Legendrian foliations on almost $\mathcal{S}$-manifolds

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Abstract

In this paper we extend the concept of Legendrian foliation to almost $\mathcal{S}$-manifolds, generalizing the definition both of Legendrian foliations on contact metric manifolds and of Lagrangian foliations on symplectic manifolds. A classification of this kind of foliations is given and a study of non-degenerate and Riemannian Legendrian foliations on almost $\mathcal{S}$-manifolds is carried out. Moreover we study bi-Legendrian structures and a canonical connection associated to them.

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1 Introduction

Legendrian foliations have been studied for the first time by M. Y. Pang in [8]. He studied these foliations in the context of contact manifolds. More precisely, given a contact manifold (M, η) of dimension $2n+1$, a *Legendrian foliation* on (M, η) is a foliation of M by n-dimensional integral submanifolds of the contact 1-form η. Pang found many interesting properties of Legendrian foliations and, in particular, presented a classification by means of a bilinear, symmetric form Π on the tangent bundle of the foliation. Later on, P. Liebermann (cf. [7]) carried out, with different methods, the study of Legendrian foliations and N. Jayne in [5] extended the work of Pang to contact metric manifolds. In this paper we give a natural generalization of the theory of Legendrian foliations to the context of almost $\mathcal{S}$-manifolds. Our definition generalizes both that one of Legendrian foliations on contact metric manifolds and that one of Lagrangian foliations on symplectic manifolds. Many of the most important results of the standard theory of Legendrian foliations are recovered. For example we generalize the definition of the form Π getting an analogous classification of Legendrian foliations on almost $\mathcal{S}$-manifolds. In particular we investigate those Legendrian foliations such that Π is non-degenerate and those which admit a bundle-like metric, the so-called Riemannian Legendrian foliations. Finally, we introduce the notion of bi-Legendrian structure and examine an important example, involving the so-called "conjugate Legendrian foliation". During the preparation of this paper the author was a guest of the Institute of Mathematics of the Jagiellonian University of Cracow, so

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2 A generalization of the concept of Legendrian foliation

An f-structure on a smooth manifold M is defined by a non-vanishing tensor field ϕ of type $(1,1)$ of constant rank $2n$ which satisfies $\phi^3+\phi=0$. It can be proved that TM splits into two complementary subbundles $Im\,(\phi)$ and $\ker(\phi)$ and the restriction of ϕ to $Im\,(\phi)$ determines an almost complex structure on such subbundle. When $\ker(\phi)$ is parallelizable we say that we have an f-structure with parallelizable kernel and we call it "$f\cdot pk$-structure". In this case there exist a global frame $\{\xi_1,\ldots,\xi_r\}$ for $\ker(\phi)$ and 1-forms $\eta_1,\ldots,\eta_r$, which satisfy $\eta_i(\xi_j)=\delta_{ij}$ and

$$\phi^2=-I+\sum_{i=1}^{r}\eta_i\otimes\xi_i,$$

from which it follows that

$$\phi(\xi_i)=0,\ \eta_i\circ\phi=0$$

for all $i\in\{1,\ldots,r\}$. $f\cdot pk$-structures are a generalization of almost complex and almost contact structures, according as $r=0$ and $r=1$, respectively. Furthermore, we say that an $f\cdot pk$-structure is *normal* if the tensor field

$$N=[\phi,\phi]+2\sum_{i=1}^{r}d\eta_i\otimes\xi_i \tag{2.1}$$

vanishes, where $[\phi,\phi]$ denotes the Nijenhuis torsion of ϕ. It is known that, given an $f\cdot pk$-structure (ϕ,ξ_i,η_i), there exists a Riemannian metric g on M such that

$$g(\phi V,\phi W)=g(V,W)-\sum_{i=1}^{r}\eta_i(V)\,\eta_i(W) \tag{2.2}$$

for all $V,W\in\Gamma TM$. Such a metric, in general, is not unique. If g is any metric satisfying (2.2) we say that (ϕ,ξ_i,η_i,g) is a *metric $f\cdot pk$-structure*. We denote by Φ the 2-form defined by $\Phi(V,W)=g(V,\phi W)$, for any $V,W\in\Gamma TM$. A metric $f\cdot pk$-manifold M^{2n+r} with structure (ϕ,ξ_i,η_i,g) is called *almost $\mathcal{S}$-manifold* if $d\eta_1=\cdots=d\eta_r=\Phi$. This definition reduces to that one of contact metric manifold for $r=1$ and almost Hermitian manifold for $r=0$. In an almost $\mathcal{S}$-manifold one can define, for each $i\in\{1,\ldots,r\}$, the operator $h_i=\frac{1}{2}\mathcal{L}_{\xi_i}\phi$. It can be proved that each h_i is self-adjoint, trace-free, anticommutes with ϕ and, finally, satisfies $h_i(\xi_j)=0$ and

$$\phi(N(V,W))+N(\phi(V),W)=2\sum_{i=1}^{r}\eta_i(X)\,h_i(W), \tag{2.3}$$

$$g\left(N\left(\phi\left(V\right),W\right),\xi_i\right)=0 \tag{2.4}$$

for all $V,W\in\Gamma TM$. It can be useful also to put $\overline{\xi}=\sum_{i=1}^{r}\xi_i$ and $h=\frac{1}{2}\mathcal{L}_{\overline{\xi}}\phi=\sum_{i=1}^{r}h_i$. For the proofs of these properties and more details on almost $\mathcal{S}$-manifolds, good references are [4] and [2].

We will denote by $\mathcal{H}$ the $2n$-dimensional distribution on M given by $\mathcal{H}=\bigcap_{i=1}^{r}\ker\left(\eta_i\right)$. This distribution in not integrable. Furthermore, it can be proved that the maximal dimension of any integrable subbundle of $\mathcal{H}$ is n. So we can set the following

Definition 2.1. *Let $\left(M^{2n+r},\phi,\xi_i,\eta_i,g\right)$ be an almost $\mathcal{S}$-manifold. A foliation $\mathcal{F}$ on M is called* Legendrian *if its leaves are n-dimensional integral submanifolds of $\mathcal{H}$ and*

$$\Phi\left(X,X'\right)=0 \tag{2.5}$$

for any vector fields X,X' tangent to $\mathcal{F}$. Two Legendrian foliations $\mathcal{F}$ and $\mathcal{F}'$, on the almost $\mathcal{S}$-manifolds (M,ϕ,η_i,ξ_i,g) and $(M',\phi',\eta'_i,\xi'_i,g')$, $i\in\{1,\dots,r\}$, are said to be equivalent *if there exists a diffeomorphism $f:M\longrightarrow M'$ which preserves both the structures and the foliations.*

Remark 2.2. *Note that the condition* (2.5) *in the previous definition, except for the case $r=0$, is redundant. Indeed, if $X,X'\in\Gamma L$*

$$\Phi\left(X,X'\right)=d\eta_i\left(X,X'\right)=\frac{1}{2}\left(X\left(\eta_i\left(X'\right)\right)-X'\left(\eta_i\left(X\right)\right)-\eta_i\left(\left[X,X'\right]\right)\right)=0$$

since L is involutive and $\eta_i\left(X\right)=\eta_i\left(X'\right)=0$. Nevertheless we have added (2.5) *to the definition of Legendrian foliation because in this way when $r=0$ and $d\Phi=0$ our definition reduces to that one of Lagrangian foliation in a symplectic manifold. So Definition 2.1 generalizes both the standard definition of Legendrian foliation on contact metric manifold (when $r=1$) and that one of Lagrangian foliation on symplectic manifolds (when $r=0$ and $d\Phi=0$).*

We denote by L the tangent bundle of the Legendrian foliation $\mathcal{F}$ and by $L^{\perp}$ the orthogonal bundle of L. Then, setting $Q=\mathcal{H}\cap L^{\perp}$, we obtain another n-dimensional distribution of M and we get the decomposition

$$TM=L\oplus Q\oplus\mathbb{R}\xi_1\oplus\cdots\oplus\mathbb{R}\xi_r.$$

In next pages we will often make use of the following

Lemma 2.3. *Let $\left(M^{2n+r},\phi,\eta_i,\xi_i,g\right)$ be an almost $\mathcal{S}$-manifold and $Z\in\Gamma\mathcal{H}$. Then, for all $i\in\{1,\dots,r\}$, $[Z,\xi_i]\in\Gamma\mathcal{H}$.*

Proof. Indeed, for any fixed $i,j\in\{1,\dots,r\}$, $\eta_j\left(\left[Z,\xi_i\right]\right)=-2d\eta_j\left(Z,\xi_i\right)+Z\left(\eta_j\left(\xi_i\right)\right)-\xi_i\left(\eta_j\left(Z\right)\right)=-2g\left(Z,\phi\left(\xi_i\right)\right)+Z\left(\delta_{ij}\right)-\xi_i\left(\eta_j\left(Z\right)\right)=0$, as $Z\in\Gamma\mathcal{H}$. So $[Z,\xi_i]\in\Gamma\mathcal{H}$. □

There are many properties similar to the "contact" case described in [8] or [5]. For example we have the following

Proposition 2.4. *Let $\mathcal{F}$ be a Legendrian foliation on an almost $\mathcal{S}$-manifold $(M, \phi, \xi_i, \eta_i, g)$, $i \in \{1, \dots, r\}$. Then*

(i) $\phi(L) = Q$ *and* $\phi(Q) = L$;

(ii) *for any* $Y, Y' \in \Gamma Q$, $[Y, Y'] \in \Gamma\mathcal{H}$.

Proof. For any $X, X' \in L$ and for any $i \in \{1, \dots, r\}$ we have $g(X, \phi(X')) = d\eta_i(X, X') = \frac{1}{2}(X(\eta_i(X')) - X'(\eta_i(X)) - \eta_i([X, X'])) = -\frac{1}{2}\eta_i([X, X']) = 0$, since L is integrable. Hence $\phi(X') \in L^{\perp}$. But we know also that $\phi(X') \in \mathcal{H}$, so $\phi(X') \in Q$. Since $\dim(\phi(L)) = \dim(L) = n = \dim(Q)$, then $\phi(L) = Q$. In a similar way one can prove that $\phi(Q) = L$. In order to prove (ii), take $Y, Y' \in \Gamma Q$. Then, applying (i), we have, for each $i \in \{1, \dots, r\}$, $\eta_i([Y, Y']) = -2d\eta_i(Y, Y') = -2g(Y, \phi(Y')) = 0$. □

The last proposition states that in general the distribution Q is not integrable. Another consequence is the possibility of constructing very useful local frames for M.

