \epsilon_l)$$

whenever $j \neq l$.

Example 2 As in Example 1 consider pentagon with the given side lengths $a_j = 2 + .1j,\ j = \overline{1,5}$. (There is shown that β_5 has to be positive!). Then using the equations

$$\beta_1 + \cdots + \beta_5 = (5 - 2(j + \nu))\frac{\pi}{2}, \tag{15}$$

we find that

j	ν	negative angle	$2\rho \in$
1	0	none	$(3.9, 4)$
2	0	none	$(3.50, 2.51)$
1	1	β_1	$(2.806, 2.807)$
1	1	β_2	$(2.750, 2.760)$
1	1	β_3	$(2.680, 2.690)$
1	1	β_4	$(2.604, 2.605)$

But, if we put e.g. $a_j = 3 + .1j,\ j = \overline{1,5}$, then $\beta_5 < 0$ is acceptable as well. Namely, then there is $3.6 < 2\rho < 3.7$ such that

$$\sum_{j=1}^{5} \arccos\frac{a_i}{3.6} < \frac{\pi}{2}, \quad \sum_{j=1}^{5} \arccos\frac{a_i}{3.7} > \frac{\pi}{2}.$$

Thus by (15) we deduce that there are at least $\boldsymbol{\sigma}[5] = 7$ chordal pentagons whose sides have the lengths $a_j = 3 + .1j,\ j = \overline{1,5}$, and the corresponding radii ρ_j of the circumcircles are different.

In the case $n = 7;\ a_j = 5 + .1j,\ j = \overline{1,7}$, it can be shown that there are $\boldsymbol{\sigma}[7] = 38$ chordal heptagons with different corresponding radii of its circumcircles. As we can see by (13) the number $\boldsymbol{\sigma}[n]$ increases with the n growing: $\boldsymbol{\sigma}[5] = 7,\ \boldsymbol{\sigma}[6] = 8,\ \boldsymbol{\sigma}[7] = 38,\ \boldsymbol{\sigma}[8] = 47,\ \boldsymbol{\sigma}[9] = 187$, etc.

4 Classes of related polygons and their main equations

In this section we consider certain relationships between polygons which possess the sides with same lengths. Firstly we introduce the term *related polygons.*

Definition 4 *Let $a_1, \cdots, a_n$ be given lengths and let $\mathbf{X} = X_1 \cdots X_n,\ \mathbf{Y} = Y_1 \cdots Y_n$ be chordal polygons with the property that the lengths of their sides satisfy*

$$x_i = y_i = a_i, \qquad i = \overline{1,n}. \tag{16}$$

Then we say that $\mathbf{X}$ and $\mathbf{Y}$ are related polygons *(with respect to their sides); the set consisting from all related polygons with respect to given $a_1, \cdots, a_n$ we denote with the symbol $C_n(a_1, \cdots, a_n)$.*

In our next considerations we deal according to the Notice 1; also whenever we consider a polygon from $C_n(a_1, \cdots, a_n)$ we will assume that no β_j vanishes (Notice 3).

Let $a_1, \cdots, a_n$ be already known and let $\mathbf{X} \in C_n(a_1, \cdots, a_n)$ and choose the angles $\beta_j,\ j = \overline{1,n}$ so that $\mathbf{X} \equiv P(a_1, \cdots, a_n; \beta_1, \cdots, \beta_n)$. If j is the index of $\mathbf{X}$ and $\nu = \sharp\{m|\ \beta_m < 0\}$, then by (3) we get

$$\beta_1 + \cdots + \beta_n = (n - 2(j + \nu))\frac{\pi}{2}.$$

Hence we have the following equalities

$$\begin{aligned} &(17) \qquad & \cos(\beta_1 + \cdots + \beta_n) &= 0, \qquad n \text{ odd},\\ &(18) \qquad & \sin(\beta_1 + \cdots + \beta_n) &= 0, \qquad n \text{ even}. \end{aligned}$$

Using the symbol $\mathfrak{S}_p^n$ we can transform the above two equalities into

$$\begin{aligned} &(19) \qquad & \mathfrak{S}_0^n - \mathfrak{S}_2^n + \mathfrak{S}_4^n - \cdots + (-1)^{\theta_1}\mathfrak{S}_{n-1}^n &= 0, \qquad n \text{ odd}\\ &(20) \qquad & \mathfrak{S}_1^n - \mathfrak{S}_3^n + \mathfrak{S}_5^n - \cdots + (-1)^{\theta_2}\mathfrak{S}_{n-1}^n &= 0, \qquad n \text{ even} \end{aligned}$$

where $\theta_1 = (1 + 3 + 5 + \cdots + n) + 1,\ \theta_2 = (1 + 3 + 5 + \cdots + (n-1)) + 1$. Now, the following steps will be done in (19) and (20). Replace $\sin\beta_j$ with

$$\sqrt{1 - \left(\frac{a_j}{2\rho}\right)^2},$$

and put $a_j/(2\rho)$ instead of $\cos\beta_j$. Then rationalizing and simplifying (19) and (20) in ρ these equations become

$$\begin{aligned} &(21) \qquad & F_1(a_1, \cdots, a_n; \rho) &= 0, \qquad n \text{ odd}\\ &(22) \qquad & F_2(a_1, \cdots, a_n; \rho) &= 0, \qquad n \text{ even}, \end{aligned}$$

where $F_m(a_1, \cdots, a_n; \rho)$ are polynomials in $\rho,\ m = 1, 2$.

Definition 5 *The equation* (21) *or* (22) *is said to be the* Main Equation *concerning the k - inscribed polygons in $C_n(a_1, \cdots, a_n)$ for each k admissible in the sense of* Definition 2.

