“Strong” extrema of functionals defined on Riemannian 2-manifolds

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Abstract

Here we generalize the one-dimensional notion of derivative fields in order to get a suitable notion of gradient fields for smooth functionals of the form $\int_V J(x, u, \nabla u) dV$, defined on a -compact- Riemannian 2-manifold V. By the way, we establish a close relationship between such gradient fields and the corresponding Hamilton-Jacobi equation. By relaxing some hypotheses, we have been able to define a proper notion of Hilbert's invariant integral on Riemannian surfaces. The main consequence of this theoretical setting is the generalization of Weierstrass' Theorem stating a sufficient and necessary condition for the existence of extrema in the class of functionals under consideration. Also, we illustrate a way to apply the principal result here to the functional ruling the conformal deformation of the underlying surface.

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1 Introduction

1.1

In what follows, V is a simple compact subset of an oriented Riemannian 2-manifold whose boundary ∂V is smooth enough, simple and oriented. We consider functionals $F : A \longrightarrow \mathbb{R}$, where A is an admissible space, say $A = \{u \in C^1(\mathring{V}) \cap C(\partial V) \mid u \equiv f \text{ en } \partial V\}$, for certain differentiable function f along ∂V. Furthermore, we assume F has the form

$$F(u) = \int_V J(x, u, \nabla u)\, dV, \tag{1.1}$$

where $J : V \times A \times TV \longrightarrow \mathbb{R}$ denotes a smooth enough function of its arguments. It is not hard to see this implies F itself is differentiable. Additionally, we must deal with a Banach space B to which the variations h of F belong, say $C_0^1(V)$.

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1.2

Let $u^* \in A$ be a singular point of F, that is to say,the Euler-Lagrange equation

$$\partial_u J(x, u^*, \nabla u^*)) - div(\partial_{\nabla u} J(x, u^*, \nabla u^*))) = 0$$

holds in $\mathring{V}$. Within the framework of Hamilton-Jacobi theory, we will suppose the subset

$$M = \Big\{ u \in A \mid F(u) = F(u^*),\ u \text{ singular} \Big\}$$

is a topological space connected by arcs and possesses a structure of differentiable manifold. For obvious reasons, B will be often identified to $T_u(M)$.

1.3

In the first place, we will motivate and define the notion of gradient field for the functional F. Then, we shall characterize such a gradient field by means of a set of sufficient and necessary conditions which guarantee its existence. This will lead us to Hamilton-Jacobi equation. Afterwards, we are going to relax the definition of gradient field and show that this delicate procedure can be accomplished with not too much harm to the previous theory. Actually, we obtain a pair of exact differential forms which constitute the ground for the notion of Hilbert's invariant integral. With it, we will be ready to prove our version of Weierstrass' Theorem.

2 Gradient fields

By the methods used in [1] and [2], it is possible to find the Euler-Lagrange equation of F.

Theorem 2.1. *The derivative of F at $u \in A$ with variations $h \in B$ is given by*

$$dF(u)h = \int\limits_V [\partial_u J - div(\partial_{\nabla u} J)] h dV + \int\limits_{\partial V} h < \partial_{\nabla u} J, dl > .$$

Proof. In

$$F(u+h) - F(u) = \int\limits_V [\partial_u J h + < \partial_{\nabla u} J, \nabla h >] dV + o(\|h\|)$$

we integrate by parts the second term to get

$$\int\limits_V < \partial_{\nabla u} J, \nabla h > dV = \int\limits_{\partial V} h < \partial_{\nabla u} J, dl > - \int\limits_V h\ div(\partial_{\nabla u} J) dV.$$

$\square$

The first part of the derivative carries the Euler-Lagrange term

$$\partial_u J - \mathrm{div}(\partial_{\nabla u} J) = 0. \tag{2.1}$$

In this way, the notion of gradient field is stated as follows.

Definition 2.2. A smooth map $\Psi : V \times A \longrightarrow TV$ is a gradient field of (1.1) if it is a gradient field of the differential equation (2.1), i. e., if $\Psi(x, u) = \nabla u$ and

$$\partial_u J(x, u, \Psi(x, u)) - \operatorname{div}(\partial_{\nabla u} J(x, u, \Psi(x, u))) = 0.$$

The idea behind the discovery of the conditions yielding the existence of Ψ is the boundary value problem

$$\begin{aligned} \partial_u J(x, u, \nabla u) - \operatorname{div}(\partial_{\nabla u} J(x, u, \nabla u)) &= 0 \qquad \text{in } \overset{o}{V}; \\ \nabla u &= \psi(x, u) \quad \text{on } \partial V. \end{aligned}$$

In this way,Ψ can be regarded as a extension of ψ on ∂V to the whole surfaces V. We also introduce a "potential" map g of such a field.

