Einstein-Yang Mills equations for gauge transformations of second order

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Abstract

Lately a big attention has been paid to the gauge transformations and their applications. The gauge theory of second order was studied by Gh. Munteanu in [16], [17] , [18]. In [7] and [8] some generalizations are given. Fundamental results in gauge theory can be found in [1], [2] etc. The transformations of type (1.1) were studied in [9], [10], [17], [18]. Some other types of transformation with more variables and their applications were studied in [3], [9], [13], [14], [15], [16].

Here in the tangent space TF such an adapted basis is constructed, that the horizontal and the two vertical distributions with respect to the given metric structure are mutualy orthogonal. The torsion free generalized connection is determined and its coefficients are obtained under condition that the metric structure is parallel or recurrent. The Einstein-Yang Mills equations are also given.

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1. Adapted basis in TF

Let F be an $n+m+l$ dimensional C^∞ manifold. Some point $u \in F$ has coordinates (x^i, y^a, z^p) and the allowable coordinate transformations are given by the equations

$$\begin{array}{ll} x^{i'} = x^{i'}(x) & i,j,h,k = 1,\ldots,n, \\ y^{a'} = y^{a'}(x,y) & a,b,c,d,e = n+1,\ldots,n+m, \\ z^{p'} = z^{p'}(x,z) & p,q,r,s,t = n+m+1,\ldots,n+m+l, \end{array} \tag{1.1}$$

where

$$\operatorname{rank}\left[\frac{\partial x^{i'}}{\partial x^i}\right] = n,\ \operatorname{rank}\left[\frac{\partial y^{a'}}{\partial y^a}\right] = m,\ \operatorname{rank}\left[\frac{\partial z^{p'}}{\partial z^p}\right] = l.$$

Proposition 1.1. *The coordinate transformations of type (1.1) form a pseudo group.*

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If the functions $\mathcal{N}_{i'}^{b'}(x',y')$ and $\mathcal{M}_{i'}^{p'}(x',z')$ satisfy the following law of transformation ([18]):

$$\mathcal{N}_i^b(x,y) = \mathcal{N}_{i'}^{b'}(x',y')\frac{\partial x^{i'}}{\partial x^i}\,\frac{\partial y^b}{\partial y^{b'}} + \frac{\partial y^{a'}}{\partial x^i}\,\frac{\partial y^b}{\partial y^{a'}}, \tag{1.2}$$

$$\mathcal{M}_i^p(x,z) = \mathcal{M}_{i'}^{p'}(x',z')\frac{\partial x^{i'}}{\partial x^i}\,\frac{\partial z^p}{\partial z^{p'}} + \frac{\partial z^{p'}}{\partial x^i}\,\frac{\partial z^p}{\partial z^{p'}}, \tag{1.3}$$

then the adapted basis of TF is $B(\mathcal{N},\mathcal{M}) = \left\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^a}, \frac{\partial}{\partial z^p}\right\}$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \mathcal{N}_i^b(x,y)\frac{\partial}{\partial y^b} - \mathcal{M}_i^p(x,z)\frac{\partial}{\partial z^p}. \tag{1.4}$$

Let us denote by T_HF, $T_{V_1}F$, $T_{V_2}F$ the subspaces of TF spanned by $\{\frac{\delta}{\delta x^i}\}$, $\{\frac{\partial}{\partial y^a}\}$, $\{\frac{\partial}{\partial z^p}\}$ respectively, then

$$TF = T_HF \oplus T_{V_1}F \oplus T_{V_2}F.$$

Theorem 1.1. *The horizontal distribution T_HF is integrable if*

$$\mathcal{N}_{i\ j}^{\ c} = \left(\frac{\partial \mathcal{N}_i^c}{\partial x^j} - \mathcal{N}_j^b\frac{\partial \mathcal{N}_i^c}{\partial y^b}\right) - (i,j) = 0, \tag{1.5}$$

$$\mathcal{M}_{i\ j}^{\ q} = \left(\frac{\partial \mathcal{M}_i^q}{\partial x^j} - \mathcal{M}_j^p\frac{\partial \mathcal{M}_i^q}{\partial z^p}\right) - (i,j) = 0, \tag{1.6}$$

and (i,j) is the expression in the previous bracket, in which i and j change their places. T_{V_1} and T_{V_2} are integrable distributions.

Proof. By direct calculation using the abbreviations $\delta_i = \frac{\delta}{\delta x^i}$, $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_a = \frac{\partial}{\partial y^a}$, $\partial_p = \frac{\partial}{\partial z^p}$, we get

$$[\delta_i,\delta_j] = \mathcal{N}_{i\ j}^{\ c}\partial_c + \mathcal{M}_{i\ j}^{\ q}\partial_q \tag{1.7}$$

and this vector is in T_HF only if $\mathcal{N}_{i\ j}^{\ c} = 0$ and $\mathcal{M}_{i\ j}^{\ q} = 0$. On the other side it is obvious, that

$$[\partial_a,\partial_b] = 0,\ [\partial_p,\partial_q] = 0.$$

Putting

$$\delta y^a = dy^a + \mathcal{N}_i^a(x,y)dx^i, \tag{1.8}$$

$$\delta z^p = dz^p + \mathcal{M}_i^p(x,z)dx^i, \tag{1.9}$$

the adapted basis $B^*(\mathcal{N},\mathcal{M}) = \{dx^i, \delta y^a, \delta z^p\}$ of T^*F is formed.

There are so many adapted basis $B(\mathcal{N},\mathcal{M})$ and $B^*(\mathcal{N},\mathcal{M})$ as many solutions have the equations (1.2) and (1.3). In the next section we shall determine such $\mathcal{N}$ and $\mathcal{M}$ ((2.9)), that T_HF, $T_{V_1}F$ and $T_{V_2}F$ are mutually orthogonal subbundles with respect to the given metric G.