Example 3 In this example we consider the main equation of chordal n-gons in $C_n(1, \cdots, 1)$, when n is odd. Take $n = 9$. Let $P(1; \beta_1(k,\nu), \cdots, \beta_9(k,\nu))$ be k - inscribed enneagon with ν negative angles. Of course, it could be $\nu \in \{0, 1, 2, 3\}$ and it is unessential which ν angles are negative, since the depeneding radii of corresponding circumcircles equals in length.

As $|\beta_1(k,\nu)| = \cdots = |\beta_9(k,\nu)|$, let $\beta(k,\nu) = |\beta_1(k,\nu)| > 0$. Then

$$\begin{aligned} \beta_1(k,\nu) + \cdots + \beta_9(k,\nu) &= (9 - 2\nu)\beta(k,\nu)\\ &= ((9 - 2\nu) - 2k)\tfrac{\pi}{2} = (9 - 2(k+\nu))\tfrac{\pi}{2}. \end{aligned}$$

Consequently

$$\begin{array}{lcll} \beta(k,0) & = & (9-2k)\frac{\pi}{18}, & k=1,2,3,4, \\ \beta(k,1) & = & (7-2k)\frac{\pi}{14}, & k=1,2,3, \\ \beta(k,2) & = & (5-2k)\frac{\pi}{10}, & k=1,2, \\ \beta(1,3) & = & \frac{\pi}{6}. & \end{array}$$

Hence we have

$$\begin{array}{llcll} (23) & \cos 9\beta(k,0) & = & 0, & k=1,2,3,4, \\ (24) & \cos 7\beta(k,1) & = & 0, & k=1,2,3, \\ (25) & \cos 5\beta(k,2) & = & 0, & k=1,2, \\ (26) & \cos 3\beta(1,3) & = & 0. & \end{array}$$

Now, from(23-26) using the well - known trigonometric equality

$$\cos n\alpha = \cos^n \alpha - \binom{n}{2} \cos^{n-2}\alpha \sin^2\alpha + \binom{n}{4} \cos^{n-4}\alpha \sin^4\alpha - \cdots,$$

it is easy to see that

1. $x_k^0 = \cos\beta(k,0),\ k=1,2,3,4$ are the positive roots of the equation

$$(27) x^9 - 36x^7(1-x^2) + 126x^5(1-x^2)^2 - 42x^3(1-x^2)^3 + 9x(1-x^2)^4 = 0,$$

2. $x_k^1 = \cos\beta(k,1),\ k=1,2,3$ are the positive roots of the equation

$$(28) \qquad x^7 - 21x^5(1-x^2) + 35x^3(1-x^2)^2 - 7x(1-x^2)^3 = 0,$$

3. $x_k^2 = \cos\beta(k,2),\ k=1,2$ are the positive roots of the equation

$$(29) \qquad x^5 - 10x^3(1-x^2) + 5x(1-x^2)^2 = 0,$$

4. $x_1^3 = \cos\beta(1,3)$ is the unique positive root of

$$(30) \qquad x^3 - 3x(1-x^2) = 0.$$

Let the left hand sides of the equations (27-30) be denoted by $f_j(x),\ j=9,7,5,3$ respectively. Then the main equation of the chordal enneagons from $C_9(1,\cdots,1)$ of the form

$$(31) \qquad F(x) = f_9(x)f_7(x)f_5(x)f_3(x),$$

where $x = 1/(2\rho)$. Its positive roots are

$$x_{k\nu} = \frac{1}{2\rho_{k\nu}}, \qquad \begin{array}{ll} \nu=0 & ,\ k=1,2,3,4, \\ \nu=1 & ,\ k=1,2,3, \\ \nu=2 & ,\ k=1,2, \\ \nu=3 & ,\ k=1, \end{array}$$

where $\rho_{k\nu}$ is the circumcircle radius of $P(1;\beta_1(k,\nu,\cdots,\beta_9(k,\nu))$.

Analogous results hold for all n odd. With respect to this question we can remark that the coefficients of the partial polynomials $f_n(x)$ could be expressed as

$$(32)\qquad c_{n-2j}=(-1)^j\sum_{i=0}^{(n-2j-1)/2}\binom{n}{2(i+j)}\binom{i+j}{i},\qquad j=\overline{0,(n-1)/2}.$$

Now, when $n=9$, it is

$$\begin{aligned}
f_9(x) &= c_9x^9+c_7x^7+c_5x^5+c_3x^3+c_1x \qquad \text{where}\\
c_9 &= 1+\binom{9}{2}+\binom{9}{4}+\binom{9}{6}+\binom{9}{8}=2^8=256,\\
-c_7 &= \binom{9}{2}+\binom{9}{4}\binom{2}{1}+\binom{9}{6}\binom{3}{2}+\binom{9}{8}\binom{4}{3}=576,\\
c_5 &= \binom{9}{4}+\binom{9}{6}\binom{3}{1}+\binom{9}{8}\binom{4}{2}=432,\\
-c_3 &= \binom{9}{6}+\binom{9}{8}\binom{4}{1}=120,\\
c_1 &= \binom{9}{8}=9.
\end{aligned}$$

Generally, if c_n is the leading coefficient of $f_n(x)$, n odd, then $c_n=2^{n-1}$.

So the main equation of chordal n-gons in $C_n(1,\cdots,1), n$ odd may be written in the form

$$(33)\qquad f_n(x)f_{n-2}(x)\cdots f_5(x)f_3(x)=0.$$

Of course, $f_j(x)$ in (31) are the same as in (33). The number of positive roots in the equation (33) is at most $\mathbf{s}[n]=\frac{n^2-1}{8}$. Also, each factor $f_j(x)$ in (33) has a root $x=0$. It is true because of $\cos n\frac{\pi}{2}=0$.