Proposition 2.5. *Let $\left(M^{2n+r}, \phi, \xi_i, \eta_i, g\right)$ be an almost $\mathcal{S}$-manifold and $\mathcal{F}$ a Legendrian foliation on M. If $\{X_1, \dots, X_n\}$ is a local g-orthonormal frame for L, then $\{X_1, \dots, X_n, \phi X_1, \dots, \phi X_n, \xi_1, \dots, \xi_r\}$ is a local g-orthonormal frame for TM and its corresponding local coframe $\{\varepsilon_1, \dots, \varepsilon_n, \theta_1, \dots, \theta_n, \eta_1, \dots, \eta_r\}$ in T^*M, where $\varepsilon_\alpha = \frac{1}{2} i_{\phi(X_\alpha)}\Phi$, $\theta_\alpha = -\frac{1}{2} i_{X_\alpha}\Phi$, satisfies the following relations:*

(i) $\varepsilon_\alpha \in \Gamma L^*$ *and* $\theta_\alpha \in \Gamma Q^*$, $\alpha \in \{1, \dots, n\}$,

(ii) $\Phi = 2 \sum_{\alpha=1}^{n} \theta_\alpha \wedge \varepsilon_\alpha$,

(iii) $g|_L = \sum_{\alpha=1}^{n} \varepsilon_\alpha \otimes \varepsilon_\alpha$, $g|_Q = \sum_{\alpha=1}^{n} \theta_\alpha \otimes \theta_\alpha$ *and* $g|_{\mathbb{R}\xi_i} = \eta_i \otimes \eta_i$, $i \in \{1, \dots, r\}$.

Proof. For any $W \in \Gamma TM$ we have

$$\begin{aligned}\varepsilon_\alpha(W) &= \frac{1}{2}\left(i_{\phi(X_\alpha)}\Phi\right) W = \Phi(\phi(X_\alpha), W) = g(\phi(X_\alpha), \phi(W)) \\ &= g(X_\alpha, W) - \sum_{i=1}^{r} \eta_i(X_\alpha)\eta_i(W) = g(W, X_\alpha)\end{aligned}$$

and

$$\theta_\alpha(W) = -\frac{1}{2}(i_{X_\alpha}\Phi) W = -\Phi(X_\alpha, W) = g(W, \phi(X_\alpha))$$

from which we obtain that $\varepsilon_\alpha \in \Gamma L^*$ and $\theta_\alpha \in \Gamma L^*$. Moreover, for any $\alpha, \beta \in \{1, \dots, n\}$, $\theta_\alpha(\phi(X_\beta)) = g(\phi(X_\alpha), \phi(X_\beta)) = g(X_\alpha, X_\beta) = \delta_{\alpha\beta}$ and $\varepsilon_\alpha(X_\beta) = g(X_\alpha, X_\beta) = \delta_{\alpha\beta}$, so that:

$$\begin{aligned}\sum_{\alpha=1}^{n} \theta_\alpha \otimes \theta_\alpha(\phi(X_\gamma), \phi(X_\beta)) &= \sum_{\alpha=1}^{n} \theta_\alpha(\phi(X_\gamma))\theta_\alpha(\phi(X_\beta)) \\ &= \delta_{\gamma\beta} = g(\phi(X_\gamma), \phi(X_\beta)).\end{aligned}$$

In the same way one can completes the proof of (iii). It remains to verify (ii). Indeed, for any $\gamma, \beta \in \{1, \ldots, n\}$,

$$2\sum_{\alpha=1}^{n}\theta_\alpha\wedge\varepsilon_\alpha\left(\phi\left(X_\gamma\right),X_\beta\right)=\sum_{\alpha=1}^{r}\left(\theta_\alpha\left(\phi\left(X_\gamma\right)\right)\varepsilon_\alpha\left(X_\beta\right)-\theta_\alpha\left(X_\beta\right)\varepsilon_\alpha\left(\phi\left(X_\gamma\right)\right)\right)$$
$$=\sum_{\alpha=1}^{n}\delta_{\alpha\gamma}\delta_{\alpha\beta}=\delta_{\gamma\beta}=\Phi\left(\phi\left(X_\gamma\right),X_\beta\right)$$

and the proposition is proved. □

3 The invariants Π and G

In this section we will give a classification of Legendrian foliations on almost $\mathcal{S}$-manifolds, which generalizes that one given in [8] in the contact case. First we need the following

Definition 3.1. *Let $\mathcal{F}$ be a Legendrian foliation on the almost $\mathcal{S}$-manifold $\left(M^{2n+r},\phi,\xi_i,\eta_i,g\right)$. We define, for any $X, X' \in \Gamma L$,*

$$\Pi\left(X,X'\right)=-\sum_{i=1}^{r}\left(\mathcal{L}_X\mathcal{L}_{X'}\eta_i\right)\left(\xi_i\right).$$

Proposition 3.2. Π *is a symmetric bilinear form on L. Furthermore, for any $X, X' \in \Gamma L$,*

$$\Pi\left(X,X'\right)=-\sum_{i=1}^{r}\eta_i\left(\left[X',\left[X,\xi_i\right]\right]\right)$$

or, equivalently,

$$\Pi\left(X,X'\right)=2g\left(\left[\overline{\xi},X\right],\phi\left(X'\right)\right).$$

Proof. From the definition of Π we have, as $[X,\xi_i]$ and $[X',\xi_i]$ belong to $\Gamma\mathcal{H}$,

$$\Pi\left(X,X'\right)=-\sum_{i=1}^{r}\left(\mathcal{L}_X\mathcal{L}_{X'}\eta_i\right)\left(\xi_i\right)=-\sum_{i=1}^{r}X\left(\left(\mathcal{L}_{X'}\eta_i\right)\left(\xi_i\right)\right)$$
$$+\sum_{i=1}^{r}\left(\mathcal{L}_{X'}\eta_i\right)\left(\left[X,\xi_i\right]\right)=-\sum_{i=1}^{r}X\left(X'\left(\eta_i\left(\xi_i\right)\right)-\eta_i\left(\left[X',\xi_i\right]\right)\right)$$
$$+\sum_{i=1}^{r}X'\left(\eta_i\left(\left[X,\xi_i\right]\right)\right)-\sum_{i=1}^{r}\eta_i\left(\left[X',\left[X,\xi_i\right]\right]\right)=-\sum_{i=1}^{r}\eta_i\left(\left[X',\left[X,\xi_i\right]\right]\right).$$

Furthermore, $\Pi\left(X,X'\right) = -\sum_{i=1}^{r}\eta_i\left(\left[X',\left[X,\xi_i\right]\right]\right) = \sum_{i=1}^{r}2d\eta_i\left(X',\left[X,\xi_i\right]\right) = 2g\left(\left[\overline{\xi},X\right],\phi\left(X'\right)\right)$ and, using this formula, it is easy to check that Π is a symmetric bilinear form on L. □

Π is an invariant of the Legendrian foliation $\mathcal{F}$ and it can be used for classifying Legendrian foliations on almost $\mathcal{S}$-manifolds in the following way:

Definition 3.3. *A Legendrian foliation on the almost $\mathcal{S}$-manifold M^{2n+r} is called:*

(i) flat *if and only if* $\Pi = 0$*;*

(i) degenerate *if and only if* Π *is degenerate;*

(iii) non-degenerate *if and only if* Π *is non-degenerate;*

(iv) positive definite *if and only if* Π *is positive definite.*

Proposition 3.2 and Proposition 2.4 enable us to give a geometrical meaning of the previous classification:

Proposition 3.4. *Let $\mathcal{F}$ be a Legendrian foliation on the almost $\mathcal{S}$-manifold $\left(M^{2n+r}, \phi, \xi_i, \eta_i, g\right)$. Then,*

(i) *$\mathcal{F}$ is flat if and only if $\overline{\xi}$ is a projectable (or foliate) vector field for the foliation $\mathcal{F}$, i.e. $\left[\overline{\xi}, X\right] \in \Gamma L$ for all $X \in \Gamma L$;*

(ii) *$\mathcal{F}$ is degenerate if and only if there exists $X \in \Gamma L$, $X \neq 0$, such that $p_Q\left(\left[\overline{\xi}, X\right]\right) = 0$;*

(iii) *$\mathcal{F}$ is non-degenerate if and only if $p_Q\left(\left[\overline{\xi}, X\right]\right) \neq 0$ for all $X \in \Gamma L$;*

(iv) *$\mathcal{F}$ positive definite if and only if $g\left(\left[\overline{\xi}, X\right], \phi\left(X\right)\right) > 0$ for all $X \in \Gamma L$, $X \neq 0$.*

Proof. First of all note that, Lemma 2.3 implies $\left[\overline{\xi}, Z\right] \in \Gamma\mathcal{H}$ for all $Z \in \Gamma\mathcal{H}$. So, in order to prove (i), it is sufficient to prove that $\Pi \equiv 0$ if and only if $\left[\overline{\xi}, X\right] \in \Gamma Q^{\perp}$ for all $X \in \Gamma L$, but this is a consequence of Proposition 3.2 and Proposition 2.4. Now we prove (ii). Suppose that there exists $X \in \Gamma L$, $X \neq 0$, such that $p_Q\left(\left[\overline{\xi}, X\right]\right) = 0$. Then, for such $X \in \Gamma L$, we have $\Pi\left(X, X\right) = 2g\left(\left[\overline{\xi}, X\right], \phi\left(X\right)\right) = 0$, that is Π is degenerate. Vice versa, suppose that $\mathcal{F}$ is degenerate. Then there exists $0 \neq X \in \Gamma L$ such that $\Pi\left(X, X'\right) = 0$ for all $X' \in \Gamma L$. Hence, using again Proposition 3.2, we have that, for all $X' \in \Gamma L$, $g\left(\left[\overline{\xi}, X\right], \phi\left(X'\right)\right) = 0$ and so, applying Proposition 2.4, $p_Q\left(\left[\overline{\xi}, X\right]\right) = 0$. Similarly one can prove (iii) and (iv). $\square$

Corollary 3.5. *If $\xi_1, \ldots, \xi_r$ are foliate vector fields, then $\mathcal{F}$ is flat.*

The previous corollary justifies the

Definition 3.6. *A Legendrian foliation on an almost $\mathcal{S}$-manifold $(M, \phi, \xi_i, \eta_i, g)$ is called* strongly flat *if $\xi_1, \ldots, \xi_r$ are foliate vector fields.*

Now we introduce a second invariant trilinear form on L, which is very useful in the study of Riemannian Legendrian foliations.