Theorem 2.3. *Let $g : \partial V \times A \longrightarrow TV$ be differentiable enough and $G : A \longrightarrow \mathbb{R}$ the functional defined by*

$$G(u) = F(u) - \int\limits_{\partial V} < g, dl > .$$

Then, the derivative of G with variations $h \in B$ is

$$dG(u)h = \int\limits_{V} h[\partial_u J - div(\partial_{\nabla u} J)]dV + \int\limits_{\partial V} h < \partial_{\nabla u} J - \partial_u g, dl > .$$

Proof. It suffices to compute the derivative of the second term.

$$\int\limits_{\partial V} < g(x, u + h), dl > - \int\limits_{\partial V} < g(x, u), dl > = \int\limits_{\partial V} < \partial_u g h, dl > + o(\|h\|).$$

$\square$

To attain a vanishing first derivative, we ought to have at once the following two conditions.

$$\begin{aligned} \partial_u J - \operatorname{div}(\partial_{\nabla u} J) &= 0 \qquad \text{in } \overset{o}{V} \\ \partial_{\nabla u} J - \partial_u g &= 0 \qquad \text{on } \partial V. \end{aligned}$$

The first equation is just but Euler-Lagrange equation and it is sometimes called consistency condition. The second equation suggests to understand g as defined in all V and demand that

$$\partial_{\nabla u} J = \partial_u g \qquad \text{in } V.$$

By analogy with the one-dimensional case, this last equation will be called self-adjointness condition.

Theorem 2.4. *Let V, F, J be as above. Let also J be such that the tensor field $\partial^2_{\nabla u \nabla u} J$ is nonsingular. Assume that $\partial_u J - div(\partial_{\nabla u} J) = 0$ in $\overset{\circ}{V}$ and that there is a smooth map $g : V \times A \longrightarrow TV$ with $\partial_{\nabla u} J = \partial_u g$ in V. Then, there exists a gradient field Ψ for the functional 1.1.*

Proof. It follows immediately from The Implicit Function Theorem. □

Conversely, the existence of a gradient field implies the self-adjoint and consistency conditions. However, we have to add an extra hypothesis, cf. [3].

Corolary 2.5. *Suppose $\partial^2_{\nabla u \nabla u} J$ is non singular and that every $h \in B = T_u(M)$ has no points conjugate to ∂V. Ψ is a gradient field of 1.1 if and only if the consistency and self-adjointness conditions are satisfied.*

Proof. The construction of g runs exactly as in the first theorem of section 4. □

3 Hamilton-Jacobi equation

Once one assumes the selfadjointness condition, the consistency is logically equivalent to the validity of the Hamilton-Jacobi PDE.

Theorem 3.1. *We suppose $\partial^2_{\nabla u \nabla u} J$ is nonsingular and let the selfadjointness condition hold for certain potential map g. Then, the implicitly defined map $\Psi(x, u) = \nabla u$ is consistent if and only if g is a solution of the Hamilton-Jacobi equation*

$$\partial iv g - J + < \Psi, \partial_u g >= 0,$$

where the symbol $\partial iv g$ stands for the partial divergence of g.

Proof. By hypothesis,

$$\partial iv g = J(x, u, \Psi(x, u)) - < \Psi(x, u), \partial_u g(x, u) > .$$

The right-hand side of this equation is usually called Hamiltonian of 1.1. After differentiation with respect to u, we obtain

$$\begin{aligned} \partial_u(\partial iv g) &= \partial_u J + < \partial_{\nabla u} J, \partial_u \Psi > - < \partial_u \Psi, \partial_u g > - < \Psi, \partial^2_{uu} g > \\ &= \partial_u J - < \Psi, \partial^2_{uu} g > . \end{aligned}$$

Also,

$$\partial iv(\partial_{\nabla u} J) = \mathrm{div}(\partial_{\nabla u} J) - < \partial_u(\partial_{\nabla u} J), \Psi >= \mathrm{div}(\partial_{\nabla u} J) - < \partial^2_{uu} g, \Psi > .$$

Hence,

$$\partial_u J - \mathrm{div}(\partial_{\nabla u} J) = 0.$$

The process can be reversed to get Hamilton-Jacobi equation from Euler-Lagrange equation as explain in next section. □

4 Approaching fields by pseudofields

One of the central contributions of Weierstrass and Hilbert has been to drop (momentarily) the condition on the nonsigularity of $\partial^2_{\nabla u \nabla u} J$. So, it is no longer true that $\Psi(x, u) = \nabla u$ (Implicit Function Theorem, cf. [4]). The idea of this relaxation is then to approach the gradient field by means of "pseudofields".