Finally, as the interesting consequence of the Example 1, we get some combinatorial/trigonometrical formulæ. From $f_n(x)x^{-1}=0$, using Viète's formulæ, it follows that

$$\begin{aligned}
\cos^2\frac{\pi}{2n}+\cos^2\frac{3\pi}{2n}+\cdots+\cos^2\frac{(n-2)\pi}{2n} &= -\frac{c_{n-2}}{2^{n-1}},\\
\cos^2\frac{\pi}{2n}\cos^2\frac{3\pi}{2n}\cdots\cos^2\frac{(n-2)\pi}{2n} &= \frac{n}{2^{n-1}},
\end{aligned}$$

where c_{n-2} is given by (32).

Example 4 Here we consider the main equation of chordal n-gons in $C_n(1,\cdots,1)$, n even. At first concentrate to the octagon, i.e. $n=8$. In the same way as in Example 3 it can be found that

$$\begin{aligned}
\beta(k,0) &= (4-k)\frac{\pi}{8}, && k=1,2,3,\\
\beta(k,1) &= (3-k)\frac{\pi}{6}, && k=1,2,\\
\beta(1,2) &= \frac{\pi}{4},\\
\sin 8\beta(k,0) &= 0, && k=1,2,3,\\
\sin 6\beta(k,1) &= 0, && k=1,2,\\
\sin 4\beta(1,2) &= 0.
\end{aligned}$$

Now, using the identity

$$\sin n\alpha = \binom{n}{1} \cos^{n-1}\alpha \sin\alpha - \binom{n}{3} \cos^{n-3}\alpha \sin^3\alpha + \cdots,$$

we clearly get

1. $x_k^0 = \cos\beta(k,0),\ k = 1,2,3$ are the positive roots of the equation

$$x^6 - 7x^4(1-x^2) + 7x^2(1-x^2)^2 - (1-x^2)^3 = 0, \tag{34}$$

2. $x_k^1 = \cos\beta(k,1),\ k = 1,2$ are the positive roots of the equation

$$3x^4 - 10x^2(1-x^2) + 3x^4(1-x^2)^3 = 0, \tag{35}$$

3. $x_1^2 = \cos\beta(1,2)$ is the unique positive root of

$$2x^2 - 1 = 0. \tag{36}$$

Let the left hand side of equations (34-36) be denoted by $f_j(x),\ j = 8,6,4$ respectively.

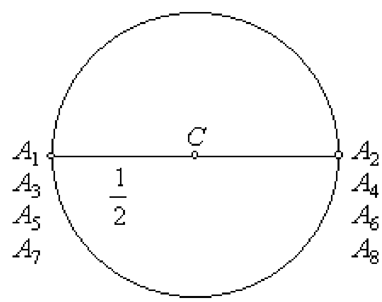

Figure 7. Degenerated $C_8(1,\cdots,1)$ octagon with $\beta(k,\nu) = 0$

Then the main equation of the chordal octagons from $C_8(1,\cdots,1)$ can be written as

$$f_8(x)f_6(x)f_4(x) = 0, \qquad x = \frac{1}{2\rho}, \tag{37}$$

excluding the polygon with circumcircle which possesses radius equal to 1/2. The positive roots of (37) are now given by

$$x_k^\nu = \frac{1}{2\rho_k^\nu}, \qquad \begin{array}{ll} \nu = 0 & ,\ k = 1,2,3, \\ \nu = 1 & ,\ k = 1,2 \\ \nu = 2 & ,\ k = 1. \end{array}$$

At this point we have to discuss the case of the circumcircle $\mathbf{C_A}$ which possesses radius $\rho = 1/2$. As

$$\sin n\alpha = \cos\alpha \sin\alpha \left\{ \binom{n}{1} \cos^{n-2}\alpha - \binom{n}{3} \cos^{n-4}\alpha \sin^2\alpha + \cdots \right\},$$

there is a chordal octagon with $x = 1$, that means $2\rho = 1$ (compare Figure 7). Indeed, by

$$(8 - 2\nu)\beta(k,\nu) = (4 - k - \nu)\pi; \qquad k + \nu = 4,$$

it is $\sin\beta(k,\nu) = 0,\ \cos\beta(k,\nu) = 1$.

So we don't need Notice 3 in this example. Hence, instead of (37) the main equation of the considered octagon becomes $(x-1)f_4(x)f_6(x)f_8(x) = 0$. On the other hand $\cos\beta(k,\nu) = 0$ results with $\beta(k,\nu) = \pi/2$, see Figure 8.

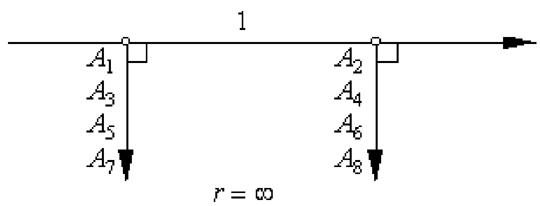

Figure 8. Degenerated $C_8(1,\cdots,1)$ octagon with $\beta(k,\nu) = \frac{\pi}{2}$

In general, analogous holds for all n-gons from the class $C_n(1,\cdots,1),\ n \geq 4,\ n$ even. Therefore the coefficients γ_j of the factors $f_n(x)$ in the main equation can be expressed in the following form

$$(38)\ \ c_{n-2(j+1)} = (-1)^j \sum_{i=0}^{(n-2j-2)/2} \binom{n}{2(i+j)+1} \binom{i+j}{i}, \quad j = \overline{0, n/2-1}.$$