Definition 3.7. *For all $X, X', X'' \in \Gamma L$ we define*

$$\begin{aligned} G\left(X, X', X''\right) = {} & \frac{1}{2}\left(X\left(\Pi\left(X', X''\right)\right) + X'\left(\Pi\left(X, X''\right)\right) + X''\left(\Pi\left(X, X'\right)\right)\right) \\ & + \frac{1}{2}\left(\sum_{i=1}^{r}\left(\mathcal{L}_{X'}\mathcal{L}_X\mathcal{L}_{X''}\eta_i + \mathcal{L}_{X''}\mathcal{L}_X\mathcal{L}_{X'}\eta_i\right)\left(\xi_i\right)\right). \end{aligned}$$

We prove some lemmas in order to find a more convenient expression for G.

Lemma 3.8. *For any $X, X', X'' \in \Gamma L$ we have:*

$$G(X, X', X'') = -\frac{1}{2}\left(X\left(\Pi\left(X', X''\right)\right) + X'\left(\Pi\left(X, X''\right)\right) + X''\left(\Pi\left(X, X'\right)\right)\right)$$
$$-\frac{1}{2}\sum_{i=1}^{r}\eta_i\left([X',[X,[X'',\xi_i]]] + [X'',[X,[X',\xi_i]]]\right).$$

Proof. Indeed we have, for each $i \in \{1, \ldots, r\}$,

$$\begin{aligned}(\mathcal{L}_{X'}\mathcal{L}_X\mathcal{L}_{X''}\eta_i)(\xi_i) &= X'\left(\mathcal{L}_X\mathcal{L}_{X''}\eta_i(\xi_i)\right) - X\left((\mathcal{L}_{X''}\eta_i)([X',\xi_i])\right)\\ +(\mathcal{L}_{X''}\eta_i)([X,[X',\xi_i]]) &= X'\left(\mathcal{L}_X\mathcal{L}_{X''}\eta_i(\xi_i)\right) + X\left(\eta_i([X'',[X',\xi_i]])\right)\\ &+X''\left(\eta_i([X,[X',\xi_i]])\right) - \eta_i\left([X'',[X,[X',\xi_i]]]\right),\end{aligned}$$

so, taking the sum over all i,

$$\sum_{i=1}^{r}(\mathcal{L}_{X'}\mathcal{L}_X\mathcal{L}_{X''}\eta_i)(\xi_i) = -X'\left(\Pi\left(X, X''\right)\right) - X\left(\Pi\left(X', X''\right)\right)$$
$$-X''\left(\Pi\left(X, X'\right)\right) - \sum_{i=1}^{r}\eta_i\left([X'',[X,[X',\xi_i]]]\right).$$

Hence, applying the definition of G we get the result. □

Using Proposition 3.2 and the symmetry of Π, an easy computation allows us to state the following

Lemma 3.9. *For any $X, X', X'' \in \Gamma L$ we have*

$$G(X, X', X'') = -X\left(g\left(\left[\overline{\xi}, X'\right], \phi(X'')\right)\right) + g\left(\left[X, \left[\overline{\xi}, X''\right]\right], \phi(X')\right)$$
$$+g\left(\left[X, \left[\overline{\xi}, X'\right]\right], \phi(X'')\right).$$

4 Non-degenerate Legendrian foliations on almost $\mathcal{S}$-manifolds

For a non-degenerate Legendrian foliation on a contact manifold Liebermann showed in [7] the existence of a tensor field which is very useful in the study of this type of the Legendrian foliations. Now we want to generalize this construction to the case of almost $\mathcal{S}$-manifolds.

Proposition 4.1. *Let $\mathcal{F}$ be a non-degenerate Legendrian foliation on an almost $\mathcal{S}$-manifold $\left(M^{2n+r}, \phi, \xi_i, \eta_i, g\right)$. Then there exists a tensor field λ of type $(1,1)$ such that*

$$\Pi(\lambda(W), X) = 4r\Phi(W, X)$$

for any $X \in \Gamma L$, $W \in \Gamma TM$. Moreover λ satisfies the relation

$$\lambda\left(\left[\overline{\xi}, X\right]\right) = 2rX \tag{4.6}$$

for all $X \in \Gamma L$.

Proof. Consider the isomorphism $\upsilon : \mathcal{H} \longrightarrow \mathcal{H}^*$, $X \mapsto 2ri_X\Phi$ and let π denote the projection of TM onto $\mathcal{H}$. Since L is a subbundle of $\mathcal{H}$, there exists a surjective morphism $p : \mathcal{H}^* \longrightarrow L^*$. Let $\Pi^\sharp : L^* \longrightarrow L$ be the inverse of the morphism $\Pi^\flat : L \longrightarrow L^*$ associated with the non-degenerate bilinear form Π. Then we can put $\lambda := \Pi^\sharp \circ p \circ \upsilon \circ \pi$. Note that, for all $W \in TM$,

$$\begin{aligned}\Pi\left(\lambda\left(W\right), X\right) &= \Pi\left(\Pi^\sharp\left(p\left(\upsilon\left(\pi\left(W\right)\right)\right)\right), X\right) = \Pi^\flat\left(\Pi^\sharp\left(p\left(\upsilon\left(\pi\left(W\right)\right)\right)\right)\right)\left(X\right)\\ &= \left(p\left(\upsilon\left(\pi\left(W\right)\right)\right)\right)\left(X\right) = 2r\left(i_{\pi(W)}\Phi\right)\left(X\right) = 4r\Phi\left(\pi\left(W\right), X\right) = 4r\Phi\left(W, X\right).\end{aligned}$$

Moreover we have, for all $X, X' \in \Gamma L$,

$$\Pi\left(X, X'\right) = 2g\left(\left[\overline{\xi}, X\right], \phi\left(X'\right)\right) = \frac{1}{2r}\Pi\left(\lambda\left(\left[\overline{\xi}, X\right]\right), X'\right)$$

and so $\lambda\left[\overline{\xi}, X\right] = 2rX$. □

Remark 4.2. *By the definition of λ, $L \subset \ker(\lambda)$ and $\lambda^2 = 0$.*

Corollary 4.3. *Let $\mathcal{F}$ be a non-degenerate Legendrian foliation on the almost $\mathcal{S}$-manifold $\left(M^{2n+r}, \phi, \xi_i, \eta_i, g\right)$. If $\{X_1, \ldots, X_n\}$ is a local frame for L in an open subset U on M, then $\{X_1, \ldots, X_n, \left[\overline{\xi}, X_1\right], \ldots, \left[\overline{\xi}, X_n\right]\}$ is a local frame for $\mathcal{H}$ in U.*

Proof. The previous proposition shows that λ maps, except for some multiplicative constants, $\left[\overline{\xi}, X_1\right], \ldots, \left[\overline{\xi}, X_n\right]$ onto the linearly independent vector fields $X_1, \ldots, X_n$, hence the vector fields $\left[\overline{\xi}, X_1\right], \ldots, \left[\overline{\xi}, X_n\right]$ are linearly independent in U. Moreover, by Proposition 3.4, for all $i \in \{1, \ldots, r\}$, $\left[\overline{\xi}, X_i\right]$ does not belong to L, so $\left\{X_1, \ldots, X_n, \left[\overline{\xi}, X_1\right], \ldots, \left[\overline{\xi}, X_n\right]\right\}$ generate $\mathcal{H}$ in U. □

For a non-degenerate Legendrian foliation, the tensor field λ enables us to extend the invariant Π to a symmetric, bilinear form $\overline{\Pi}$ on TM, as follows:

$$\overline{\Pi}\left(W, W'\right) = \Pi\left(\lambda\left(W\right), \lambda\left(W'\right)\right),$$

and to introduce a family of almost $\mathcal{S}$-structures (ϕ, ξ_i, η_i, g) as shown in the following

Lemma 4.4. *Let $\mathcal{F}$ be a non-degenerate Legendrian foliation on the almost $\mathcal{S}$-manifold $\left(M^{2n+r}, \phi, \xi_i, \eta_i, g\right)$. Then there exists an almost $\mathcal{S}$-structure $(\phi', \xi_i, \eta_i, g')$ on M, where g' is a semi-Riemannian metric, such that*

$$g'|_L = \frac{1}{4r}\Pi.$$

Proof. Let $\{X_1, \ldots, X_n\}$ be a local orthonormal frame for L with respect to $\frac{1}{4r}\Pi$ and $\{Y_1, \ldots, Y_n\}$ be a local frame of $\mathcal{H} \cap L'$ such that $\lambda(Y_\alpha) = X_\alpha$ for all $\alpha \in \{1, \ldots, n\}$, where L' is the complement set of L in TM. Put $Q' = span\{Y_1, \ldots, Y_n\}$, so that $\lambda|_{Q'} : Q' \longrightarrow L$ is an isomorphism, and define the tensor field ϕ' of type (1,1) as follows:

$$\phi'|_L = \left(\lambda|_{Q'}\right)^{-1}, \ \phi'|_{Q'} = -\lambda|_{Q'}, \ \phi'|_{\mathbb{R}\xi_1} = \cdots = \phi'|_{\mathbb{R}\xi_r} = 0.$$

By definition ϕ' satisfies the following relations:

$$\phi'(X_\alpha) = Y_\alpha,\ \phi'(Y_\alpha) = -X_\alpha,\ \phi'(\xi_1) = \cdots = \phi'(\xi_r) = 0.$$

Finally we define a semi-Riemannian metric g' on M by setting

$$g'|_L = \frac{1}{4r}\Pi,\ g'|_{Q'} = \frac{1}{4r}(\lambda^*\Pi)|_{Q'},\ g'|_{\mathbb{R}\xi_i} = \eta_i \otimes \eta_i,\ g' = 0 \text{ otherwise.}$$

It is easy to verify that $(\phi', \xi_i, \eta_i, g')$ is a metric $f \cdot pk$-structure for M, so we check only that $\Phi' = d\eta_i$ for all $i \in \{1, \dots, r\}$. Indeed, for $X \in \Gamma L$ and $Y \in \Gamma Q'$, we have $\Phi'(X,Y) = g'(X, \phi'(Y)) = \frac{1}{4r}\Pi(X, \phi'(Y)) = -\frac{1}{4r}\Pi(\lambda(Y), X) = d\eta_i(X,Y) = \Phi(X,Y)$. The other cases are obvious. □

It is clear that the almost $\mathcal{S}$-structure on M considered in Lemma 4.4 is not unique. In analogy to [5] we call the family of all almost $\mathcal{S}$-structures such that $g|_L = \frac{1}{4r}\Pi$ the *canonical family of almost $\mathcal{S}$-structures* for (M, L). Moreover the given structure (ϕ, ξ_i, η_i, g) in Lemma 4.4 does not belong necessarily to the canonical family. In fact we have

Lemma 4.5. *Let $\mathcal{F}$ be a Legendrian foliation on the almost $\mathcal{S}$-manifold $\left(M^{2n+r}, \phi, \xi_i, \eta_i, g\right)$. Then the following statements are equivalent:*

(i) $g|_L = \frac{1}{4r}\Pi$;

(ii) $p_{L^\perp}\left(\left[\bar{\xi}, X\right]\right) = p_Q\left(\left[\bar{\xi}, X\right]\right) = 2r\phi(X)$ *for all* $X \in \Gamma L$.