Some d-tensor gauge T on F in the bases B and B^* is expressed in the form:

$$T = T^{\ldots i\ \ldots a\ \ldots r\ \ldots}_{\ldots j\ldots\ b\ldots\ s\ldots}\frac{\delta}{\delta x^i}\otimes dx^j\cdots\frac{\partial}{\partial y^a}\otimes\delta y^b\cdots\frac{\partial}{\partial z^r}\otimes\delta z^s\cdots$$

The components of d-tensor gauge T, with respect to the coordinate transformations (1.1) are transformed in the following way:

$$T^{\dots i' \dots a' \dots r' \dots}_{\dots j' \dots b' \dots s' \dots} = T^{\dots i \dots a \dots r \dots}_{\dots j \dots b \dots s \dots} \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^j}{\partial x^{j'}} \cdots \frac{\partial y^{a'}}{\partial y^a} \frac{\partial y^b}{\partial y^{b'}} \cdots \frac{\partial z^{r'}}{\partial z^r} \frac{\partial z^s}{\partial z^{s'}} \cdots$$

Theorem 1.2. *The adapted bases B and B^* are dual to each other.*

2. Orthogonality of the subspaces of TF

The metric tensor G in F is a symmetric, positive definite tensor of type $(0,2)$. In the natural basis of T^*F, $\bar{B}^* = \{dx^i, dy^a, dz^p\}$, G has the form:

$$\begin{aligned} G &= \bar{g}_{ij} dx^i \otimes dx^j + \bar{g}_{ib} dx^i \otimes dy^b + \bar{g}_{iq} dx^i \otimes dz^q + \\ &\quad \bar{g}_{aj} dy^a \otimes dx^j + \bar{g}_{ab} dy^a \otimes dy^b + \bar{g}_{aq} dy^a \otimes dz^q + \\ &\quad \bar{g}_{pj} dz^p \otimes dx^j + \bar{g}_{pb} dz^p \otimes dy^b + \bar{g}_{pq} dz^p \otimes dz^q. \end{aligned} \tag{2.1}$$

In the adapted basis $B^* = \{dx^i, \delta y^a, \delta y^b\}$ of T^*F the metric tensor G has the following components:

$$\begin{aligned} G &= g_{ij} dx^i \otimes dx^j + g_{ib} dx^i \otimes \delta y^b + g_{iq} dx^i \otimes \delta z^q + \\ &\quad g_{aj} \delta y^a \otimes dx^j + g_{ab} \delta y^a \otimes \delta y^b + g_{aq} \delta y^a \otimes \delta z^q + \\ &\quad g_{pj} \delta z^p \otimes dx^j + g_{pb} \delta z^p \otimes \delta y^b + g_{pq} \delta z^p \otimes \delta z^q. \end{aligned} \tag{2.2}$$

Proposition 2.1. *The components of the metric tensor G expressed in the bases $\bar{B}^*$ and B^* are connected by formulae:*

$$\begin{aligned} g_{ij} &= \bar{g}_{ij} - \bar{g}_{cj}\mathcal{N}^c_i - \bar{g}_{ic}\mathcal{N}^c_j - \bar{g}_{rj}\mathcal{M}^r_i - \bar{g}_{ir}\mathcal{M}^r_j + \\ &\quad \bar{g}_{ab}\mathcal{N}^a_i\mathcal{N}^b_j + \bar{g}_{aq}\mathcal{N}^a_i\mathcal{M}^q_j + \bar{g}_{pb}\mathcal{M}^p_i\mathcal{N}^b_j + \bar{g}_{pq}\mathcal{M}^p_i\mathcal{M}^q_j, \\ g_{ib} &= \bar{g}_{ib} - \bar{g}_{ab}\mathcal{N}^a_i - \bar{g}_{pb}\mathcal{M}^p_i \\ g_{aj} &= \bar{g}_{aj} - \bar{g}_{ab}\mathcal{N}^b_j - \bar{g}_{aq}\mathcal{M}^q_j \\ g_{iq} &= \bar{g}_{iq} - \bar{g}_{aq}\mathcal{N}^a_i - \bar{g}_{pq}\mathcal{M}^p_i \\ g_{pj} &= \bar{g}_{pj} - \bar{g}_{pb}\mathcal{N}^b_j - \bar{g}_{pq}\mathcal{M}^q_r \end{aligned} \tag{2.3}$$

$$g_{ab} = \bar{g}_{ab},\ g_{aq} = \bar{g}_{aq},\ g_{pb} = \bar{g}_{pb},\ g_{pq} = \bar{g}_{pq}. \tag{2.4}$$

The proof follows from (1.8), (1.9), (2.1) and (2.2).

Proposition 2.2. *If T_HF, $T_{V_1}F$ and $T_{V_2}F$ are mutually orthogonal spaces with respect to the metric tensor G, then (2.3) has the form:*

$$\begin{aligned} g_{ij} &= \bar{g}_{ij} - \bar{g}_{cj}\mathcal{N}^c_i - \bar{g}_{ic}\mathcal{N}^c_j - \bar{g}_{rj}\mathcal{M}^r_i - \bar{g}_{ir}\mathcal{M}^r_j + \\ &\quad g_{pq}\mathcal{M}^p_i\mathcal{M}^q_j + g_{ab}\mathcal{N}^a_i\mathcal{N}^b_j, \end{aligned} \tag{2.5}$$

$$0 = \bar{g}_{ib} - g_{ab}\mathcal{N}^a_i,\ 0 = \bar{g}_{aj} - g_{ab}\mathcal{N}^b_j, \tag{2.6}$$

$$0 = \bar{g}_{iq} - g_{pq}\mathcal{M}^p_i,\ 0 = \bar{g}_{pj} - g_{pq}\mathcal{M}^q_j. \tag{2.7}$$

Proof. FromFrom the orthogonallity of T_HF, $T_{V_1}F$ and $T_{V_2}F$ with respect to the metric tensor G, follows that in (2.2) we have:

$$g_{ib}=0,\ g_{iq}=0,\ g_{aj}=0,\ g_{aq}=0,\ g_{pj}=0,\ g_{pb}=0. \tag{2.8}$$

Substituting (2.4) and (2.8) into (2.3) we obtain (2.5), (2.6) and (2.7).

Theorem 2.1. *If T_HF, $T_{V_1}F$ and $T_{V_2}F$ are mutually orthogonal with respect to the metric tensor G given by (2.1), then:*

$$\mathcal{N}_i^c=\bar{g}_{ib}g^{bc},\ \mathcal{M}_j^r=\bar{g}_{pj}g^{pr}, \tag{2.9}$$

where (g^{bc}) and (g^{pr}) are the inverse matrices of (g_{ab}) and (g_{rs}) respectively. If T_{V_1} is orthogonal to T_{V_2} with respect to G and (2.9) are satisfied, then T_H is orthogonal to T_{V_1} and to T_{V_2}.