Now, in our case we get

$$\begin{aligned}
f_8(x) &= c_6x^6 + c_4x^4 + c_2x^2 + c_0 \qquad \text{where} \\
c_6 &= \binom{8}{1} + + \binom{8}{3} + \binom{8}{5} + \binom{8}{7} = 2^7 = 128, \\
-c_4 &= \binom{8}{3} + \binom{8}{5}\binom{2}{1} + \binom{8}{7}\binom{3}{2} = 192, \\
c_2 &= \binom{8}{5} + \binom{8}{7}\binom{3}{1} = 80, \\
-c_0 &= \binom{8}{7} = 8.
\end{aligned}$$

We see that the leading coefficient c_{n-2} of $f_n(x)$ is equal to 2^{n-1} similarly to the odd n case. Therefore the main equation of the chordal n-gons in the class $C_n(1,\cdots,1),\ n$ even could be written in the form

$$(x-1)f_4(x)f_6(x)\cdots f_{n-2}(x)f_n(x)=0.$$

Finally as an application of (38) and the the Viète's formulæ we derive the identity

$$\cos^2\left(\frac{n}{2}-1\right)\frac{\pi}{n}\cdot\cos^2\left(\frac{n}{2}-2\right)\frac{\pi}{n}\cdot\cos^2\left(\frac{n}{2}-3\right)\frac{\pi}{n}\cdot\ldots\cdot\cos^2\frac{\pi}{n}=\frac{n}{2^{n-1}}.$$

Example 5 Let a_1, a_2, a_3, a_4 be given lengths. The main equation of the chordal quadrangles in $C_4(a_1, a_2, a_3, a_4)$ will be considered. Since

$$\beta_1+\beta_2+\beta_3+\beta_4=(4-2(j+\nu))\frac{\pi}{2},$$

we recognize three different cases; namely $(j,\nu)\in\{(1,0),(1,1),(0,2)\}$. For $j=1,\nu=0$ it is $\beta_1+\beta_2+\beta_3+\beta_4=\pi$. So, by some heavy but straightforward trigonometry we deduce

$$\begin{gathered}((\cos\beta_1\cos\beta_2+\cos\beta_3\cos\beta_4)^2-\sin^2\beta_1\sin^2\beta_2\\ -\sin^2\beta_3\sin^2\beta_4)^2=4\sin^2\beta_1\sin^2\beta_2\sin^2\beta_3\sin^2\beta_4.\end{gathered}$$

Putting $a_j/(2\rho)$ instead of $\cos\beta_j$ and $\sqrt{1-(a_j/(2\rho))^2}$ instead of $\sin\beta_j$, by rationalizing we get the equation in ρ reads as follows

$$R_1\rho^2-Q_1=0, \tag{39}$$

with

$$\begin{aligned} R_1 &= -a_1^4-a_2^4-a_3^4-a_4^4+2(a_1^2a_2^2+a_1^2a_3^2+a_1^2a_4^2+a_2^2a_3^2\\ &+ a_2^2a_4^2+a_3^2a_4^2)+8a_1a_2a_3a_4,\\ Q_1 &= a_1a_2a_3a_4(a_1^2+a_2^2+a_3^2+a_4^2)+a_1^2a_2^2(a_3^2+a_4^2)+a_3^2a_4^2(a_1^2+a_2^2).\end{aligned}$$

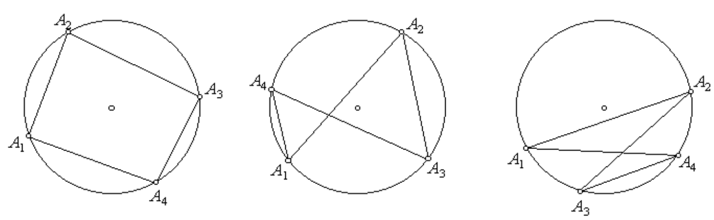

Figure 9. Three possible cases of $C_4(a_1, a_2, a_3, a_4)$ - quadrangles

Similarly for $(j,\nu)\in\{(1,1),(0,2)\}$ we have the main equation in the form

$$R_2\rho^2-Q_2=0, \tag{40}$$

where

$$\begin{aligned} R_2 &= a_1^4+a_2^4+a_3^4+a_4^4-2(a_1^2a_2^2+a_1^2a_3^2+a_1^2a_4^2\\ &+ a_2^2a_3^2+a_2^2a_4^2+a_3^2a_4^2)+8a_1a_2a_3a_4,\\ Q_2 &= a_1a_2a_3a_4(a_1^2+a_2^2+a_3^2+a_4^2)-a_1^2a_2^2(a_3^2+a_4^2)-a_3^2a_4^2(a_1^2+a_2^2).\end{aligned}$$

Let us remark that $R_2=Q_2=0$ when the quadrangle is equilateral. In this case (40) has infinitely many solutions, compare the Figure 10.

The cases $(j,\nu) = (1,1),(0,2)$ are mutually exclusive, cannot be satisfied simultaneously. Therefore just two different circumcircles exists, consult Figure 9 where the second circle realizes in the case $j=1, \nu=1$, while the third circle happens for $j=0,\ \nu=2$.

