Contact transformation of a presymplectic form with Quasi-Sasakian structure

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Abstract

Geometrical and structural properties are proved for manifolds which are structured by the presence of a presymplectic form inducing a quasi-Sasakian structure.

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1 Principal vector fields

Let $M(\phi, \Omega, \xi, \eta, g)$ be a $2m+1$-dimensional Riemannian manifold M, endowed withe the structure tensors $(\phi, \Omega, \xi, \eta)$ consisting of a (1.1)-tensor field, a 2-form, a Reeb vector field and a Reeb covector field respectively, and as usual, g designs the metric tensor. As it is known, these fields satisfy

(1.1) $$\begin{cases} \phi^2 = -\mathrm{Id} + \eta \otimes \xi \,, & \eta(\xi) = 1 \,, \qquad \phi\xi = 0 \,, \\ \Omega(Z\,, Z') = g(\phi Z\,, Z') \,, & \text{and} \qquad \Omega^m \wedge \eta \neq 0 \,. \end{cases}$$

Next the canonical vector valued 1-form of M associated with (1.1) is

(1.2) $$dp = \sum_{A=0}^{2m} \omega^A \otimes e_A \,,$$

which is also called the soldering form of M [2]. Let ∇ be the covariant differential operator defined by the metric tensor. We assume in the sequel that the connection ∇ is symmetric. We recall that under this condition the identity

(1.3) $$d^\nabla(dp) = 0$$

is valid. Consequently,

(1.4) $$\mathcal{O} = \mathrm{vect}\{e_A | A = 0, \cdots 2m\}$$

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means an adapted local field of orthonormal frames over M, and

$$\mathcal{O}^* = \text{covect}\{\omega^A | A = 0, \cdots 2m\} \tag{1.5}$$

its associated coframe.

We also remind that E. Cartan's structure equations can be written as

$$\nabla e_A = \sum_{B=0}^{2m} \theta_A^B \otimes e_B \,, \tag{1.6}$$

$$d\omega^A = -\sum_{B=0}^{2m} \theta_B^A \wedge \omega^B \,, \tag{1.7}$$

$$d\theta_B^A = -\sum_{C=0}^{2m} \theta_B^C \wedge \theta_C^A + \Theta_B^A \,. \tag{1.8}$$

In the above equations θ (respectively Θ) are the local connection forms in the tangent bundle TM (respectively the curvature 2-forms on M).

If we denote by θ_a^b $(a, b \in \{1, \cdots 2m\})$ the horizontal connection forms, then it is well known that they satisfy the Kähler relations

$$\theta_j^i = \theta_{j^*}^{i^*} \,, \qquad \theta_j^{i^*} = \theta_i^{j^*} \,, \qquad i^* = i + m \,. \tag{1.9}$$

Next,

$$\Omega = \sum_{i=1}^{m} \omega^i \wedge \omega^{i^*} \,, \qquad i^* = i + m \,, \tag{1.10}$$

defines the local cosymplectic structure 2-form on M. In the case under consideration, we assume that the Reeb vector field ξ is defined by

$$\nabla \xi = \sum_{i=1}^{m} (\omega^i \otimes e_{i^*} - \omega^{i^*} \otimes e_i) = \phi dp \tag{1.11}$$

and we agree to call the vector field

$$C = \sum_{a=1}^{2m} C^a e_a + C^0 \xi \tag{1.12}$$

the principal vector field on M. By the Levi-Civita connection ∇ one has

$$\nabla C = \sum_{A=0}^{2m} dC^A \otimes e_A + C^A \otimes \nabla e_A \,, \tag{1.13}$$

and we assume that

$$\nabla C = \rho dp + \lambda \nabla \xi = \rho dp + \lambda \phi dp \tag{1.14}$$

where ρ is a scalar (the principal scalar associated with C) and λ is constant.

By (1.10) one derives that

$$i_C\Omega = \sum_{i=1}^{m}(C^i\omega^{i^*} - C^{i^*}\omega^i) \tag{1.15}$$

and by (1.14) one finds that

$$d\Omega = 0\,, \qquad d\eta = 2\Omega\,, \tag{1.16}$$

which shows that Ω is a local presymplectic form (also called a relative Cartan form).