Proof. The first assertion follows from (2.6) and (2.7). The existence of inverse matrices follows from the fact, that G is positive definite. From the symmetry of the metric tensor and (2.9) follow (2.6) and (2.7). From the orthogonality of T_{V_1} and T_{V_2} we have $g_{aq}=\bar{g}_{aq}=0$, $g_{pb}=\bar{g}_{pb}=0$. From these relations, (2.6) and (2.9), using (2.3), we obtain $g_{ib}=0$, $g_{aj}=0$, $g_{iq}=0$, $g_{pj}=0$.

Proposition 2.3. *The nonlinear connections $\mathcal{N}_i^c$ and $\mathcal{M}_j^r$ determined by (2.9) satisfy the transformation laws (1.2) and (1.3).*

Proof. Using the relations

$$dx^{i'}=\frac{\partial x^{i'}}{\partial x^i}dx^i,\ dy^{a'}=\frac{\partial y^{a'}}{\partial x^i}dx^i+\frac{\partial y^{a'}}{\partial y^a}dy^a,$$
$$dz^{p'}=\frac{\partial z^{p'}}{\partial x^i}dx^i+\frac{\partial z^{p'}}{\partial z^p}dz^p$$

and the expression similar to (2.1) for the metric tensor G in $\bar{B}^{*'}=\{dx^{i'},dy^{a'},dz^{p'}\}$ we get

$$\bar{g}_{ib}=\bar{g}_{i'b'}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial y^{b'}}{\partial y^b}+\bar{g}_{a'b'}\frac{\partial y^{a'}}{\partial x^i}\frac{\partial y^{b'}}{\partial y^b}+\bar{g}_{p'b'}\frac{\partial z^{p'}}{\partial x^i}\frac{\partial y^{b'}}{\partial y^b} \tag{2.10}$$

$$\bar{g}_{ab}=g_{a'b'}\frac{\partial y^{a'}}{\partial y^a}\frac{\partial y^{b'}}{\partial y^b}. \tag{2.11}$$

From

$$g_{ib}=g_{i'b'}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial y^{b'}}{\partial y^b}\text{ and }g_{ib}=\bar{g}_{ib}-\bar{g}_{ab}\mathcal{N}_i^a$$

follows

$$\bar{g}_{ib}-\bar{g}_{ab}N_i^a=(\bar{g}_{i'b'}-\bar{g}_{a'b'}N_{i'}^{a'})\frac{\partial x^{i'}}{\partial x^i}\frac{\partial y^{b'}}{\partial y^b}. \tag{2.12}$$

Subsituting (2.10) and (2.11) into (2.12) and using the fact that T_{V_1} is orthogonal to T_{V_2}, i.e. $g_{pb}=g_{p'b'}=0=\bar{g}_{p'b'}$ we get

$$g_{a'b'}\frac{\partial y^{a'}}{\partial x^i}\frac{\partial y^{b'}}{\partial y^b}-g_{a'b'}\frac{\partial y^{a'}}{\partial y^a}\frac{\partial y^{b'}}{\partial y^b}N_i^a= \tag{2.13}$$
$$-g_{a'b'}N_{i'}^{a'}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial y^{b'}}{\partial y^b}.$$

If we multiply (2.13) with $\frac{\partial y^b}{\partial y^{c'}} g^{c'd'} \frac{\partial y^a}{\partial y^{d'}}$ we obtain (1.2). In the similar way we can obtain (1.3).

Proposition 2.4. *If the adapted basis $B = \{\delta_i, \partial_a, \partial_p\}$ is formed by $\mathcal{N}_i^c$ and $\mathcal{M}_j^r$ determined by (2.9), then the horizontal distribution $T_H F$ is integrable iff:*

$$\mathcal{N}_i{}^c_j = [\partial_j \bar{g}_{ia} g^{ac}) - \bar{g}_{jd} g^{db} \partial_b (\bar{g}_{ia} g^{ac})] - [i,j] = 0$$
$$\mathcal{M}_i{}^q_j = [\partial_j (\bar{g}_{ip} g^{pq}) - \bar{g}_{jp} g^{pr} \partial_r (\bar{g}_{is} g^{sq})] - [i,j] = 0.$$

The proof follows from Theorem 1.1 and (2.9).

The connection between adapted basis and metric structure for the generalized Finsler space was studied in [5], [6].

3. Gauge covariant derivatives of the second order

We shall suppose that on F the metric tensor G is given by (2.1).

If we form the adapted basis B^* using the nonlinear connection coefficients $\mathcal{N}_i^c$ and $\mathcal{M}_j^r$ determined by (2.9), as functions of the metric tensor G and suppose that T_{V_1} is orthogonal to T_{V_2}, then according to Theorem 2.1 it follows that $T_H F$, $T_{V_1} F$, $T_{V_2} F$ are mutually orthogonal subbundles with respect to G and in this basis the metric tensor instead of (2.2) has the form:

$$G = g_{ij} dx^i \otimes dx^j + g_{ab} \delta y^a \otimes \delta y^b + g_{pq} \delta z^p \otimes \delta z^q. \tag{3.1}$$

From now on we shall always choose such adapted bases B and B^* in which $\mathcal{N}_i^c$ and $\mathcal{M}_j^r$ are determined by (2.9).

Definition 3.1. *Let $\nabla : TF \times TF \to TF$ ($\times$ is the Descarte's product) be a usual linear connection, such that $\nabla : (X, Y) \to \nabla_X Y \in TF$, $\forall X, Y \in TF$. The operator ∇ is called generalized gauge connection of the second order.*

A generalized gauge connection ∇ of the second order locally is expressed by

$$\nabla_{\partial_\alpha} \partial_\beta = F_\beta{}^\gamma_\alpha \partial_\gamma, \tag{3.2}$$

where $\alpha, \beta, \gamma, \ldots = 1, \ldots, n+m+l$ and ∂_α are elements of the basis B.

It is called d-gauge connection of second order if $\nabla_X Y$ is in $T_H F$, $T_{V_1} F$ or $T_{V_2} F$ if Y is in $T_H F$, $T_{V_1} F$ or $T_{V_2} F$ respectively, $\forall X \in TF$. It has been studied by many authors, mostly romanian geometers ([2], [16], [17], [18]). The generalized connection in K-Hamilton spaces and in dual vector bundles were studied in [3] and [4].