By $\beta_1+\beta_2+\beta_3+\beta_4 = (2-j-\nu)\pi$ the main equation concerning chordal polygons living in $C_4(a_1,a_2,a_3,a_4)$ we obtain using $\sin(\beta_1+\beta_2+\beta_3+\beta_4)=0$. Repeating the procedure explained about the equation (39), we get

$$(R_1\rho^2 - Q_1)(R_2\rho^2 - Q_2) = 0. \tag{41}$$

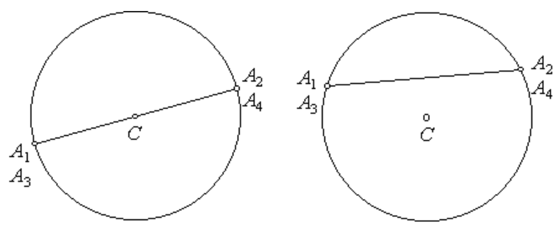

Figure 10. Trivial degenerated $C_4(a,a,a,a)$ - quadrangles

Example 6 Let a_1,a_2,a_3,a_4,a_5 be given. Then

$$\beta_1+\beta_2+\beta_3+\beta_4+\beta_5 = (5-2(j+\nu))\pi/2,$$

so we begin the main equation derivation transforming e.g.

$$\cos(\beta_1+\beta_2+\beta_3) = \pm\sin(\beta_4+\beta_5).$$

After hard, but obvious computation we deduce

$$G\sin\beta_1\sin\beta_2 + H\sin\beta_2\sin\beta_3 + K\sin\beta_3\sin\beta_1 = L \tag{42}$$

where

$$\begin{aligned}
G &= -64a_1^3a_2^3a_3^4x^{10} + 16(4a_1^3a_2^3a_3^2 + 2a_1^3a_2a_3^4 + 2a_1a_2^3a_3^4 - a_1a_2a_3^2a_4^2a_5^2)x^8 \\
&+ 8(-4a_1^3a_2a_3^2 - 4a_1a_2^3a_3^2 - 2a_1a_2a_3^4 - a_1^3a_2^3 + a_1a_2a_3^2a_4^2 + a_1a_2a_3^2a_5^2 \\
&+ a_1a_2a_4^2a_5^2)x^6 + 4(a_1a_2^3 + a_1^3a_2 + 3a_1a_2a_3^2 - a_1a_2a_4^2 - a_1a_2a_5^2)x^4,
\end{aligned}$$

$$\begin{aligned}
H &= -64a_1^4a_2^3a_3^3x^{10} + 16(4a_1^2a_2^3a_3^3 + 2a_1^4a_2a_3^3 + 2a_1^4a_2^3a_3 - a_1^2a_2a_3a_4^2a_5^2)x^8 \\
&+ 8(-4a_1^2a_2a_3^3 - 4a_1^2a_2^3a_3 - 2a_1^4a_2a_3 - a_2^3a_3^3 + a_1^2a_2a_3a_4^2 + a_1^2a_2a_3a_5^2 \\
&+ a_2a_3a_4^2a_5^2)x^6 + 4(a_2a_3^3 + a_2^3a_3 + 3a_1^2a_2a_3 - a_2a_3a_4^2 - a_2a_3a_5^2)x^4,
\end{aligned}$$

$$\begin{aligned}
K &= -64a_1^3a_2^4a_3^3x^{10} + 16(4a_1^3a_2^2a_3^3 + 2a_1a_2^4a_3^3 + 2a_1^3a_2^4a_3 - a_1a_2^2a_3a_4^2a_5^2)x^8 \\
&+ 8(-4a_1a_2^2a_3^3 - 4a_1^3a_2^2a_3 - 2a_1a_2^4a_3 - a_1^3a_3^3 + a_1a_2^2a_3a_4^2 + a_1a_2^2a_3a_5^2 \\
&+ a_1a_3a_4^2a_5^2)x^6 + 4(a_1a_3^3 + a_1^3a_3 + 3a_1a_2^2a_3 - a_1a_3a_4^2 - a_1a_3a_5^2)x^4,
\end{aligned}$$

$$
\begin{array}{rcl}
L & = & -64a_1^4a_2^4a_3^4x^{12} + 16(4a_1^2a_2^4a_3^4 + 4a_1^4a_2^2a_3^4 + 4a_1^4a_2^4a_3^2 - a_1^2a_2^2a_3^2a_4^2a_5^2)x^{10} \\
& + & 8(-8a_1^2a_2^2a_3^4 - 8a_1^2a_2^4a_3^2 - 8a_1^4a_2^2a_3^2 - a_1^4a_2^4 - a_1^4a_3^4 - a_2^4a_3^4 \\
& + & a_1^2a_2^2a_3^2a_4^2 + a_1^2a_2^2a_3^2a_5^2 + a_1^2a_2^2a_4^2a_5^2 + a_1^2a_3^2a_4^2a_5^2 + a_2^2a_3^2a_4^2a_5^2)x^8 \\
& + & 4(2a_1^2a_2^4 + 2a_1^4a_2^2 + 2a_1^2a_3^4 + 2a_1^4a_3^2 + 2a_2^2a_3^4 + 2a_2^4a_3^2 + 15a_1^2a_2^2a_3^2 \\
& - & a_1^2a_2^2a_4^2 - a_1^2a_2^2a_5^2 - a_1^2a_3^2a_4^2 - a_1^2a_3^2a_5^2 - a_1^2a_4^2a_5^2 - a_2^2a_3^2a_4^2 - a_2^2a_3^2a_5^2 \\
& - & a_2^2a_4^2a_5^2 - a_3^2a_4^2a_5^2)x^6 + (-a_1^4 - a_2^4 - a_3^4 - a_4^4 - a_5^2 - 6a_1^2a_2^2 - 6a_2^2a_3^2 \\
& - & 6a_1^2a_3^2 + 2a_1^2a_4^2 + 2a_1^2a_5^2 + 2a_2^2a_4^2 + 2a_2^2a_5^2 + 2a_3^2a_4^2 + 2a_3^2a_5^2 + 2a_4^2a_5^2)x^4,
\end{array}
$$

and the abreviation $x = 1/(2\rho)$ is used.