Taking the covariant differential of (1.14) one calculates that

$$\nabla^2 C = (d\rho - \eta) \wedge dp\,, \tag{1.17}$$

which shows that C is an exterior concurrent vector field [4]. Consequently, it follows from the above that

$$d\rho - \eta = -\frac{1}{2m-1} \quad \text{Ric}\, C\,. \tag{1.18}$$

Further, one also has that

$$\phi C = \sum_{i=1}^{m}(C^i\omega^{i^*} - C^{i^*}\omega^i)\,, \tag{1.19}$$

and taking the Lie differential of η with respect to ϕC one derives that

$$\mathcal{L}_{\phi C}\eta = 0\,, \tag{1.20}$$

which shows that ϕC defines a Pfaffian transformation. Next, defining the q-th covariant derivative inductively by

$$\nabla^q Z = d^{\nabla}(\nabla^{q-1}Z)\,,$$

for $Z \in \Xi(M)$, and so one gets from (1.17)

$$\nabla^3 C = -2\Omega \wedge dp\,, \qquad \nabla^4 C = 0\,. \tag{1.21}$$

Consequently, the principal vector field C is 3-exterior concurrent.

2 Distributions generated by fundamental vector fields

In the present section, we discuss various properties of the distributions generated by the vector fields C, ϕC, and ξ. By applying the Lie bracket, one derives that

$$
\begin{aligned}
&(2.22) && [\xi\,,C] &&= \rho\xi - \phi C\,,\\
&(2.23) && [C\,,\phi C] &&= ((C^0)^2 + C^0(1-\lambda))\xi\,,\\
&(2.24) && [\xi\,,\phi C] &&= \nabla_\xi \phi C = C^0\xi - C
\end{aligned}
$$

which shows that $\{\xi\,,C\,,\phi C\}$ defines a 3-foliation and ρ is the principal scalar.

Acting now with the Lie differential, one gets

$$\mathcal{L}_C\Omega = d(\phi C^\flat) \tag{2.25}$$

and consequently

$$d(\mathcal{L}_C\Omega) = 0\,. \tag{2.26}$$

The principal 2-form Ω is therefore relative conformal with respect tot the principal vector field C, and by reference to the definition of te divergence

$$\text{div } Z = \sum_{A=0}^{2m} < \nabla_{e_A} Z\,, e_A >$$

one obtains in the case under consideration that

$$\text{div } C = (2m+1)\rho\,. \tag{2.27}$$

Further, by (1.11) one gets

$$\nabla^2\xi = -\eta \wedge dp\,, \tag{2.28}$$

and also

$$\nabla_C\xi = \sum_{i=1}^{m}(C^i e_{i^*} - C^{i^*} e_i) = \phi C\,. \tag{2.29}$$

Hence, by reference to [5] it follows that the manifold M under consideration is endowed with a quasi Sasakian structure. Operating now consecutively on the vector fields C, ξ, and ϕC, by the operator ∇, one derives

$$\nabla^4 C = 0\,, \qquad \nabla^4\xi = 0\,, \qquad \text{and} \qquad \nabla^4\phi C = 0\,. \tag{2.30}$$

Consequently, we conclude that the triple of vector fields C, ξ, and ϕC defines a 3-distribution.

Let Σ be the exterior differential system which defines the vector field C. By (1.10), (1.13), (2.26), and by reference to [1] one sees that the characteristic numbers of Σ are

$$s_0 = 3\,, \qquad s_1 = 1\,, \qquad \text{and} \qquad r = 4\,.$$

Therefore, since $r = s_0 + s_1$, it is proved that Σ is involutive (in the sense of E. Cartan).

Finally, by Yano's formula [6] one also gets that

(2.31) $$2(\rho^2+\lambda^2)-(2m+1)\ \mathrm{div}\ \rho C=\mathcal{R}(C,C)\,,$$

where $\mathcal{R}$ denotes the Ricci tensor.