Theorem 3.1. *If the vector fields X, Y expressed in B have the form*

$$X = X^\alpha \partial_\alpha = X^i \delta_i + X^a \partial_a + X^p \partial_p,$$
$$Y = Y^\beta \partial_\beta = Y^j \delta_j + Y^b \partial_b + Y^q \partial_q,$$

then

$$\nabla_Y X = X^\alpha_{|\beta} Y^\beta \partial_\alpha, \tag{3.3}$$

where

$$X^\alpha_{|\beta} = \partial_\beta X^\alpha + F_\gamma{}^\alpha_\beta X^\gamma = \partial_\beta X^\alpha + F_i{}^\alpha_\beta X^i + F_a{}^\alpha_\beta X^a + F_p{}^\alpha_\beta X^p. \tag{3.4}$$

Theorem 3.2. *The covariant derivatives are transformed as tensors if all connection coefficients are transformed as tensors except*

$$(a)\quad F_{j\,i}^{\ k} = F_{j'\ i'}^{\ k'}\frac{\partial x^{i'}}{\partial x^i}\,\frac{\partial x^k}{\partial x^{k'}}\,\frac{\partial x^{j'}}{\partial x^j} + \frac{\partial^2 x^{k'}}{\partial x^i \partial x^j}\,\frac{\partial x^k}{\partial x^{k'}}$$

$$(b)\quad F_{b\,i}^{\ c} = F_{b'\ i'}^{\ c'}\frac{\partial x^{i'}}{\partial x^i}\,\frac{\partial y^{b'}}{\partial y^b}\,\frac{\partial y^c}{\partial y^{c'}} + \frac{\partial^2 y^{c'}}{\partial x^i \partial y^b}\,\frac{\partial y^c}{\partial y^{c'}} - \mathcal{N}_i^a\frac{\partial^2 y^{c'}}{\partial y^b \partial y^a}\,\frac{\partial y^c}{\partial y^{c'}}$$

$$(c)\quad F_{q\,i}^{\ r} = F_{q'\ i'}^{\ r'}\frac{\partial x^{i'}}{\partial x^i}\,\frac{\partial z^{q'}}{\partial z^q}\,\frac{\partial z^r}{\partial z^{r'}} + \frac{\partial^2 z^{r'}}{\partial x^i \partial z^q}\,\frac{\partial z^r}{\partial z^{r'}} - \mathcal{M}_i^s\frac{\partial^2 z^{r'}}{\partial z^s \partial z^q}\,\frac{\partial z^r}{\partial z^{r'}}$$

$$(d)\quad F_{b\,a}^{\ c} = F_{b'\ a'}^{\ c'}\frac{\partial y^{b'}}{\partial y^b}\,\frac{\partial y^{a'}}{\partial y^a}\,\frac{\partial y^c}{\partial y^{c'}} + \frac{\partial^2 y^{c'}}{\partial y^a \partial y^b}\,\frac{\partial y^c}{\partial y^{c'}}$$

$$(e)\quad F_{q\,p}^{\ r} = F_{q'\ p'}^{\ r'}\frac{\partial z^{q'}}{\partial z^q}\,\frac{\partial z^{p'}}{\partial z^p}\,\frac{\partial z^r}{\partial z^{r'}} + \frac{\partial^2 z^{r'}}{\partial z^q \partial z^p}\,\frac{\partial z^r}{\partial z^{r'}}$$

The torsion tensor $T(X,Y)$ is defined in the usual way by:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{3.5}$$

Theorem 3.3. *The torsion tensor for the generalized gauge connection of the second order has the form:*

$$T(X,Y) = T^k\delta_k + T^c\partial_c + T^r\partial_r, \tag{3.6}$$

where

$$\begin{aligned} T^\alpha = \quad & T_{j\,i}^{\ \alpha}\,Y^j X^i + T_{j\,b}^{\ \alpha}\,Y^j X^b + T_{j\,q}^{\ \alpha}\,Y^j X^q + \\ & T_{b\,i}^{\ \alpha}\,Y^b X^i + T_{b\,a}^{\ \alpha}\,Y^b X^a + T_{b\,q}^{\ \alpha}\,Y^b X^q + \\ & T_{p\,i}^{\ \alpha}\,Y^p X^i + T_{p\,b}^{\ \alpha}\,Y^p X^b + T_{p\,q}^{\ \alpha}\,Y^p X^q, \end{aligned} \tag{3.7}$$

where $\alpha = k$ or $\alpha = c$ or $\alpha = r$. The components of the torsion tensor are expressed as the difference of the corresponding connection coefficients for instance

$$T_{j\,i}^{\ k} = F_{j\,i}^{\ k} - F_{i\,j}^{\ k} \quad T_{j\,b}^{\ k} = F_{j\,b}^{\ k} - F_{b\,j}^{\ k}, \ldots$$

except the following

$$\begin{aligned} T_{j\,i}^{\ c} &= F_{j\,i}^{\ c} - F_{i\,j}^{\ c} - \mathcal{N}_{i\,j}^{\ c} \\ T_{j\,b}^{\ c} &= F_{j\,b}^{\ c} - F_{b\,j}^{\ c} + (\partial_b \mathcal{N}_j^c) \\ T_{b\,i}^{\ c} &= F_{b\,i}^{\ c} - F_{i\,b}^{\ c} - (\partial_b \mathcal{N}_i^c) \end{aligned}$$

$$\begin{aligned} T_{j\,i}^{\ r} &= F_{j\,i}^{\ r} - F_{i\,j}^{\ r} - \mathcal{M}_{i\,j}^{\ r} \\ T_{j\,q}^{\ r} &= F_{j\,q}^{\ r} - F_{q\,j}^{\ r} + \partial_q \mathcal{M}_j^r \\ T_{p\,i}^{\ r} &= F_{p\,i}^{\ r} - F_{i\,p}^{\ r} - \partial_p \mathcal{M}_i^r \end{aligned}$$

Theorem 3.4. *The generalized gauge connection of the second order is torsion free if all connection coefficients are symmetric in the lower indices except the following:*

$$
\begin{aligned}
&(a)\quad F_j{}^c_{b} = F_b{}^c_{j} - \partial_b \mathcal{N}^c_j, \quad F_j{}^r_{q} = F_q{}^r_{j} - \partial_q \mathcal{M}^r_j \\
&(b)\quad F_j{}^c_{i} = F_i{}^c_{j} + \mathcal{N}_i{}^c_{j} \\
&(c)\quad F_j{}^r_{i} = F_i{}^r_{j} + \mathcal{M}_i{}^r_{j}
\end{aligned} \tag{3.8}
$$

As $\mathcal{N}_i{}^c_{j}$ and $\mathcal{M}_i{}^r_{j}$ are tensors, so all F's appeared in (3.8)(b) and (3.8)(c) are also tensors. When the horizontal distribution is integrable, then $\mathcal{N}_i{}^c_{j} = 0$ and $\mathcal{M}_i{}^r_{j} = 0$ (Theorem 1.1) and then $F_j{}^c_{i} = F_i{}^c_{j}$ and $F_j{}^r_{i} = F_i{}^r_{j}$.