Now, transforming once more (42), writing $s_j = \sin\beta_j$, we have

$$
4s_1^2s_2^2(HKs_3^2 + GL)^2 = (L^2 + G^2s_1^2s_2^2 - H^2s_2^2s_3^2 - K^2s_1^2s_3^2)^2. \tag{43}
$$

The equation (43) can be written in the form $f(x) = 0$, where f is a polynomial in x. To write this polynomial explicitely we need few pages therefore it is omitted. We shall here restrict ourselves to the use of the form (43) and consider the following special cases.

(i) Let $a_1 = a_2 = a_3 = 1, a_4 = a_5 = \sqrt{2}$. In this case there are pentagons where $\sin\beta_j \sin\beta_k,\ j,k \in \{1,2,3\},\ j \neq k$ are all positive. Then, (43) becomes

$$
256x^8 - 512x^6 + 352x^4 - 92x^2 + 9 = 0, \tag{i}
$$

where $x_{1,2} = \cos\frac{\pi}{3}$ is its double root (compare the first and the second pentagon on the Figure 11). Let us remark that $x_3 = \cos\beta_1 = \frac{\sqrt{7}}{4}$ is not the root of **(i)** but the root of $(x^2-1)G = L$ which becomes $16x^2 - 7 = 0$. Thus $2\rho_3 = \dfrac{4}{\sqrt{7}}$ is the diameter of the circumcircle of the third pentagon. Finally, it is not hard to see that if three consecutive sides of the chordal pentagon have the same lengths, then at most three different circumcircles could arise.

(ii) Let $a_1 = a_2 = a_3 = 1, a_4 = 2, a_5 = 3$. Then there is only one pentagon whose sides have given lengths (Figure 11). Since

$$
\begin{array}{lcl}
G = H = K & = & -64x^{10} - 448x^8 + 304x^6 - 32x^4, \\
L & = & -64x^{12} - 384x^{10} + 752x^8 - 480x^6 + 32x^4,
\end{array}
$$

the equation (43) becomes now

$$
16x^8 - 96x^6 + 188x^4 - 93x^2 + 8 = 0.
$$

we find that $3.0364 < 2\rho < 3.0365$ since

$$
\begin{array}{lcl}
3\arccos\frac{1}{3.0364} + \arccos\frac{2}{3.0364} + \arccos\frac{3}{3.0364} = 269.99^0 & < & \frac{3\pi}{2}, \\
3\arccos\frac{1}{3.0365} + \arccos\frac{2}{3.0365} + \arccos\frac{3}{3.0365} = 270.01^0 & > & \frac{3\pi}{2}.
\end{array}
$$

Let us remark that using the equation $(x^2-1)G = M$ we deduce $x = 0$ or $\rho = \infty$, compare the second pentagon on Figure 11.

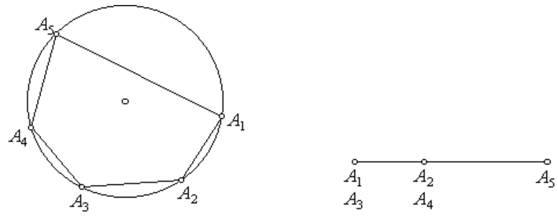

Figure 11. $P(1,1,1,2,3)$ pentagons with $2\rho_1 \approx 3.0364,\ 2\rho_2 = \infty$

(iii) Now, let $a_1 = a_2 = 1, a_3 = 2, a_4 = 3, a_5 = 4$. Then two pentagons are there whose sides have these given lengths, see Figure 12. Since

$$\begin{array}{rcl} G & = & -1024x^{10} - 7936x^8 + 1432x^6 - 44x^4, \\ H & = & -512x^{10} - 3776x^8 + 2288x^6 - 136x^4 = K, \\ L & = & -1024x^{12} - 6912x^{10} + 9368x^8 - 3780x^6 + 179x^4, \end{array}$$

then (43) can be written as

$$\begin{array}{rcl} \pm 2(1-x^2)(HK(1-4x^2) + GL) & = & L^2 + G^2(1-x^2)^2 \\ & - & (H^2+K^2)(1-x^2)(1-4x^2), \end{array}$$

which means

iii $$4(1-x^2)(1-4x^2)H^2 = (L - G(1-x^2))^2, \qquad L + (1-x^2)G = 0,$$

where the first equation in **(iii)** corresponds to the sign $+$, and the second one to sign $-$ in its initial equation.

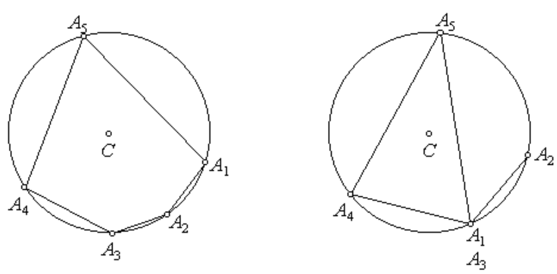

Figure 12. Pentagons $P(1,1,2,3,4)$ with $2\rho_1 \approx 4.12004,\ 2\rho_2 \approx 4.13118$

From the first equation we obtain $4.12004 < 2\rho_1 < 4.12005$, and for the second one it is $2\rho_2 \approx 4.13118$ (it is the diameter of the circumcircle of the the triangle $A_3A_4A_5$ on the Figure 12).

(iv) Let $a_j = j,\ j = \overline{1,5}$. Then there are only two different radii - the second and the third pentagon have equal diameters, compare Figure 13.