Proof. The proof follows directly by Proposition 3.2. □

Lemma 4.6. *Let $\mathcal{F}$ be a Legendrian foliation on the almost $\mathcal{S}$-manifold M^{2n+r}. If (ϕ, ξ_i, η_i, g) and $(\phi', \xi_i, \eta_i, g')$ are two almost $\mathcal{S}$-structures on M such that $g|_L = \frac{1}{4r}\Pi = g'|_L$, then there exist smooth functions $c_{\alpha\beta}$ such that*

$$\phi'(X_\alpha) = \phi(X_\alpha) + \sum_{\beta=1}^{n} c_{\alpha\beta} X_\beta,$$

where $\{X_1, \dots, X_n\}$ is a local orthonormal frame for L with respect to $\frac{1}{4r}\Pi$.

Proof. Since $\{X_1, \dots, X_n, \phi(X_1), \dots, \phi(X_n)\}$ is a local frame for $\mathcal{H}$ and, for all $\alpha \in \{1, \dots, n\}$, $\phi' X_\alpha \in \Gamma\mathcal{H}$, there exist smooth functions $b_{\alpha\beta}$ and $c_{\alpha\beta}$ such that

$$\phi'(X_\alpha) = \sum_{\beta=1}^{n} \left(b_{\alpha\beta}\phi(X_\beta) + c_{\alpha\beta}X_\beta\right).$$

Thus, for all $\alpha, \beta \in \{1, \dots, n\}$ we have

$$\Pi(\lambda(\phi'(X_\alpha)), X_\beta) = 4r\Phi(\phi'(X_\alpha), X_\beta) = 4rg(X_\alpha, X_\beta) = \Pi(X_\alpha, X_\beta)$$

and, in the same way,

$$\Pi(\lambda(\phi(X_\alpha)), X_\beta) = \Pi(X_\alpha, X_\beta),$$

from which we obtain that $\lambda(\phi'(X_\alpha)) = X_\alpha = \lambda(\phi(X_\alpha))$ for all $\alpha \in \{1, \dots, n\}$. So, for all $\alpha, \beta \in \{1, \dots, n\}$, since $L \subset \ker(\lambda)$,

$$X_{\alpha} = \lambda\left(\phi'\left(X_{\alpha}\right)\right) = \lambda\left(\sum_{\beta=1}^{n}\left(b_{\alpha\beta}\phi\left(X_{\beta}\right) + c_{\alpha\beta}X_{\beta}\right)\right)$$
$$= \sum_{\beta=1}^{n} b_{\alpha\beta}\lambda\left(\phi\left(X_{\beta}\right)\right) = \sum_{\beta=1}^{n} b_{\alpha\beta}X_{\beta}$$

and this implies that $b_{\alpha\beta} = \delta_{\alpha\beta}$. $\square$

In Section 2 we have considered the tensor field $h = \frac{1}{2}\mathcal{L}_{\overline{\xi}}\phi$. It is clear that, in general, h does not transform vectors tangent to $\mathcal{F}$ into vectors tangent to $\mathcal{F}$. However, if $\mathcal{F}$ is non-degenerate, we can find an almost $\mathcal{S}$-structure in the canonical family such that h sends L into itself, as the following theorem shows.

Theorem 4.7. *Let $\mathcal{F}$ be a non-degenerate Legendrian foliation on an almost $\mathcal{S}$-manifold M^{2n+r} with structure $(\phi_0, \xi_i, \eta_i, g_0)$. Then there exists a unique almost $\mathcal{S}$-structure (ϕ, ξ_i, η_i, g) such that*

(i) $g|_L = \frac{1}{4r}\Pi$,

(ii) $h : L \longrightarrow L$,

Proof. Following [7] we consider the tensor field $S : \mathcal{H} \longrightarrow \mathcal{H}$ such that

$$S\left(X\right) = 0,$$

$$S\left(\left[\overline{\xi}, X\right]\right) = \frac{1}{2r}\left(\left[\overline{\xi}, X\right] - \frac{1}{4r}\lambda\left(\left[\overline{\xi}, \left[\overline{\xi}, X\right]\right]\right)\right),$$

for all $X \in \Gamma L$. The definition is well posed in view of Corollary 4.3. Moreover, for all $X \in \Gamma L$, we get

$$\lambda\left(S\left[\overline{\xi}, X\right]\right) = \frac{1}{2r}\left(\lambda\left(\left[\overline{\xi}, X\right]\right) - \frac{1}{4r}\lambda^2\left(\left[\overline{\xi}, \left[\overline{\xi}, X\right]\right]\right)\right) = \frac{1}{2r}\lambda\left(\left[\overline{\xi}, X\right]\right) = X.$$

Now let $\{X_1, \dots, X_n\}$ be a local orthonormal frame for L with respect to $\frac{1}{4r}\Pi$ and, with the notation of Lemma 4.4, choose a $Q \subset \mathcal{H} \cap L'$ such that $Q = span\left\{Y_1, \dots, Y_n\right\}$, where $Y_{\alpha} = S\left(\left[\overline{\xi}, X_{\alpha}\right]\right)$. Since, for all $\alpha \in \{1, \dots, n\}$, $\lambda\left(Y_{\alpha}\right) = \lambda\left(S\left(\left(\left[\overline{\xi}, X_{\alpha}\right]\right)\right)\right) = X_{\alpha}$, following the proof of Lemma 4.4, we can define ϕ and g as follows:

$$\phi|_L = \left(\lambda|_Q\right)^{-1}, \ \phi|_Q = -\lambda|_Q, \ \phi|_{\mathbb{R}\xi_1} = \cdots = \phi|_{\mathbb{R}\xi_r} = 0$$

and

$$g|_L = \frac{1}{4r}\Pi, \ g|_Q = \frac{1}{4r}\left(\lambda^*\Pi\right)|_Q, \ g|_{\mathbb{R}\xi_i} = \eta_i \otimes \eta_i, \ g = 0 \text{ otherwise},$$

$i \in \{1, \dots, r\}$. Thus we obtain an almost $\mathcal{S}$-structure (ϕ, ξ_i, η_i, g) such that the condition (i) of the theorem is satisfied. Now we have to verify (ii). First of all note that from (2.3) it follows that, for all $X \in \Gamma L$,

$$h_i(X) = \frac{1}{2}\phi\left(N\left(\xi_i, X\right)\right),$$

for all $i \in \{1, \dots, r\}$. So it is sufficient to show that $\phi\left(N\left(\bar{\xi}, X\right)\right) \in L$. Lemma 4.5 implies

$$p_Q\left(\left[\bar{\xi}, X_\alpha\right]\right) = 2r\phi\left(X_\alpha\right) = 2rY_\alpha,$$

from which we have

$$\begin{aligned}
\lambda\left(\left[\bar{\xi}, Y_\alpha\right]\right) &= \lambda\left(\left[\bar{\xi}, S\left(\left[\bar{\xi}, X_\alpha\right]\right)\right]\right) \\
&= \frac{1}{2r}\lambda\left(\left[\bar{\xi}, \left[\bar{\xi}, X_\alpha\right]\right] - \frac{1}{4r}\left[\bar{\xi}, \lambda\left(\left[\bar{\xi}, \left[\bar{\xi}, X_\alpha\right]\right]\right)\right]\right) \\
&= \frac{1}{2r}\lambda\left(\left[\bar{\xi}, \left[\bar{\xi}, X_\alpha\right]\right] - \frac{1}{2}\left[\bar{\xi}, \left[\bar{\xi}, X_\alpha\right]\right]\right) \\
&= \frac{1}{4r}\lambda\left(\left[\bar{\xi}, p_L\left(\left[\bar{\xi}, X_\alpha\right]\right)\right] + \left[\bar{\xi}, p_Q\left(\left[\bar{\xi}, X_\alpha\right]\right)\right]\right) = \frac{1}{2}\left(p_L\left(\left[\bar{\xi}, X_\alpha\right]\right) + \lambda\left(\left[\bar{\xi}, Y_\alpha\right]\right)\right)
\end{aligned}$$

so that

$$\lambda\left(\left[\bar{\xi}, Y_\alpha\right]\right) = p_L\left(\left[\bar{\xi}, X_\alpha\right]\right).$$