Using (1.2), (1.3) and (b), (c) from Theorem 3.2 it can be proved that $F_j{}^c_{i}$ and $F_j{}^r_{q}$ given by (3.8)(a) are transformed as tensors, as was stated in Theorem 3.2. By the proof the relation

$$
\begin{aligned}
&(\partial_{c'}\partial_{b'}y^c)(\partial_b y^{b'}) + (\partial_b\partial_a y^{b'})(\partial_{c'}y^a)(\partial_{b'}y^c) = \\
&\partial_{c'}[(\partial_{b'}y^c)(\partial_b y^{b'}] = \partial_{c'}\delta^c_b = 0
\end{aligned}
$$

(similar for z) is used.

The proofs of Theorems 3.2, 3.3 and 3.4 are given in [8] and the curvature theory of ∇ is given in [7].

4. Recurrent gauge connection of second order

As before we shall use such adapted basis B^* in which the nonlinear connections are given by (2.9) and the metric tensor G has the form (3.1).

Definition 4.1. *The generalized gauge connection ∇ of second order is recurrent (metric) if*

$$g_{\alpha\beta|\gamma} = w_\gamma g_{\alpha\beta} \quad (g_{\alpha\beta|\gamma} = 0), \tag{4.1}$$

where

$$w = w_h dx^h + w_d \delta y^d + w_s \delta z^s$$

is a 1-form in T^*F and

$$g_{\alpha\beta|\gamma} = \partial_\gamma g_{\alpha\beta} - F_\alpha{}^\kappa_\gamma g_{\kappa\beta} - F_\beta{}^\kappa_\gamma g_{\alpha\kappa}. \tag{4.2}$$

Theorem 4.1. *The connection coefficients of the recurrent gauge connection of the second order are determined by*

$$2F_\alpha{}^\gamma_\beta = g^{\kappa\gamma}(\gamma_{\alpha\kappa\beta} - w_{\alpha\kappa\beta} + \tilde{T}_{\alpha\kappa\beta}), \tag{4.3}$$

where

$$\gamma_{\alpha\kappa\beta} = \partial_\beta g_{\alpha\kappa} + \partial_\alpha g_{\kappa\beta} - \partial_\kappa g_{\alpha\beta} \tag{4.4}$$

$$w_{\alpha\gamma\beta} = w_\alpha g_{\gamma\beta} + w_\beta g_{\alpha\gamma} - w_\gamma g_{\alpha\beta}, \tag{4.5}$$

$$\tilde{T}_{\alpha\gamma\beta} = \bar{T}_{\alpha\ \gamma}^{\ \rho} g_{\rho\beta} + \bar{T}_{\beta\ \gamma}^{\ \rho} g_{\rho\alpha} + \bar{T}_{\alpha\ \beta}^{\ \rho} g_{\rho\gamma}, \tag{4.6}$$

$$\bar{T}_{\alpha\ \gamma}^{\ \rho} = F_{\alpha\ \gamma}^{\ \rho} - F_{\gamma\ \alpha}^{\ \rho}. \tag{4.7}$$

From (3.1) and (4.1) follows

$$g_{aj|\gamma} = 0,\ g_{pj|\gamma} = 0,\ g_{aq|\gamma} = 0, \tag{4.8}$$

where

$$\gamma = i,\ \text{or}\ \gamma = b,\ \text{or}\ \gamma = s.$$

Theorem 4.2. *The connection coefficients of the metric gauge connection of the second order are given by*

$$2F_{\alpha\ \beta}^{\ \gamma} = g^{\kappa\gamma}(\gamma_{\alpha\kappa\beta} + \tilde{T}_{\alpha\gamma\beta}). \tag{4.9}$$

Theorem 4.3. *The connection coefficients of the recurrent torsion free gauge connections of the second order are given by (4.3) in which $\tilde{T}_{\alpha\gamma\beta} = 0$ except when in (4.6)*

$$\begin{array}{lll} \bar{T}_{j\ b}^{\ c} = -\partial_b \mathcal{N}_i^c = -\bar{T}_{b\ j}^{\ c} & , & \bar{T}_{j\ q}^{\ r} = -\partial_q \mathcal{M}_j^r = -\bar{T}_{q\ j}^{\ r} \\ \bar{T}_{j\ i}^{\ c} = \mathcal{N}_{i\ j}^{\ c} & , & \bar{T}_{j\ i}^{\ r} = \mathcal{M}_{i\ j}^{\ r}. \end{array} \tag{4.10}$$

appear.

The proof follows from Theorem 3.4.

If the horizontal distribution $T_H F$ is integrable, then according to the Theorem 1.1 for the torsion free connection (in (4.10)) we have $\bar{T}_{j\ i}^{\ c} = 0$ and $\bar{T}_{j\ i}^{\ r} = 0$.

From (4.3) we can obtain all 3^3 types of connection coefficients. In the following we shall give some explicite expressions for (4.3). In all calculations it is important that such an adapted basis B^* is used in which the metric tensor G is determined by (3.1).

Case 1. For $(\alpha, \beta, \gamma) = (i, j, a)$ we get

$$2F_{i\ j}^{\ a} = g^{ab}(\gamma_{ibj} - \omega_{ibj} + \tilde{T}_{ibj}), \tag{4.11}$$

where

$$\gamma_{ibj} = -\partial_b g_{ij},\ \omega_{ibj} = -\omega_b g_{ij},$$
$$\tilde{T}_{ibj} = \bar{T}_{i\ b}^{\ h}\, g_{hj} + \bar{T}_{j\ b}^{\ h}\, g_{ih} + \bar{T}_{i\ j}^{\ c}\, g_{bc}.$$