Definition 4.1. A smooth map $\Psi : V \times M \longrightarrow TV$ is a pseudofield of 1.1 if it satisfy (only) the (pseudo-)consistency condition

$$\partial_u J(x, u, \Psi(x, u)) - div(\partial_{\nabla u} J(x, u, \Psi(x, u))) = 0.$$

In this way, every field is a pseudofield. The converse is not always valid.

Pseudofields preserve nice properties of fields. This is achieved with the help of Hamilton-Jacobi Theory, cf. [3].

Theorem 4.2. *Let u^*, M be as above and Ψ be a pseudofield. If for all $u \in M$ the tangent vectors $h \in T_u M$ (no identically zero) do not have conjugate points to ∂V, then the map $g : V \times M \longrightarrow TV$ defined by the line integral*

$$g(x, u) = \int_{u^*}^{u} \partial_{\nabla u} J(x, \upsilon, \Psi(x, \upsilon)) d\upsilon$$

satisfies the (pseudo-)selfadjointness condition

$$\partial_u g(x, u) = \partial_{\nabla u} J(x, u, \Psi(x, u)).$$

Proof. In order to compute $\partial_u g$, we set

$$g(x, u + h) - g(x, u = \int_{u}^{u+h} \partial_{\nabla u} J(x, \upsilon, \Psi(x, \upsilon)) d\upsilon,$$

along a curve in M joining u with $u + h$, properly $h = \exp(h)$. By virtue of the Mean Value Theorem, there is a $\hat{u}$ such that, for small h,

$$g(x, u + h) - g(x, u) = \partial_{\nabla u} J(x, \hat{u}, \Psi(x, \hat{u}))h.$$

By the hypothesis on the conjugate points of h it is now possiblel to pass to the limit and get

$$\partial_u g(x, u) = \partial_{\nabla u} J(x, u, \Psi(x, u)).$$

This very last equation guarantees g is independent of the selected curve and so, it is well-defined. $\square$

There is also a surface integral involving an exact differential of a function depending on a pseudofield.

Theorem 4.3. *Additionally to the assumptions and notations in the previous theorem, we suppose now that Ψ satisfies the (pseudo-)Hamilton - Jacobi equation*

$$\partial ivg(x, u) + H(x, u, \Psi(x, u)) = 0.$$

Then,

$$\int_{\partial V} < g(x, u), \hat{n} dl >= - \int_{V} H(x, u, \Psi(x, u)) dV,$$

in which $H(x, u, \Psi(x, u)) = -J(x, u, \Psi(x, u)) + < \Psi(x, u), \partial_u g(x, u) >$ denotes the Hamiltoniann of 1.1. $\hat{n}$ is the normal unit field along ∂V in the chosen orientation.

Proof. By the Divergence Theorem, cf. [1],

$$\int\limits_V \partial iv_x g(x,u)dV = -\int\limits_V H(x,u,\Psi(x,u))dV = \int\limits_{\partial V} < g(x,u), \hat{n}dl > .$$

□

It is important for what follows that the pseudofields actually approach certain field. To do so, we will call K the solution set of the Ψ accomplishing

$$\partial_u J(x,u,\Psi(u,x)) - div(\partial_{\nabla u} J(x,u,\Psi(u,x)))) = 0,$$

that is, a set of pseudofields. Formally, we need

Definition 4.4. Let u^*, M y K be a before, $u \in M$ is embedded in K if
1. $\partial^2_{\nabla u \nabla u} J$ is nonsingular at u. Being so, there exists $\Psi(x,u) = \nabla u$.
2. Each $h \not\equiv 0 \in T_v M$ does not have conjugate points to ∂V for all v in a neighborhood of u in M.

From now on, we will suppose that if u is embedded in K, we will be able to approach it as close as we wish by a proper choice of a pseudo field $\Psi \in K$.

5 Hilbert's invariant integral

Now, returning to our functional, we know that for a fixed u^* and given u, for all pseudogradient field $\Psi(x,u)$ the integrals $g(x,u)$ and $\int_{\partial V} < g(x,u), \hat{n}dl >$ are independent of the path joining u^* to u. This motivates the following definition.