Thus, for all $\alpha \in \{1, \dots, n\}$, since $L \subset \ker(\lambda)$ we have

$$\begin{aligned}
\phi\left(N\left(\bar{\xi}, X_\alpha\right)\right) &= -\phi\left(\left[\bar{\xi}, X_\alpha\right]\right) + \left[\bar{\xi}, \phi\left(X_\alpha\right)\right] = -\phi\left(\left[\bar{\xi}, X_\alpha\right]\right) + \left[\bar{\xi}, Y_\alpha\right] \\
&= -\left(\lambda|_Q\right)^{-1}\left(\lambda\left(\left[\bar{\xi}, Y_\alpha\right]\right)\right) - \phi\left(p_Q\left(\left[\bar{\xi}, X_\alpha\right]\right)\right) + \left[\bar{\xi}, Y_\alpha\right] \\
&= -\left(\lambda|_Q\right)^{-1}\left(\lambda\left(p_Q\left(\left[\bar{\xi}, Y_\alpha\right]\right)\right)\right) - \phi\left(\lambda\left(p_L\left(\left[\bar{\xi}, Y_\alpha\right]\right)\right)\right) - \phi\left(p_Q\left(\left[\bar{\xi}, X_\alpha\right]\right)\right) \\
&\quad + p_Q\left(\left[\bar{\xi}, Y_\alpha\right]\right) + p_L\left(\left[\bar{\xi}, Y_\alpha\right]\right),
\end{aligned}$$

and we can conclude that

$$h\left(X_\alpha\right) = \frac{1}{2}\phi\left(N\left(\bar{\xi}, X_\alpha\right)\right) = \frac{1}{2}\phi\left(p_Q\left(\left[\bar{\xi}, X_\alpha\right]\right)\right) + \frac{1}{2}p_L\left(\left[\bar{\xi}, Y_\alpha\right]\right) \in L.$$

It remains to prove the uniqueness of such a structure. For this purpose, let $(\phi', \xi_i, \eta_i, g')$ denote any almost $\mathcal{S}$-structure satisfying (i) and (ii). By Lemma 4.6 there exist smooth functions $c_{\alpha\beta}$ such that $\phi'(X_\alpha) = \phi(X_\alpha) + \sum_{\beta=1}^{n} c_{\alpha\beta}X_\beta$. ¿From the relations

$$p_Q\left(\left[\bar{\xi}, X_\alpha\right]\right) = 2r\phi\left(X_\alpha\right) \text{ and } p_{Q'}\left(\left[\bar{\xi}, X_\alpha\right]\right) = 2r\phi'\left(X_\alpha\right)$$

we deduce that there exist functions $a_{\alpha\beta}$ such that

$$\begin{aligned}
\left[\bar{\xi}, X_\alpha\right] &= \sum_{\beta=1}^{n} a_{\alpha\beta}X_\beta + 2r\phi'\left(X_\alpha\right) = \sum_{\beta=1}^{n} a_{\alpha\beta}X_\beta + 2r\phi\left(X_\alpha\right) + 2r\sum_{\beta=1}^{n} c_{\alpha\beta}X_\beta = \\
&= \sum_{\beta=1}^{n}\left(a_{\alpha\beta} + 2rc_{\alpha\beta}\right)X_\beta + 2r\phi\left(X_\alpha\right).
\end{aligned}$$

Moreover, the condition (ii) of the theorem implies that

$$0 = p_Q\left(h\left(X_\alpha\right)\right) = \frac{1}{2}\left(p_Q\left(\left[\overline{\xi}, \phi\left(X_\alpha\right)\right]\right) - p_Q\left(\phi\left(\left[\overline{\xi}, X_\alpha\right]\right)\right)\right)$$

and so

$$p_Q\left(\left[\overline{\xi}, \phi\left(X_\alpha\right)\right]\right) = p_Q\left(\phi\left(\left[\overline{\xi}, X_\alpha\right]\right)\right) = \sum_{\beta=1}^{n}\left(a_{\alpha\beta} + 2rc_{\alpha\beta}\right)\phi\left(X_\beta\right).$$

Also $h' : L \longrightarrow L$, then:

$$\begin{aligned} 0 &= g'\left(h'\left(X_\alpha\right), \phi'\left(X_\beta\right)\right) = \Phi'\left(h'\left(X_\alpha\right), X_\beta\right) \\ &= \Phi\left(h'\left(X_\alpha\right), X_\beta\right) = g\left(h'\left(X_\alpha\right), \phi\left(X_\beta\right)\right) \end{aligned}$$

which, together Lemma 4.6 and the previous formula, implies

$$\begin{aligned} 0 = p_Q\left(h'\left(X_\alpha\right)\right) &= \frac{1}{2}p_Q\left(\left[\overline{\xi}, \phi'\left(X_\alpha\right)\right] - \phi'\left(\left[\overline{\xi}, X_\alpha\right]\right)\right) \\ &= \frac{1}{2}p_Q\left(\left[\overline{\xi}, \phi\left(X_\alpha\right)\right] + \sum_{\beta=1}^{n} c_{\alpha\beta}\left[\overline{\xi}, X_\beta\right] - \phi'\left(\left[\overline{\xi}, X_\alpha\right]\right)\right) = \\ \frac{1}{2}\sum_{\beta=1}^{n}\left(\left(a_{\alpha\beta} + 2rc_{\alpha\beta}\right)\phi\left(X_\beta\right) + 2rc_{\alpha\beta}\phi\left(X_\beta\right) - a_{\alpha\beta}\phi\left(X_\beta\right)\right) &= 2r\sum_{\beta=1}^{n} c_{\alpha\beta}\phi\left(X_\beta\right). \end{aligned}$$

Hence, for all $\alpha, \beta \in \{1, \ldots, n\}$, $c_{\alpha\beta} = 0$ and $\phi = \phi'$. Finally, from $\Phi = \Phi'$ and $\phi = \phi'$ we deduce that $g = g'$. □

5 Riemannian Legendrian foliations

A Legendrian foliation $\mathcal{F}$ on an almost $\mathcal{S}$-manifold $\left(M^{2n+r}, \phi, \xi_i, \eta_i, g\right)$ is called a *Riemannian Legendrian foliation* if the metric g is *bundle-like* with respect to $\mathcal{F}$, that is

$$\mathcal{L}_X g|_{L^\perp} = 0$$

for all $X \in \Gamma L$.

Theorem 5.1. *Let $\mathcal{F}$ be a Riemannian Legendrian foliation on the almost $\mathcal{S}$-manifold $\left(M^{2n+r}, \phi, \xi_i, \eta_i, g\right)$. Then we have:*

(i) *$\mathcal{F}$ is a non-degenerate Legendrian foliation;*

(ii) $g|_L = \frac{1}{4r}\Pi$.

Proof. (i). Suppose by absurd that $\mathcal{F}$ is degenerate. Then there exists $X \in \Gamma L$, $X \neq 0$, such that $\Pi\left(X, X\right) = 0$. Then:

$$0=(\mathcal{L}_X g)\left(\phi(X),\overline{\xi}\right)=\sum_{i=1}^{r}X\left(g\left(\phi(X),\xi_i\right)\right)-\sum_{i=1}^{r}g\left(\left[X,\phi(X)\right],\xi_i\right)$$
$$-g\left(\left[X,\overline{\xi}\right],\phi(X)\right)=-\sum_{i=1}^{r}\eta_i\left(\left[X,\phi(X)\right]\right)+\frac{1}{2}\Pi(X,X)$$
$$=\sum_{i=1}^{r}2d\eta_i\left(X,\phi(X)\right)=-2rg(X,X)$$

and this leads to a contradiction.
(ii) For all $X\in\Gamma L$ and $Y\in\Gamma Q$ we have:

$$0=(\mathcal{L}_X g)\left(Y,\overline{\xi}\right)=-\sum_{i=1}^{r}g\left(\left[X,Y\right],\xi_i\right)+g\left(\left[\overline{\xi},X\right],Y\right)$$
$$=-\sum_{i=1}^{r}\eta_i\left(\left[X,Y\right]\right)-\frac{1}{2}\Pi\left(X,\phi(Y)\right)=\sum_{i=1}^{r}2d\eta_i(X,Y)-\frac{1}{2}\Pi\left(X,\phi(Y)\right)$$
$$=2rg\left(X,\phi(Y)\right)-\frac{1}{2}\Pi\left(X,\phi(Y)\right),$$

so $g|_L=\frac{1}{4r}\Pi$. □

Corollary 5.2. *Any Riemannian Legendrian foliation on an almost $\mathcal{S}$-manifold M is positive definite.*

Now we deal with the problem of finding an obstruction to the fact that a Legendrian foliation is Riemannian. First we need the following

Lemma 5.3. *Let $\mathcal{F}$ be a non-degenerate Legendrian foliation on an almost $\mathcal{S}$-manifold $\left(M^{2n+r},\phi,\xi_i,\eta_i,g\right)$ with $g|_L=\frac{1}{4r}\Pi$. Then, for all $X,X',X''\in\Gamma L$,*

$$G\left(X,X',X''\right)=2rg\left(\nabla_{\phi(X')}\phi\left(X''\right)+\nabla_{\phi(X'')}\phi\left(X'\right),X\right)$$

where ∇ is the Levi Civita connection of (M,g).

Proof. By Lemma 3.9, for all $X,X',X''\in\Gamma L$,

$$G\left(X,X',X''\right)=-X\left(g\left(p_Q\left[\overline{\xi},X'\right],\phi\left(X''\right)\right)\right)$$
$$+g\left(p_Q\left(\left[X,\left[\overline{\xi},X''\right]\right]\right),\phi\left(X'\right)\right)+g\left(p_Q\left(\left[X,\left[\overline{\xi},X'\right]\right]\right),\phi\left(X''\right)\right).$$

Now,

$$\begin{aligned}p_Q\left(\left[X,\left[\overline{\xi},X''\right]\right]\right)&=p_Q\left(\left[X,p_Q\left(\left[\overline{\xi},X''\right]\right)\right]\right)+p_Q\left(\left[X,p_L\left(\left[\overline{\xi},X''\right]\right)\right]\right)=\\&=p_Q\left(\left[X,2r\phi\left(X''\right)\right]\right)\end{aligned}$$

and, in the same way, $p_Q\left(\left[X,\left[\overline{\xi},X'\right]\right]\right)=p_Q\left(\left[X,2r\phi\left(X'\right)\right]\right)$. So

$$G\left(X,X',X''\right)=-2rX\left(g\left(\phi\left(X'\right),\phi\left(X''\right)\right)\right)+2rg\left(\left[X,\phi\left(X''\right)\right],\phi\left(X'\right)\right)$$
$$+2rg\left(\left[X,\phi\left(X'\right)\right],\phi\left(X''\right)\right)=-2rX\left(g\left(\phi\left(X'\right),\phi\left(X''\right)\right)\right)$$
$$+2rg\left(\nabla_X\phi\left(X''\right),\phi\left(X'\right)\right)-2rg\left(\nabla_{\phi(X'')}X,\phi\left(X'\right)\right)+2rg\left(\nabla_X\phi\left(X'\right),\phi\left(X''\right)\right)$$
$$-2rg\left(\nabla_{\phi(X')}X,\phi\left(X''\right)\right)=2rg\left(\nabla_{\phi(X')}\phi\left(X''\right)+\nabla_{\phi(X'')}\phi\left(X'\right),X\right)$$

and the lemma is proved. □

Theorem 5.4. *If $\mathcal{F}$ is a Riemannian Legendrian foliation on the almost $\mathcal{S}$-manifold $\left(M^{2n+r}, \phi, \xi_i, \eta_i, g\right)$, then $G \equiv 0$.*

Proof. Indeed, we know that, for all $X \in \Gamma L$, $\mathcal{L}_X g|_{L^\perp} = 0$. In particular this condition implies that

$$(\mathcal{L}_X g)(Y, Y') = 0$$

for all $X \in \Gamma L$ and $Y, Y' \in \Gamma Q$. The previous formula is equivalent to

$$g\left(\nabla_Y Y' + \nabla_{Y'} Y, X\right) = 0,$$

where ∇ denotes the Levi Civita connection on (M, g). Therefore, applying the previous lemma, we get $G \equiv 0$. □

Remark 5.5. *In general for $r > 1$ the converse in the statement of Theorem 5.4 is not true. Indeed a simple computation shows that, for each $i \in \{1, \dots, r\}$ and for all $X \in \Gamma L$, $Y \in \Gamma Q$,*

$$(\mathcal{L}_X g)(Y, \xi_i) = g\left(Y, \frac{1}{r}\left[X, \overline{\xi}\right] - [X, \xi_i]\right)$$

which is zero if and only if $r = 1$.