Using (4.7) and (4.10) for the torsion free connection the above equation has the form

$$\tilde{T}_{ibj} = \mathcal{N}_{i\ j}^{\ c}\, g_{bc},$$

because in this case

$$\bar{T}_{i\ b}^{\ h} = F_{i\ b}^{\ h} - F_{b\ i}^{\ h} = T_{i\ b}^{\ h} = 0.$$

If the horizontal distribution $T_H F$ is integrable, then $\tilde{T}_{ibj} = 0 (\mathcal{N}_{i\ j}^{\ c} = 0)$. In this case for the torsion free metric connection (4.11) reduces to the form

$$2F_{i\ j}^{\ a} = -g^{ab}\partial_b g_{ij}.$$

Case 2. For $(\alpha, \beta, \gamma) = (i, j, r)$ we get

$$2F_{i\ j}^{\ r} = g^{rp}(\gamma_{ipj} - \omega_{ipj} + \tilde{T}_{ipj}), \tag{4.12}$$

where

$$\gamma_{ipj} = -\partial_p g_{ij},\ \omega_{ipj} = -\omega_p g_{ij},$$
$$\tilde{T}_{ipj} = \bar{T}_{i\ p}^{\ h}\, g_{hj} + \bar{T}_{j\ p}^{\ h} g_{hi} + \bar{T}_{i\ j}^{\ q}\, g_{qp}.$$

For the torsion free connection we have

$$\bar{T}_{i\ p}^{\ h} = 0,\ \bar{T}_{j\ p}^{\ h} = 0,\ \bar{T}_{i\ j}^{\ q} = \mathcal{M}_{i\ j}^{\ q}$$

and if beside the above conditions the horizontal distribution $T_H F$ is integrable ($\mathcal{M}_{i\ j}^{\ q} = 0$), then $\bar{T}_{i\ j}^{\ q} = 0$. In this case (4.12) reduces to

$$2F_{i\ j}^{\ r} = g^{rp}(-\partial_p g_{ij} - \omega_p g_{ij}).$$

For the metric connection in the above equation $\omega_p = 0$.

Case 3. For $(\alpha, \beta, \gamma) = (j, b, c)$ we get

$$2F_{j\ b}^{\ c} = g^{cd}(\gamma_{jdb} - \omega_{jdb} + \tilde{T}_{jdb}), \tag{4.13}$$

where

$$\gamma_{jdb} = \partial_j g_{db},\ \omega_{jdb} = \omega_j g_{db},$$
$$\tilde{T}_{jdb} = \bar{T}_{j\ d}^{\ c} g_{cb} + \bar{T}_{b\ d}^{\ i} g_{ij} + \bar{T}_{j\ b}^{\ c} g_{cd}.$$

For the torsion free connection we have

$$\bar{T}_{b\ d}^{\ i} = 0,\ \bar{T}_{j\ d}^{\ c} = -\partial_d \mathcal{N}_j^c$$

and

$$\tilde{T}_{jdb} = -(\partial_d \mathcal{N}_j^c) g_{cb} - (\partial_b \mathcal{N}_j^c) g_{cd}.$$

For the torsion free metric connection (4.13) has the form:

$$2F_{j\ b}^{\ c} = g^{cd}(\partial_j g_{db} - g_{ab}\partial_d \mathcal{N}_j^a - g_{ad}\partial_b \mathcal{N}_j^a).$$

The other coefficients of the generalized recurrent second order gauge connection ∇ can be obtained in the similar manner.

5. Einstein-Yang Mills equations of the second order

Let $L(x, y, z)$ be a Lagrangian defined on the compact set $\Omega \subset R^{n+m+l}$. As it is a scalar field we have

$$L(x,y,z) = L(x',y',z'). \tag{5.1}$$

We shall suppose, that the adapted basis $B = \{\delta_i, \partial_a, \partial_p\}$ of $T(F)$ and its dual basis $B^* = \{dx^i, \delta y^a, \delta z^p\}$ are choosen in such a way, that $T_H F$, $T_{V_1} F$, $T_{V_2} F$ are mutually orthogonal with respect to G, i.e. when (2.9) is satisfied. The metric tensor in this basis has the form

$$G = g_{ij} dx^i \otimes dx^j + g_{ab} \delta y^a \otimes \delta y^b + g_{pq} \delta z^p \otimes \delta z^{pq}. \tag{5.2}$$

We have

$$g = detG(x,y,z) = |g_{ij}| \cdot |g_{ab}| \cdot |g_{pq}|. \tag{5.3}$$

As

$$g_{ij} = g_{i'j'} \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^{j'}}{\partial x^j}$$

$$g_{ab} = g_{a'b'} \frac{\partial y^{a'}}{\partial y^a} \frac{\partial y^{b'}}{\partial y^b}$$

$$g_{pq} = g_{p'q'} \frac{\partial z^{p'}}{\partial z^p} \frac{\partial z^{q'}}{\partial z^q}$$

and the determinant of the Jacobian matrix is

$$|J| = \left| \frac{D(x',y',z')}{D(x,y,z)} \right| = \left| \frac{\partial x^{i'}}{\partial x^i} \right| \left| \frac{\partial y^{a'}}{\partial y^a} \right| \left| \frac{\partial z^{p'}}{\partial z^p} \right| \tag{5.4}$$

from (5.3) we obtain

$$|G(x,y,z)| = |g_{i'j'}||g_{a'b'}||g_{p'q'}||J|^2 = |G(x',y',z')||J|^2. \tag{5.5}$$

Let us define the Lagrangian density by

$$\mathcal{L}(x,y,z) = L(x,y,z)\sqrt{G(x,y,z)} = \sqrt{g} L(x,y,z). \tag{5.6}$$

The substitution of (5.1) and (5.5) into (5.6) results

$$\mathcal{L}(x,y,z) = L(x',y',z')\sqrt{|G(x',y',z')|}|J| = \mathcal{L}(x',y',z')|J|. \tag{5.7}$$

The elementary volume element $d\omega$ in Ω is

$$d\omega(x,y,z) = dx^1 \wedge \ldots \wedge dx^n \wedge dy^{n+1} \wedge \ldots \wedge dy^{n+m} \wedge dz^{n+m+1} \wedge \ldots \wedge dz^{n+m+l}.$$

It is known, that

$$d\omega(x',y',z') = |J| d\omega(x,y,z). \tag{5.8}$$

Proposition 5.1. *The integral of action*

$$I = \int_\Omega \mathcal{L}(x,y,z) d\omega \tag{5.9}$$

does not depend on coordinate system if and only if $\mathcal{L}(x,y,z)$ satisfies the relation

$$\mathcal{L}(x,y,z) = |J|\mathcal{L}(x',y',z'). \tag{5.10}$$

Proof. I is invariant if

$$\mathcal{L}(x,y,z)d\omega(x,y,z) = \mathcal{L}(x',y',z')d\omega(x',y',z'). \tag{5.11}$$

The substitution of (5.8) into (5.11) results (5.10). The proof in the opposite direction is obvious.