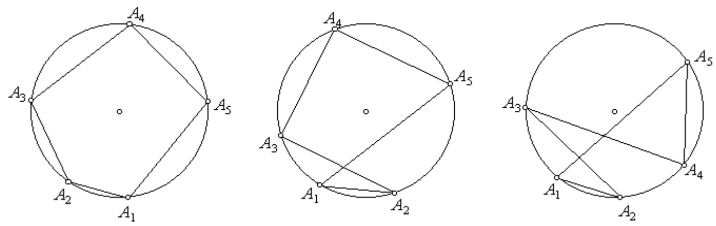

Figure 13. Pentagons $P(1,2,3,4,5)$ with $2\rho_1 \approx 5.43513,\ 2\rho_2 \approx 507762$

The diameters of these are $5.43513 < 2\rho_1 < 5.43514$ and $5.07762 < 2\rho_2 < 5.07763$.

5 On properties of main equation

It is clear that the main equations

21 $$F_1(a_1, \cdots, a_n; \rho) = 0, \qquad n \text{ odd}$$

22 $$F_2(a_1, \cdots, a_n; \rho) = 0, \qquad n \text{ even}$$

have the following properties.

1. The radius of the circumcircle of k - inscribed chordal polygon whose sides have the lengths $a_1, \cdots, a_n$ is a positive root of the equation (21) or (22).

2. The equations (21), (22) depending on $a_1, \cdots, a_n$ may have no one positive root, may have exactly one positive root or may have at least $\boldsymbol{\sigma}[n]$ positive roots if our Conjecture is true.

Here we give an another property of the main equation.

Theorem 3 *The positive roots of the main equations* (21), (22) *are the radii of the circumcircles of the chordal n-gons whose sides have the lengths $a_1, \cdots, a_n$.*

Proof. Let n be odd. We transform $\cos(\beta_1 + \cdots + \beta_n) = 0$ into the form

$$\mathfrak{S}_0^n - \mathfrak{S}_2^n + \mathfrak{S}_4^n - \cdots + (-1)^{\theta_1}\mathfrak{S}_{n-1}^n = 0,$$

where $\theta_1 = (1 + 3 + 5 + \cdots + (n-1)) + 1$, compare (19). Expressing $\sin\beta_1$ from the previous equality, we have

(44) $$U_1 \sin\beta_1 = V_1;$$

no integer ζ_1 exists that $\sin^{2\zeta_1+1}\beta_1$ is a factor of addends in U_1 or V_1 unless $\sin^{2\zeta_1+2}\beta_1$ is a factor there as well.

By squaring (44) we get $U_1^2 \sin^2\beta_1 = V_1^2$. Writing this equality as

(45) $$U_2 \sin\beta_2 = V_2,$$

we deduce that no integer ζ_2 is there that $\sin^{2\zeta_2+1}\beta_2$ is a factor of an addend in U_2 or of V_2 unless $\sin^{2\zeta_2+2}\beta_2$ is a factor as well.

Repeating this procedure we finish with

$$U_n^2 \sin^2 \beta_n = V_n^2. \tag{46}$$

Now we replace

$$\frac{a_j}{2\rho} \longrightarrow \cos\beta_j, \qquad \left(1-\left(\frac{a_j}{2\rho}\right)^2\right)^\zeta \longrightarrow \sin^{2\zeta}\beta_j \tag{47}$$

in all n equalities

$$U_l^2 \sin^2 \beta_l = V_l^2, \qquad l = \overline{1,n}.$$

This iterative procedure ends with (46), which gives us equation (21).

If ρ_m is a positive root of (21), let p be the first integer with the property that ρ_m is a root of algebaric equation obtained by the replacement procedure (47) from

$$U_p^2 \sin^2 \beta_p = V_p^2. \tag{48}$$

Then the following cases arise

- ρ_m is a root of the equation which originates back to $U_p \sin\beta_p = V_p$,
- ρ_m is a root of the equation which originates back to $U_p \sin\beta_p = -V_p$.

In both cases ρ_m is the radius of the circumcircle $\mathbf{C}_\mathbf{A}$ of the chordal n-gon $\mathbf{A}$ with side lengths $a_1, \cdots, a_n$, whose angles are not all of the same sign.

Similar holds for n even.

Remark 2 We point out that

$$U_p^2 \sin^2 \beta_p = V_p^2$$

may generate the main equation of the chordal polygon for $p < n$.

Example 7 So, consider e.g. a equilateral pentagon with unit side lengths. Then

$$\cos(4\beta + (-1)^{\epsilon_j}\beta) = 0, \tag{49}$$

since in this case only one angle may be negative. Transforming (49) we derive the equation

$$64\cos^6\beta - 128\cos^4\beta + 80\cos^2\beta - 15 = 0$$

or

$$15\rho_k^6 - 20\rho_k^4 + 80\rho_k^2 - 1 = 0, \tag{50}$$

where $\rho_k \cos\beta(k) = 1,\ k = 1,2,3$ and

$$\beta(1) = \frac{3\pi}{10},\ \ \beta(2) = \frac{\pi}{10},\ \ \beta(3) = \frac{\pi}{6}.$$

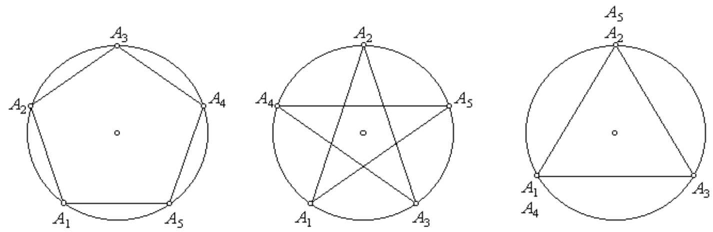

Figure 14. $P(1)$ pentagons with $\rho_1 = \frac{1}{2\cos\frac{3\pi}{10}}$, $\rho_2 = \frac{1}{2\cos\frac{\pi}{10}}$, $\rho_3 = \frac{1}{2\cos\frac{\pi}{6}}$

Adopting the expression (43) in the Example 4 to our case, it becomes

$$256x^8 - 704x^6 + 704x^4 - 300x^2 + 45 = 0, \tag{51}$$

with positive solutions

$$x_1 = \cos\frac{3\pi}{10},\;\; x_2 = \cos\frac{\pi}{10},\;\; x_3 = \cos\frac{\pi}{6}.$$

Thus $2\rho_j x_j = 1,\; j = 1, 2, 3$, see Figure 14.