6 Bott connection on Legendrian foliations

Consider the Bott connection on $L^\perp = Q \oplus \mathbb{R}\xi_1 \oplus \ldots \oplus \mathbb{R}\xi_r$, given by

$$\nabla^{L^\perp}_X Y := p_{L^\perp}([X, Y]),$$

for all $X \in \Gamma L$, $Y \in \Gamma L^\perp$. Then $\nabla^{L^\perp}$ determines a Bott partial connection $\nabla^{L^{\perp^*}}$ on the dual bundle $L^{\perp^*}$ by

$$\left(\nabla^{L^{\perp^*}}_X v\right) Y = X(v(Y)) - v\left(p_{L^\perp}([X, Y])\right) = (\mathcal{L}_X v) Y = 2dv(X, Y)$$

for $X \in \Gamma L$, $Y \in \Gamma L^\perp$, $v \in \Gamma L^{\perp^*}$. Now, if $\mathcal{F}$ is a Legendrian foliation, then $\nabla^{L^{\perp^*}}$ induces a partial connection ∇^{Q^*} defined by

$$\nabla^{Q^*}_X v := p_{Q^*}\left(\nabla^{L^{\perp^*}}_X v\right),$$

for $X \in \Gamma L$, $v \in \Gamma Q^*$. Note that, as the Bott connection $\nabla^{L^\perp}$ is flat, also the curvature of $\nabla^{L^{\perp^*}}$ vanishes. Let $\{X_1, \dots, X_n, \phi(X_1), \dots, \phi(X_n), \xi_1, \dots, \xi_r\}$ be a local orthonormal frame for M as in Proposition 2.5 and evaluate the Bott partial connection on the local orthonormal frame $\{X_1, \dots, X_n\}$ of L and the corresponding coframe $\{\theta_1, \dots, \theta_n, \eta_1, \dots, \eta_r\}$

$$\left(\nabla^{L^{\perp^*}}_{X_\alpha} \eta_i\right)(\phi(X_\beta)) = 2d\eta_i(X_\alpha, \phi(X_\beta)) = -2\theta_\alpha(\phi(X_\beta)),$$

$$\left(\nabla^{L^{\perp *}}_{X_\alpha}\eta_i\right)(\xi_j) = 2d\eta_i\left(X_\alpha,\xi_j\right) = 0 = -2\theta_\alpha\left(\xi_j\right),$$

and

$$\begin{aligned}\left(\nabla^{L^{\perp *}}_{X_\alpha}\theta_\beta\right)\left(\phi\left(X_\gamma\right)\right) &= X_\alpha\left(\theta_\beta\left(\phi\left(X_\gamma\right)\right)\right) - \theta_\beta\left(\left[X_\alpha,\phi\left(X_\gamma\right)\right]\right)\\ &= -\theta_\beta\left(\left[X_\alpha,\phi\left(X_\gamma\right)\right]\right),\end{aligned}$$

$$\left(\nabla^{L^{\perp *}}_{X_\alpha}\theta_\beta\right)(\xi_i) = X_\alpha\left(\theta_\beta\left(\xi_i\right)\right) - \theta_\beta\left(\left[X_\alpha,\xi_i\right]\right) = -\theta_\beta\left(\left[X_\alpha,\xi_i\right]\right).$$

¿From these relations it follows that

$$\nabla^{L^{\perp *}}_{X_\alpha}\eta_i = -2\theta_\alpha, \tag{6.7}$$

$$\nabla^{L^{\perp *}}_{X_\alpha}\theta_\beta = -\sum_{\gamma=1}^{n}\theta_\beta\left(\left[X_\alpha,\phi\left(X_\gamma\right)\right]\right)\theta_\gamma - \sum_{j=1}^{r}\theta_\beta\left(\left[X_\alpha,\xi_j\right]\right)\eta_j, \tag{6.8}$$

for any $\alpha,\beta\in\{1,\ldots,n\}$ and $i\in\{1,\ldots,r\}$. In particular, (6.8) implies

$$\nabla^{Q^*}_{X_\alpha}\theta_\beta = p_{Q^*}\left(\nabla^{L^{\perp *}}_{X_\alpha}\theta_\beta\right) = -\sum_{\gamma=1}^{n}\theta_\beta\left(\left[X_\alpha,\phi\left(X_\gamma\right)\right]\right)\theta_\gamma.$$

Then the partial connection 1-form of $\nabla^{L^{\perp *}}$ corresponds to the $(n+r)\times(n+r)$ matrix form $\omega = (\omega_{hk})$ determined by the relation

$$\nabla^{L^{\perp *}}\begin{pmatrix}\theta_\alpha\\ \eta_i\end{pmatrix} = -\begin{pmatrix}\omega_{\alpha\beta} & \omega_{\alpha n+j}\\ \omega_{n+i\beta} & 0\end{pmatrix}\otimes\begin{pmatrix}\theta_\beta\\ \eta_j\end{pmatrix}.$$

Hence $(\omega_{\alpha\beta})_{1\leq\alpha,\beta\leq n}$ is the relative partial connection form for ∇^{Q^*}. Comparing the last equation with (6.7) and (6.8) we get:

$$\omega_{\beta\gamma}\left(X_\alpha\right) = \theta_\beta\left(\left[X_\alpha,\phi\left(X_\gamma\right)\right]\right), \tag{6.9}$$

$$\omega_{\beta n+j}\left(X_\alpha\right) = \theta_\beta\left(\left[X_\alpha,\xi_j\right]\right) = g\left(\phi\left(X_\beta\right),\left[X_\alpha,\xi_j\right]\right), \tag{6.10}$$

$$\omega_{n+i\gamma}\left(X_\alpha\right) = 2\delta_{\gamma\alpha}. \tag{6.11}$$

As the Bott connection is flat along L, we have $d\omega+\omega\wedge\omega=0$, from which we deduce the equations:

$$d\omega_{\alpha\beta} + \sum_{\gamma=1}^{n}\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta} + \sum_{l=1}^{n}\omega_{\alpha n+l}\wedge\omega_{n+l\beta} = 0, \tag{6.12}$$

$$d\omega_{\alpha n+j} + \sum_{\gamma=1}^{n}\omega_{\alpha\gamma}\wedge\omega_{\gamma n+j} = 0, \tag{6.13}$$

$$d\omega_{n+i\beta}+\sum_{\gamma=1}^{n}\omega_{n+i\gamma}\wedge\omega_{\gamma\beta}=0, \tag{6.14}$$

$$\sum_{\gamma=1}^{n}\omega_{n+i\gamma}\wedge\omega_{\gamma n+j}=0. \tag{6.15}$$

Now we are going to define a partial connection on L whose curvature is related, as we will see, to the invariant Π. First consider the isomorphism $\Psi : L \longrightarrow Q^*$ given by $\Psi(X)=\frac{1}{2}i_X\Phi$, for any $X\in\Gamma L$. We define, for all $X,X'\in\Gamma L$,

$$\widetilde{\nabla}^L_X X' := \Psi^{-1}\left(\nabla^{Q^*}_X\Psi(X')\right).$$

$\widetilde{\nabla}^L$ is a partial connection along L. Note that in the local orthonormal frame $\{X_1,\ldots,X_n\}$ of L one has $\Psi(X_\alpha)=-\theta_\alpha$ and, using (6.8),

$$\begin{aligned}\widetilde{\nabla}^L_{X_\alpha}X_\beta &= \Psi^{-1}\left(\nabla^{Q^*}_{X_\alpha}(\Psi(X_\beta))\right)=-\Psi^{-1}\left(\nabla^{Q^*}_{X_\alpha}\theta_\beta\right)\\ &=\Psi^{-1}\left(\sum_{\delta=1}^{n}\theta_\beta([X_\alpha,\phi(X_\delta)])\,\theta_\delta\right)=-\sum_{\delta=1}^{n}\theta_\beta([X_\alpha,\phi(X_\delta)])\,X_\delta.\end{aligned}$$

Proposition 6.1. *$\widetilde{\nabla}^L$ is torsion free and has curvature given by:*

$$\widetilde{R}^L(X,X')X''=\frac{1}{2}\left(\Pi(X,X'')X'-\Pi(X',X'')X\right)$$

for all $X,X',X''\in\Gamma L$.