From (5.6) and (5.7) it follows, that one example for $\mathcal{L}$, which gives coordinate invariant integral of action is

$$\mathcal{L}(x,y,z) = \sqrt{g}L(x,y,z).$$

Proposition 5.2. *For arbitrary C^2 function $\mathcal{L}(x,y,z)$ the following relation is valid:*

$$d\mathcal{L} = (\delta_i\mathcal{L})\delta x^i + (\delta_a\mathcal{L})\delta y^a + (\delta_p\mathcal{L})\delta z^p. \tag{5.12}$$

Proof. As $\mathcal{L} = \mathcal{L}(x,y,z)$ we have:

$$d\mathcal{L} = (\partial_i\mathcal{L})dx^i + (\partial_a\mathcal{L})dy^a + (\partial_p\mathcal{L})dz^p. \tag{5.13}$$

From (1.4), (1.8) and (1.9) we have

$$\delta_i = \partial_i - N_i^b\partial_b - M_i^p\partial_p,\ \delta_a = \partial_a,\ \delta_p = \partial_p \tag{5.14}$$

$$\delta x^i = dx^i,\ \delta y^a = dy^a + N_i^a dx^i,\ \delta z^p = dz^p + M_i^p dx^i. \tag{5.15}$$

The substitution of (5.14), (5.15) into (5.13) results (5.12).

We shall suppose that the Lagrangian $L(x,y,z)$ is the function of $\phi^A(x,y,z)$, $\partial_i\phi^A(x,y,z)$, $\partial_a\phi^A(x,y,z)$ and $\partial_p\phi^A(x,y,z)$, where $\phi^A(x,y,z)$ are scalar fields, i.e.

$$\phi^A(x,y,z) = \phi^A(x',y',z') \quad A = 1,2,\ldots,p. \tag{5.16}$$

For the simplification, we shall consider only one function $\phi = \phi(x,y,z)$ and use these abbreviations:

$$\partial_i\phi = \partial_i\phi(x,y,z),\ \partial_a\phi = \partial_a\phi(x,y,z),\ \partial_p\phi = \partial_p\phi(x,y,z). \tag{5.17}$$

Now we have

$$\mathcal{L}(\phi,\partial_i\phi,\partial_a\phi,\partial_p\phi) = \sqrt{g}L(\phi,\partial_i\phi,\partial_a\phi,\partial_p\phi) \tag{5.18}$$

and the integral of action has the form

$$I(\phi) = \int\limits_\Omega \mathcal{L}(\phi,\partial_i\phi,\partial_a\phi,\partial_p\phi)d\omega. \tag{5.19}$$

We are looking for such functions ϕ, for which $I(\phi)$ has maximal or minimal value, i.e. for which $\delta I(\phi) = 0$.

For the simplicity we shall suppose that Ω is the $n+m+l$ dimensional rectangle, such that

$$\int\limits_{\Omega} d\omega = \int\limits_{a^i}^{b^i} dx^i \int\limits_{a^a}^{b^a} dy^a \int\limits_{a^p}^{b^p} dz^p \qquad \begin{array}{l} i = 1, \ldots, n \\ a = n+1, \ldots, n+m \\ l = n+m+1, \ldots, n+m+l \end{array} \tag{5.20}$$

and the variation of ϕ on the boundary of Ω is equal to zero, i.e.

$$\delta\phi(b^i, y, z) = \delta\phi(a^i, y, z) = 0,\ i = 1, 2, \ldots, n \tag{5.21}$$

$$\delta\phi(x, b^a, z) = \delta\phi(x, a^a, z) = 0,\ a = n+1, \ldots, n+m \tag{5.22}$$

$$\delta\phi(x, y, b^p) = \delta\phi(x, y, a^p) = 0,\ p = n+m+1, \ldots, n+m+l \tag{5.23}$$

where for instance

$$\int\limits_{a^i}^{b_i} dx^i = \int\limits_{a^1}^{b^1} dx^1 \int\limits_{a^2}^{b^2} dx^2 \ldots \int\limits_{a^n}^{b^n} dx^n,$$
$$\delta\phi(b^1, y, z) = \delta\phi(b^1, x^2, \ldots, x^n, y, z), \ldots$$
$$\delta\phi(x, y, b^{n+m+l}) = \delta\phi(x, y, z^{n+m+1}, \ldots, b^{n+m+l}).$$

From the variation principle we have

$$\delta I(\phi) = \int\limits_{\Omega} \delta\mathcal{L}(x, y, z) d\omega = \tag{5.24}$$
$$\int\limits_{\Omega} \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_i\phi)}\delta(\partial_i\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\delta(\partial_a\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_p\phi)}\delta(\partial p\phi) \right] d\omega.$$

From (5.19) it can be seen that $\mathcal{L}$ is function of independent variables ϕ, $\partial_i\phi$, $\partial_a\phi$, $\partial_p\phi$. To express this fact we shall write in (5.24)

$$\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)} = \frac{d\mathcal{L}}{d(\partial_i\phi)},\ \frac{\partial\mathcal{L}}{\partial(\partial_a\phi)} = \frac{d\mathcal{L}}{d(\partial_a\phi)},\ \frac{\partial\mathcal{L}}{\partial(\partial_p\phi)} = \frac{d\mathcal{L}}{d(\partial_p\phi)}. \tag{5.25}$$

From (5.14) it follows

$$\mathcal{L}(\phi, \partial_i\phi, \partial_a\phi, \partial_p\phi) = \mathcal{L}(\phi, \delta_i\phi, \partial_a\phi, \partial_p\phi), \tag{5.26}$$

$$\delta_i\phi = \partial_i\phi - N_i^a\partial_a\phi - M_i^p\partial_p\phi, \tag{5.27}$$

so we have

$$\frac{d\mathcal{L}}{d(\partial_i\phi)} = \frac{\partial\mathcal{L}}{\partial(\delta_i\phi)}\frac{\partial(\delta_i\phi)}{\partial(\partial_i\phi)} = \frac{\partial\mathcal{L}}{\partial(\delta_i\phi)} = A, \tag{5.28}$$

$$\frac{d\mathcal{L}}{d(\partial_a\phi)} = \frac{\partial\mathcal{L}}{\partial(\delta_i\phi)}\frac{\partial(\delta_i\phi)}{\partial(\partial_a\phi)} + \frac{\partial\mathcal{L}}{\partial(\partial_a\phi)} = \frac{\partial\mathcal{L}}{\partial(\delta_i\phi)}(-N_i^a) + \frac{\partial\mathcal{L}}{\partial(\partial_a\phi)} = B. \tag{5.29}$$

$$\frac{d\mathcal{L}}{d(\partial_p\phi)} = \frac{\partial\mathcal{L}}{\partial(\delta_i\phi)}\frac{\partial(\delta_i\phi)}{\partial(\partial_p\phi)} + \frac{\partial\mathcal{L}}{\partial(\partial_p\phi)} = \frac{\partial\mathcal{L}}{\partial(\delta_i\phi)}(-M_i^p) + \frac{\partial\mathcal{L}}{\partial(\partial_p\phi)} = C. \tag{5.30}$$