Also, let us remark that the equation (50) can be rewritten into

$$(5\rho_k^4 - 5\rho_k^2 + 1)(3\rho_k^2 - 1) = 0,$$

and then we have at most

$$\mathbf{s}[5] = \left[\frac{5-1}{2}\right] + \left[\frac{5-3}{2}\right] = 2 + 1 = 3$$

positive roots.

Consider at the end a new kind of n-gons. Denote $\mathbf{A}(1) \equiv \mathbf{A} = A_1 \cdots A_n$ as above, and let $\lambda \geq 2$ positive integer. Then we are interested in the λn-gon

$$\mathbf{A}(\lambda) = \underbrace{\mathbf{A}(1) \cdots \mathbf{A}(1)}_{\lambda},$$

which circumcircle $\mathbf{C}_{\mathbf{A}(\lambda)}$ coincides with $\mathbf{C}_{\mathbf{A}(1)}$. Now, the new class of λn-gons $C_{\lambda n}((a_1, \cdots, a_n)_\lambda)$, say, we build with the polygons $\mathbf{A}(\lambda)$ whose initial side lengths are $a_1, \cdots, a_n$ using it λ times.

Theorem 4 *Let $a_1, \cdots, a_n$ be given lengths and let s and t be positive integers such that $s|t$. Then*

$$F((a_1, \cdots, a_n)_s; \rho) | F((a_1, \cdots, a_n)_t; \rho),$$

where $F((a_1, \cdots, a_n)_s; \rho) = 0$ and $F((a_1, \cdots, a_n)_t; \rho) = 0$ are the main equations of the polygons in $C_{sn}((a_1, \cdots, a_n)_s)$ and $C_{tn}((a_1, \cdots, a_n)_t)$ respectively.

Proof. Firstly it could be say that the assertion of the Theorem is evident, since the polygons $\mathbf{A}(s)$, $\mathbf{A}(t)$ have equal diameters. Of course, it can be deduced the same using the properties of sine and cosine too. Namely, noting $\tau = \beta_1 + \cdots + \beta_n$, we have

$$\sin(s\tau)|\sin q(s\tau) \qquad q \text{ integer,} \tag{52}$$

$$\cos(s\tau)|\cos q(s\tau) \qquad q \text{ odd,} \tag{53}$$

$$\sin(s\tau)|\sin q(s\tau) \qquad q \text{ even.} \tag{54}$$

Starting with $\sin(s\tau) = 0$ we get $F((a_1, \cdots, a_n)_s; \rho) = 0$, and starting with $\sin q(s\tau)/\sin(s\tau)$ we finish with the equation $T((a_1, \cdots, a_n)_q; \rho) = 0$, that

$$F((a_1, \cdots, a_n)_s; \rho)T((a_1, \cdots, a_n)_q; \rho) = F((a_1, \cdots, a_n)_t; \rho).$$

Similarly we get the assertions for (53) and (54).

Example. The circumcircles of triangle $\mathbf{A}(1) = A_1A_2A_3$ and the hexagon $\mathbf{A}(2)$ have the same diameters. Using

$$\beta_1 + \beta_2 + \beta_3 + \beta_1 + \beta_2 + \beta_3 = (6 - 2(j + \nu))\frac{\pi}{2}$$

we have $\sin 2(\beta_1 + \beta_2 + \beta_3) = 0$, i.e. $\sin(\beta_1 + \beta_2 + \beta_3) = 0$, $\cos(\beta_1 + \beta_2 + \beta_3) = 0$. Starting with $\cos(\beta_1 + \beta_2 + \beta_3) = 0$, we obtain the equation $F(a_1, a_2, a_3; \rho) = 0$ and transforming $\sin(\beta_1+\beta_2+\beta_3) = 0$ we finish the computation with $T(a_1, a_2, a_3; \rho) = 0$, so

$$F((a_1, a_2, a_3)_2; \rho) = F(a_1, a_2, a_3; \rho)T(a_1, a_2, a_3; \rho) = 0.$$

The diameter of $\mathbf{C}_{\mathbf{A}(1)}$ and $\mathbf{C}_{\mathbf{A}(2)}$ is a root of the equation $F(a_1, a_2, a_3; \rho) = 0$. The diameters of the circumcircles of all other hexagons (whose sides have prescribed lengths) are the roots of $T(a_1, a_2, a_3; \rho) = 0$. In this case the equations are:

$$\begin{aligned} F(a_1, a_2, a_3; \rho) &= (4a_1^2a_2^2a_3^2x^2 - 2a_1^2a_2^2 - 2a_1^2a_3^2 - 2a_2^2a_3^2 + a_1^4 + a_2^4 + a_3^4)x^4 = 0, \\ T(a_1, a_2, a_3; \rho) &= 4a_1^2a_2^2a_3^2x^6 - (2a_1^2a_2^2 + 2a_1^2a_3^2 + 2a_2^2a_3^2 + a_1^4 + a_2^4 + a_3^4)x^4 \\ &+ 2(a_1^2 + a_2^2 + a_3^2)x^2 - 1 = 0, \end{aligned}$$

where $x = 1/(2\rho)$.

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