Proof. Indeed, to compute the torsion and the curvature of $\widetilde{\nabla}^L$ it is sufficient to evaluate them on a local orthonormal frame $\{X_1,\ldots,X_n\}$ for L.

$$\begin{aligned}\Psi\left(\widetilde{T}^L(X_\alpha,X_\beta)\right)&=-\nabla^{Q^*}_{X_\alpha}\theta_\beta+\nabla^{Q^*}_{X_\beta}\theta_\alpha-\Psi([X_\alpha,X_\beta])\\ &=\frac{1}{2}\left(\nabla^{Q^*}_{X_\alpha}\nabla^{L^{\perp^*}}_{X_\beta}\eta_i-\nabla^{Q^*}_{X_\beta}\nabla^{L^{\perp^*}}_{X_\alpha}\eta_i\right)-\Psi([X_\alpha,X_\beta])\\ &=\frac{1}{2}p_{Q^*}\left(\nabla^{L^{\perp^*}}_{X_\alpha}\nabla^{L^{\perp^*}}_{X_\beta}\eta_i-\nabla^{L^{\perp^*}}_{X_\beta}\nabla^{L^{\perp^*}}_{X_\alpha}\eta_i\right)-\Psi([X_\alpha,X_\beta])\\ &=\frac{1}{2}p_{Q^*}\left(\nabla^{L^{\perp^*}}_{X_\alpha}\nabla^{L^{\perp^*}}_{X_\beta}\eta_i-\nabla^{L^{\perp^*}}_{X_\beta}\nabla^{L^{\perp^*}}_{X_\alpha}\eta_i-\nabla^{L^{\perp^*}}_{[X_\alpha,X_\beta]}\eta_i\right)\\ &=\frac{1}{2}p_{Q^*}\left(R^{L^{\perp^*}}(X_\alpha,X_\beta)\right)=0,\end{aligned}$$

since $\nabla^{L^{\perp^*}}$ is flat. As Ψ is an isomorphism we get $\widetilde{T}^L(X_\alpha,X_\beta)=0$ for any $\alpha,\beta\in\{1,\ldots,n\}$. Now we compute the curvature.

$$\begin{aligned}\Psi\left(\widetilde{R}^L(X_\alpha,X_\beta)X_\gamma\right)&=\nabla^{Q^*}_{X_\alpha}\Psi\left(\widetilde{\nabla}^L_{X_\beta}X_\gamma\right)-\nabla^{Q^*}_{X_\beta}\Psi\left(\widetilde{\nabla}^L_{X_\alpha}X_\gamma\right)-\nabla^{Q^*}_{[X_\alpha,X_\beta]}\Psi(X_\gamma)\\ &=\nabla^{Q^*}_{X_\alpha}\nabla^{Q^*}_{X_\beta}\Psi(X_\gamma)-\nabla^{Q^*}_{X_\beta}\nabla^{Q^*}_{X_\alpha}\Psi(X_\gamma)-\nabla^{Q^*}_{[X_\alpha,X_\beta]}\Psi(X_\gamma)\\ &=\nabla^{Q^*}_{X_\alpha}\nabla^{Q^*}_{X_\beta}\theta_\gamma-\nabla^{Q^*}_{X_\beta}\nabla^{Q^*}_{X_\alpha}\theta_\gamma-\nabla^{Q^*}_{[X_\alpha,X_\beta]}\theta_\gamma=-R^{Q^*}(X_\alpha,X_\beta)\,\theta_\gamma\end{aligned}$$

So we have obtained

$$\Psi\left(\widetilde{R}^{L}\left(X_{\alpha},X_{\beta}\right)X_{\gamma}\right)=-R^{Q^{*}}\left(X_{\alpha},X_{\beta}\right)\theta_{\gamma}. \tag{6.16}$$

Using this formula and equation (6.12) we have:

$$\begin{aligned}
\widetilde{R}^{L}\left(X_{\alpha},X_{\beta}\right)X_{\gamma}&=\Psi^{-1}\left(\sum_{\sigma=1}^{n}\Omega_{\sigma\gamma}\left(X_{\alpha},X_{\beta}\right)\theta_{\sigma}\right)\\
&=\sum_{\sigma=1}^{n}\left(d\omega_{\gamma\sigma}+\sum_{\rho=1}^{n}\omega_{\gamma\rho}\wedge\omega_{\rho\sigma}\right)\left(X_{\alpha},X_{\beta}\right)X_{\sigma}\\
&=-\sum_{\sigma=1}^{n}\sum_{l=1}^{r}\left(\omega_{\gamma n+l}\wedge\omega_{n+l\sigma}\right)\left(X_{\alpha},X_{\beta}\right)X_{\sigma}\\
&=-\frac{1}{2}\sum_{\sigma=1}^{n}\sum_{l=1}^{r}\left(\omega_{\gamma n+l}\left(X_{\alpha}\right)\omega_{n+l\sigma}\left(X_{\beta}\right)-\omega_{\gamma n+l}\left(X_{\beta}\right)\omega_{n+l\sigma}\left(X_{\alpha}\right)\right)X_{\sigma}\\
&=-\sum_{\sigma=1}^{n}\sum_{l=1}^{r}\left(\theta_{\gamma}\left(\left[X_{\alpha},\xi_{l}\right]\right)\delta_{\beta\sigma}-\theta_{\gamma}\left(\left[X_{\beta},\xi_{l}\right]\right)\delta_{\alpha\sigma}\right)X_{\sigma}=\\
&=\sum_{l=1}^{r}\Phi\left(X_{\gamma},\left[X_{\alpha},\xi_{l}\right]\right)X_{\beta}-\sum_{l=1}^{r}\Phi\left(X_{\gamma},\left[X_{\beta},\xi_{l}\right]\right)X_{\alpha}=\\
&=\frac{1}{2}\left(\Pi\left(X_{\alpha},X_{\gamma}\right)X_{\beta}-\Pi\left(X_{\beta},X_{\gamma}\right)X_{\alpha}\right)
\end{aligned}$$

and so we get the result. □

Corollary 6.2. *If $\mathcal{F}$ is a flat Legendrian foliation on an almost $\mathcal{S}$-manifold M^{2n+r} then $\widetilde{R}^{L}\equiv 0$.*

So the last corollary justifies the name "flat" for a Legendrian foliation characterized by the vanishing of Π.

7 Bi-Legendrian structures

By a *bi-Legendrian structure* on the almost $\mathcal{S}$-manifold $\left(M^{2n+r},\phi,\eta_i,\xi_i,g\right)$ we mean a couple $(\mathcal{F},\mathcal{G})$ of two complementary Legendrian foliations of M. In this section we examine an important class of examples of bi-Legendrian structures. More precisely, we know that, as $\phi|_{\mathcal{H}}$ is an isomorphism, $Q=\phi\left(L\right)$ is a n-dimensional subbundle of $\mathcal{H}$, so if it is involutive it defines a Legendrian foliation which, in analogy with [5], we will call the *conjugate Legendrian foliation* of $\mathcal{F}$. In this section we deal with this very particular situation and give conditions which ensure the integrability of Q, involving the tensor field N defined in (2.1).

Theorem 7.1. *Let $\mathcal{F}$ be a non-degenerate Legendrian foliation on an almost $\mathcal{S}$-manifold $\left(M^{2n+r},\phi,\xi_i,\eta_i,g\right)$ with $g|_L=\frac{1}{4r}\Pi$. Then the bundle Q is integrable if and only if $p_L\left(N\left(X,X'\right)\right)=0$ for all $X,X'\in\Gamma L$. Moreover, if $\overline{\xi}$ is a Killing vector field and Q is integrable, then $g|_Q=\frac{1}{4r}\overline{\Pi}|_Q=\frac{1}{4r}\left(\phi^{*}\Pi\right)|_Q$ and $N\left(X,X'\right)=0$ for all $X,X'\in\Gamma L$.*

Proof. Firstly observe that, for any $X, X' \in \Gamma L$, $N(X, X') = [\phi(X), \phi(X')] - [X, X'] - \phi([\phi(X), X']) - \phi([X, \phi(X')])$. Let $\{X_1, \ldots, X_n\}$ be a local orthonormal frame for L. Then from (6.13) and (6.9), (6.10), (6.11) we have, for any $\alpha, \beta \in \{1, \ldots, n\}$, $j \in \{1, \ldots, r\}$,

$$\begin{aligned}
0 &= 2\left(d\omega_{\rho n+j} + \sum_{\gamma=1}^{n} \omega_{\rho\gamma} \wedge \omega_{\gamma n+j}\right)(X_\alpha, X_\beta) \\
&= X_\alpha\left(\omega_{\rho n+j}(X_\beta)\right) - X_\beta\left(\omega_{\rho n+j}(X_\alpha)\right) - \omega_{\rho n+j}([X_\alpha, X_\beta]) \\
&\quad + \sum_{\gamma=1}^{n}\left(\omega_{\rho\gamma}(X_\alpha)\,\omega_{\gamma n+j}(X_\beta) - \omega_{\rho\gamma}(X_\beta)\,\omega_{\gamma n+j}(X_\alpha)\right) \\
&= X_\alpha\left(\theta_\rho([X_\beta, \xi_j])\right) - X_\beta\left(\theta_\rho([X_\alpha, \xi_j])\right) - \theta_\rho\left([[X_\alpha, X_\beta], \xi_j]\right) \\
&\quad + \sum_{\gamma=1}^{n}\left(\theta_\rho([X_\alpha, \phi(X_\rho)])\,\theta_\gamma([X_\beta, \xi_j]) - \theta_\rho([X_\beta, \phi(X_\gamma)])\,\theta_\gamma([X_\alpha, \xi_j])\right).
\end{aligned}$$

Now, taking the sum over all j we have,

$$\begin{aligned}
0 &= \sum_{j=1}^{r} 2\left(d\omega_{\rho n+j} + \sum_{\gamma=1}^{n} \omega_{\rho\gamma} \wedge \omega_{\gamma n+j}\right)(X_\alpha, X_\beta) \\
&= X_\alpha\left(\theta_\rho\left([X_\beta, \overline{\xi}]\right)\right) - X_\beta\left(\theta_\rho\left([X_\alpha, \overline{\xi}]\right)\right) - \theta_\rho\left([[X_\alpha, X_\beta], \overline{\xi}]\right) \\
&\quad + \sum_{\gamma=1}^{n}\left(\theta_\rho([X_\alpha, \phi(X_\gamma)])\,\theta_\gamma\left([X_\beta, \overline{\xi}]\right) - \theta_\rho([X_\beta, \phi(X_\gamma)])\,\theta_\gamma\left([X_\alpha, \overline{\xi}]\right)\right).
\end{aligned}$$