We shall suppose

$$\delta(\partial_i\phi) = \partial_i(\delta\phi), \delta(\partial_a\phi) = \partial_a(\delta\phi), \delta(\partial_p\phi) = \partial_p(\delta\phi). \tag{5.31}$$

From (5.24)-(5.31) we obtain

$$\begin{aligned}\delta\mathcal{L} &= \frac{d\mathcal{L}}{d\phi}\delta\phi + \frac{d\mathcal{L}}{d(\partial_i\phi)}\delta(\partial_i\phi) + \frac{d\mathcal{L}}{d(\partial_a\phi)}\delta(\partial_a\phi) + \frac{d\mathcal{L}}{d(\partial_p\phi)}\delta(\partial_p\phi) = \\ &\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + A\delta(\partial_i\phi) + B\delta(\partial_a\phi) + C\delta(\partial_p\phi) = \\ &\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + A\partial_i(\delta\phi) + B\partial_a(\delta\phi) + C(\partial_p\delta\phi) = \\ &\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \partial_i(A\delta\phi) + \partial_a(B\delta\phi) + \partial_p(C\delta\phi) \\ &-(\partial_i A)\delta\phi - \partial_a(B)\delta\phi - \partial_p(C)\delta\phi.\end{aligned} \tag{5.32}$$

Using (5.20) and (5.21) we have

$$\begin{aligned}&\int_\Omega \partial_i(A\delta\phi)d\omega = \int_{a^a}^{b^a} dy^a \int_{a^p}^{b^p} dz^p \int_{a^i}^{b^i} \partial_i(A\delta\phi)dx^i = \\ &\int_{a^a}^{b^a} dy^a \int_{a^p}^{b^p} dz^p (A\delta\phi)|_{a^i}^{b^i} = 0.\end{aligned} \tag{5.33}$$

In the similar way we obtain

$$\int_\Omega \partial_a(B\delta\phi)d\omega = 0 \quad \int_\Omega \partial_p(C\delta\phi)d\omega = 0. \tag{5.34}$$

From (5.32)-(5.34) it follows

$$\delta I(\phi) = \int_\Omega \delta\mathcal{L}d\omega = \int_\Omega \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_i A - \partial_a B - \partial_p C\right)\delta\phi d\omega. \tag{5.35}$$

The extrem value of integral of action is obtained, when $\delta\mathcal{L} = 0$. If we in (5.35) substitute A, B and C from (5.28), (5.29) and (5.30) we obtain

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi} - \partial_i\left(\frac{\partial\mathcal{L}}{\partial(\delta_i\phi)}\right) - \partial_a\left(\frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\right) - \partial_p\left(\frac{\partial\mathcal{L}}{\partial(\partial_p\phi)}\right) \\ &+N_i^a\partial_a\frac{\partial\mathcal{L}}{\partial(\delta_i\phi)} + \frac{\partial\mathcal{L}}{\partial(\delta_i\phi)}\partial_a N_i^a \\ &+M_i^a\partial_p\left(\frac{\partial\mathcal{L}}{\partial(\delta_i\phi)}\right) + \frac{\partial\mathcal{L}}{\partial(\delta_i\phi)}\partial_p M_i^p.\end{aligned}$$

Theorem 5.1. *The Einstein-Yang Mills equation for the gauge transformation (1.1) are given by*

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}-\delta_i\left(\frac{\partial\mathcal{L}}{\partial(\delta_i\phi)}\right)-\partial_a\left(\frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\right)-\partial_p\left(\frac{\partial\mathcal{L}}{\partial(\partial_p\phi)}\right)\\ &\quad+\frac{\partial\mathcal{L}}{\partial(\delta_i\phi)}(\partial_a N_i^a+\partial_p M_i^p)=0.\end{aligned}\tag{5.36}$$

As from (5.18) $\mathcal{L}=\sqrt{g}L$, and $\sqrt{g}$ is not the function of ϕ, $\partial_i\phi$, $\partial_a\phi$ and $\partial_p\phi$ from (5.36) we get

$$\begin{aligned}&\frac{\partial(\sqrt{g}L)}{\partial\phi}-\delta_i\left(\sqrt{g}\frac{\partial L}{\partial(\delta_i\phi)}\right)-\partial_a\left(\sqrt{g}\frac{\partial L}{\partial(\partial_a\phi)}\right)-\partial_p\left(\sqrt{g}\frac{\partial L}{\partial(\partial_p\phi)}\right)\\ &+\sqrt{g}\frac{\partial L}{\partial(\delta_i\phi)}(\partial_a N_i^a+\partial_p M_i^p)=0.\end{aligned}\tag{5.37}$$

Theorem 5.2. *The Einstein-Yang Mills equation for the gauge transformation (1.1) expressed as function of the Lagrangian L and metric function is given by*

$$\begin{aligned}&\left[\frac{\partial L}{\partial\phi}-\delta_i\left(\frac{\partial L}{\partial(\delta_i\phi)}\right)-\partial_a\left(\frac{\partial L}{\partial(\partial_a\phi)}\right)-\partial_p\left(\frac{\partial L}{\partial(\partial_p\phi)}\right)\right.\\ &\left.+\frac{\partial L}{\partial(\delta_i\phi)}(\partial_a N_i^a+\partial_p M_i^p)\right]-\\ &\frac{1}{\sqrt{g}}\left[\frac{\partial L}{\partial(\delta_i\phi)}\delta_i\sqrt{g}+\frac{\partial L}{\partial(\partial_a\phi)}\partial_a\sqrt{g}+\frac{\partial L}{\partial(\partial_p\phi)}\partial_p\sqrt{g}\right]=0.\end{aligned}\tag{5.38}$$

Proof. The proof follows from (5.37).

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