Summarizing, we may formulate the following

Theorem 2.1. *Let $M(\phi,\Omega,\xi,\eta,g)$ be a $2m+1$-dimensional Riemannian manifold, carrying a local cosymplectic 2-form, and let C and ξ be the principal vector field and the Reeb vector field on M respectively. One has the following properties:*

(i) *Ω and $\xi^\flat$ ($\xi^\flat=\eta$) are related by*
$$d\eta=2\Omega\,;$$

(ii) *C is an exterior concurrent vector field [4], i.e.*

$$\nabla^2 C=(d\rho-\eta)\wedge dp\,,$$

where ρ is a scalar and dp is the soldering form;

(iii)
$$\mathit{div}\ C=(2m+1)\rho\,;$$

(iv) *the vector fields C, ξ, and ϕC are related by*

$$\nabla_C\xi=\phi C\,,$$

which shows that the manifold M carries a quasi-Sasakian structure;

(v) *the triple C, ξ and ϕC of vector fields define a 3-exterior distribution, i.e.*

$$\nabla^4 C=0\,,\qquad \nabla^4\xi=0\,,\qquad \textit{and}\qquad \nabla^4\phi C=0\,;$$

(vi) *the exterior differential system Σ is in involution (in the sense of E. Cartan).*

3 Lie derivatives and infinitesimal transformations

Consider now the dual form $C^\flat$ of C, i.e.

(3.32) $$C^\flat=\sum_{A=0}^{2m}C^A\omega^A\,.$$

By exterior differentiation one derives

(3.33)
$$d(\phi C^\flat)=-\sum_{i=1}^{m}\left(dC^{i^*}+\sum_{a=1}^{2m}C^a\theta_a^{i^*}\right)\wedge\omega^i+\sum_{i=1}^{m}\left(dC^{i}+\sum_{a=1}^{2m}C^a\theta_a^{i}\right)\wedge\omega^{i^*}+\eta\wedge C^\flat\,,$$

and one obtains

(3.34) $$d\left(\phi C^\flat\right)=-2\rho\Omega\,.$$

Then taking the Lie differential of η by ϕC yields

$$\mathcal{L}_{\phi C^\flat}\eta = 0\,. \tag{3.35}$$

This shows that the vector field ϕC defines an infinitesimal Pfaffian transformation of η [3]. In a similar way one calculates that

$$d\,(\mathcal{L}_C\eta) = d\rho \wedge \eta + 2\rho\Omega\,, \tag{3.36}$$

which shows that C is an infinitesimal quasi-conformal transformation of η. Further, since one may verify that

$$i_\xi \phi C^\flat = 0\,, \tag{3.37}$$

one can say that ϕC is a semi-basic vector field.

Finally, one also gets that

$$\mathcal{L}_{\rho C}\Omega = \rho\mathcal{L}_C\Omega + d\rho \wedge (\phi C)^\flat\,, \tag{3.38}$$

and since Ω is a closed 2-form, then all vector fields Z such that

$$i_Z\Omega = 0,\ , \qquad \mathcal{E}_Z = \{Z \in \Xi(M)\,, i_Z\Omega = 0\}$$

form a Lie algebra and M receives a foliation.

Summarizing, we may formulate the following

Theorem 3.1. *Let $C^\flat$ be the dual form of the principal vector field C on M. Then we have proved the following properties:*

(i) *The vector field ϕC defines a Pfaffian transformation of the Reeb covector $\eta = \xi^\flat$, i.e.*

$$\mathcal{L}_{\phi C}\eta = 0\,;$$

(ii)

$$d\,(\mathcal{L}_C\eta) = d\rho \wedge \eta + 2\rho\Omega\,,$$

i.e. C is an infinitesimal quasi-conformal transformation of η;

(iii)

$$\mathcal{L}_{\rho C}\Omega = \rho\mathcal{L}_C\Omega + d\rho \wedge (\phi C)^\flat\,,$$

i.e. all $Z \in \Xi(M)$ such that $i_Z\Omega = 0$, form a Lie algebra and M receives a foliation.

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