Definition 5.1. Assume u^*, u, M and Ψ as in Theorems 4.2, 4.3 and Definition 4.4. The Hilbert's invariant integral of Ψ associated to 1.1 is

$$\begin{aligned}
\gamma(u,\Psi) &= g(x,u) - \int\limits_{\partial V} < g(x,u), \hat{n}dl > \\
&= \int\limits_V < \partial_{\nabla u} J(x,u,\Psi(x,u)), \nabla u > dV \\
&+ \int\limits_V [J(x,u,\Psi(x,u)) - < \partial_{\nabla u} J(x,u,\Psi(x,u)), \Psi(x,u) >] dV \\
&= \int\limits_V [J(x,u,\Psi(x,u)) - < \partial_{\nabla u} J(x,u,\Psi(x,u)), \nabla u - \Psi(x,u) >] dV
\end{aligned}$$

Clearly, if Ψ is a gradient field, $\gamma(u,\Psi) = F(u)$. For this reason, we should understand Hilbert's invariant integral like the perturbation of functional F resulting from computing its value in a close enough pseudofield instead of in the actual gradient field. This claim is clarified in the following section.

6 Weierstrass' condition

We keep the previous notations and definitions.

Definition 6.1. Let $F : M \subset A \longrightarrow \mathbb{R}$, be a functional of the form

$$F(u) = \int_V J(x, u, \nabla u)\; dV,$$

with J differentiable enough. The Weierstrass' E map of F is the map $E : V \times M \times TV \times TV \longrightarrow \mathbb{R}$ defined by the expression

$$E(x, u, z, w) = J(x, u, w) - J(x, u, z) - < w - z, \partial_{\nabla u} J(x, u, z) > .$$

This means E is just but Taylor's residue of J, understood as a function of ∇u. An alternative definition would have been

$$E(x, u, z, w) = \frac{1}{2}(w - z)^t [\partial^2_{\nabla u \nabla u} J(x, u, z + \tau(w - z))](w - z),$$

for some $\tau \in (0, 1)$.

The underlying importance of this definition lies in the following result, which states a sufficient condition for the existence of a minimum (maximum).

Theorem 6.2. *Let u^*, M and K be as before and suppose u is embedded in K. If $E(x, u, z, w) \geq 0$ (≤ 0) for all $w \in T_u(M) \cong \mathbb{R}^2$, then F attains a local minimum (maximum) at u.*

Proof. For $\Psi \in K$, we compute the increment

$$\begin{aligned} F(u) - \gamma(u, \Psi) &= \int_V [J(x, u, \nabla u) - J(x, u, \Psi) - < \Psi - \nabla u, \partial_{\nabla u} J(x, u, \Psi) >]\; dV \\ &= \int_V E(x, u, \Psi(x, u), \nabla u)\; dV \geq 0 \quad (\leq 0). \end{aligned}$$

□

Conversely, it can be proved that Weirstrass' condition is also necessary for the existence of the extrema. The idea of the proof is a generalization of the method described in [2].

Theorem 6.3 (Weierstrass). *If 1.1 possesses a minimum (maximum) at $u \in M$, then $E(x, u, \nabla u, w) \geq 0$, for all $w \in T_u(V) \cong \mathbb{R}^2$.*

Example 6.4. The conformal Gauss curvature functional, cf. [5],

$$F(u) = \int_V \left(\frac{1}{2}\langle \nabla u, \nabla u \rangle - \frac{K}{2} e^{2u} + ku \right) dV,$$

whose Euler-Lagrange equation is

$$-div(\nabla u) - Ke^{2u} + k = 0,$$

has Weierstrass' E map

$$\begin{aligned} E(x,u,z,w) &= \frac{1}{2}\langle z,z\rangle - \frac{K}{2}e^{2u} + ku - \frac{1}{2}\langle w,w\rangle + \frac{K}{2}e^{2u} - ku - \langle w-z,z\rangle \\ &= \frac{1}{2}(3\langle z,z\rangle - \langle w,w\rangle) - \langle w,z\rangle. \end{aligned}$$

We notice it is independent from the curvatures K, k.

7 Concluding remark

The focal point for determining the extrema is shifted here from the space of admissible functions A to the space K of pseudofields at the critical point u, that is why these type of extremum is known under the name of "strong" in classical literature, cf. [2].

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