Note that, using Lemma 4.5, we see that $\theta_\beta\left([X_\alpha, \overline{\xi}]\right) = g\left(\phi(X_\beta), [X_\alpha, \overline{\xi}]\right) = 2rg(\phi(X_\beta), \phi(X_\alpha)) = 2rg(X_\alpha, X_\beta) = 2r\delta_{\alpha\beta}$. So the last equation becomes:

$$\begin{aligned}
0 &= -2rg([X_\alpha, X_\beta], X_\rho) \\
&\quad + 2r\sum_{\gamma=1}^{n}\left(g([X_\alpha, \phi(X_\gamma)], \phi(X_\delta))\,\delta_{\gamma\beta} - g([X_\beta, \phi(X_\rho)], \phi(X_\rho))\,\delta_{\gamma\alpha}\right) \\
&= -2r\left(g([X_\alpha, X_\beta], X_\rho) - g(\phi([X_\alpha, \phi(X_\beta)]), X_\delta) + g(\phi([X_\beta, \phi(X_\alpha)]), X_\rho)\right)
\end{aligned}$$

that is

$$g\left([X_\alpha, X_\beta] + \phi([X_\alpha, \phi(X_\beta)]) + \phi([\phi(X_\alpha), X_\beta]), X_\rho\right) = 0$$

for all $\alpha, \beta, \delta \in \{1, \ldots, n\}$. In particular it follows that, for all $X, X' \in \Gamma L$,

$$p_L\left([X, X'] + \phi([X, \phi(X')]) + \phi([\phi(X), X'])\right) = 0,$$

and so $p_L(N(X, X')) = p_L([\phi(X), \phi(X')])$. Therefore Q is integrable if and only if $p_L(N(X, X')) = 0$.
Now we pass to the second part of the theorem. First of all note that, since $\overline{\xi}$ is Killing we have $h = 0$, hence, for all $X \in \Gamma L$, $[\overline{\xi}, \phi(X)] = \phi\left([\overline{\xi}, X]\right)$ and then

$$p_L\left([\overline{\xi}, \phi(X)]\right) = p_L\left(\phi\left([\overline{\xi}, X]\right)\right) = \phi\left(p_Q\left([\overline{\xi}, X]\right)\right). \tag{7.17}$$

Now suppose that Q is integrable and take $Y \in \Gamma Q$. There exists a unique $X \in \Gamma L$ such that $Y = \phi(X)$. Thus, by Lemma 4.5, $p_L\left(\left[\overline{\xi}, Y\right]\right) = p_L\left(\left[\overline{\xi}, \phi(X)\right]\right) = \phi(2r\phi(X)) = -2rX = 2r\phi(Y)$ and so, using again Lemma 4.5, $g|_Q = \frac{1}{4r}\Pi^Q$, where Π^Q is the invariant of the Legendrian foliation Q defined in Section 3. But in this case we can prove that, for all $X, X' \in \Gamma L$, $\Pi^Q(\phi(X), \phi(X')) = \Pi(X, X')$. Indeed, since $\overline{\xi}$ is Killing,

$$\begin{aligned}\Pi^Q(\phi(X), \phi(X')) &= 2g\left(\left[\overline{\xi}, \phi(X)\right], \phi^2(X')\right) = -2g\left(\left[\overline{\xi}, \phi(X)\right], X'\right)\\ &= -2g\left(\phi\left(\left[\overline{\xi}, X\right]\right), X'\right) = 2g\left(\left[\overline{\xi}, X\right], \phi(X')\right) = \Pi(X, X').\end{aligned}$$

Moreover Π^Q coincides also with $\overline{\Pi}|_Q$, because, using equation (4.6)

$$\begin{aligned}&\overline{\Pi}(\phi(X), \phi(X')) = \Pi(\lambda(\phi(X)), \lambda(\phi(X')))\\ &= \Pi\left(\frac{1}{2r}\lambda\left(\left[\overline{\xi}, X\right]_Q\right), \frac{1}{2r}\lambda\left(\left[\overline{\xi}, X'\right]_Q\right)\right) = \Pi\left(\frac{1}{2r}\lambda\left(\left[\overline{\xi}, X\right]\right), \frac{1}{2r}\lambda\left(\left[\overline{\xi}, X'\right]\right)\right)\\ &= \Pi(X, X') = \Pi^Q(\phi(X), \phi(X')).\end{aligned}$$

It remains to prove that $N(X, X') = 0$ for all $X, X' \in \Gamma L$. The first part of the theorem implies that $p_L(N(X, X')) = 0$ and $p_Q(N(\phi(X), \phi(X'))) = 0$. But, $N(\phi(X), \phi(X')) = \left[\phi^2(X), \phi^2(X')\right] - [\phi(X), \phi(X')] - \phi\left(\left[\phi^2(X), \phi(X')\right]\right) - \phi\left(\left[\phi(X), \phi^2(X')\right]\right) = [X, X'] - [\phi(X), \phi(X')] + \phi([X, \phi(X')]) + \phi([\phi(X), X']) = -N(X, X')$, from which we conclude that $N(X, X') = 0$ because for all $i \in \{1, \dots, r\}$, $g(N(X, X'), \xi_i) = 0$. □

Remark 7.2. *Let $\mathcal{F}$ be a Legendrian foliation such that its conjugate exists. Then, applying twice the last theorem, we conclude that $N(X, X') = 0$ for all $X, X' \in \Gamma L$ and $N(Y, Y') = 0$ for all $Y, Y' \in \Gamma Q$. In particular we can prove the following*

Theorem 7.3. *Let $\mathcal{F}$ be a non-degenerate Legendrian foliation on the almost $\mathcal{S}$-manifold $(M, \eta_i, \xi_i, \phi, g)$ with $g|_L = \frac{1}{4r}\Pi$. Then $(M, \eta_i, \xi_i, \phi, g)$ is an $\mathcal{S}$-manifold, i.e. $N \equiv 0$, if and only if each ξ_i is a Killing vector field and Q is integrable.*

Proof. Suppose that Q is integrable and, for all $i \in \{1, \dots, r\}$, ξ_i is a Killing vector field. In particular $\overline{\xi}$ is a Killing vector field and we can apply Theorem 7.1, obtaining $N(X, X') = 0$ and $N(Y, Y') = 0$ for all $X, X' \in \Gamma L$, $Y, Y' \in \Gamma Q$. Moreover, since each ξ_i is Killing we have $h_i = 0$ for all $i \in \{1, \dots, r\}$. So, applying (2.3), we get $\phi(N(X, X')) + N(\phi(X), X') = 0$ for all $X, X' \in \Gamma L$, from which we deduce $N(X, Y) = 0$ for all $X \in \Gamma L$ and $Y \in \Gamma Q$ and we can conclude that $N(Z, Z') = 0$ for all $Z, Z' \in \Gamma\mathcal{H}$. Finally, from (2.4) we deduce that $g(N(Z, \xi_j), \xi_i) = 0$ and, from (2.3), $\phi(N(Z, \xi_j)) = 0$ for all $Z \in \Gamma\mathcal{H}$. Thus $N(Z, \xi_j) \in \mathcal{H} \cap span\{\xi_1, \dots, \xi_r\}$ and so vanishes. Since, obviously, $N(\xi_i, \xi_j) = 0$ we conclude that $N \equiv 0$ and $(M, \eta_i, \xi_i, \phi, g)$ is an $\mathcal{S}$-manifold. On the contrary, if the tensor field N vanishes identically, then each ξ_i is Killing and, applying Theorem 7.1, we deduce that Q is integrable. □

Directly by (7.17) and by Proposition 3.4 we deduce the following theorem, which elucidates relations between a Legendrian foliation and its conjugate.

Theorem 7.4. *Let $\left(M^{2n+r}, \phi, \xi_i, \eta_i, g\right)$ be an almost $\mathcal{S}$-manifold such that $\overline{\xi}$ is a Killing vector field and $\mathcal{F}$ a Legendrian foliation on M such that the conjugate Legendrian foliation exists. Then the conjugate belongs to the same class of $\mathcal{F}$ as in Proposition 3.4.*

Remark 7.5. *Assuming that each ξ_i is a Killing vector field, the statement of Theorem 7.4 applies also to strongly flat Legendrian foliations.*

The last theorem enables us to define a connection adapted to a bi-Legendrian structure $(\mathcal{F},\mathcal{G})$ on an almost $\mathcal{S}$-manifold M. Let L and Q denote the tangent bundles of $\mathcal{F}$ and $\mathcal{G}$, respectively, and $E=\bigoplus_{i=1}^{r}\mathbb{R}\xi_i$. Consider the connection $\widetilde{\nabla}^L$ introduced in Section 6 and let ∇' be the partial connection along L defined by:

$$\nabla'_X V := \begin{cases} \widetilde{\nabla}^L_X V, & \text{if } V\in\Gamma L \\ \nabla^{L^\perp}_X V, & \text{if } V\in\Gamma L^\perp \end{cases}$$

for all $X\in\Gamma L$. The same construction can be repeated for Q, so we have a partial connection ∇'' along Q, given by

$$\nabla''_Y Z := \begin{cases} \widetilde{\nabla}^Q_Y V, & \text{if } V\in\Gamma Q \\ \nabla^{Q^\perp}_Y V, & \text{if } V\in\Gamma Q^\perp \end{cases}$$

for all $Y\in\Gamma Q$. Finally we define a partial connection ∇''' along E setting

$$\nabla'''_Z V := \begin{cases} p_L\left([Z,V]\right), & \text{if } V\in\Gamma L \\ p_Q\left([Z,V]\right), & \text{if } V\in\Gamma Q \\ \sum_{j=1}^{r} Z\left(\eta_j\left(V\right)\right)\xi_j, & \text{if } V\in\Gamma E \end{cases}$$

for all $Z\in\Gamma E$. So we can define a global connection $\overline{\nabla}$ on M putting, for all $V,W\in\Gamma TM$,

$$\overline{\nabla}_W V := \nabla'_{W_L}V + \nabla''_{W_Q}V + \nabla'''_{W_E}V, \tag{7.18}$$

where we have decomposed W as $W=W_L+W_Q+W_E$, using the decomposition $TM=L\oplus Q\oplus E$. This construction can be done without any hypothesis on the bi-Legendrian structure $(\mathcal{F},\mathcal{G})$. In particular, it is very interesting the case of strongly flat bi-Legendrian structures. Indeed we have

Theorem 7.6. *Let $(\mathcal{F},\mathcal{G})$ be a bi-Legendrian structure on an almost $\mathcal{S}$-manifold M such that both $\mathcal{F}$ and $\mathcal{G}$ are strongly flat. Then for the connection $\overline{\nabla}$ adapted to $(\mathcal{F},\mathcal{G})$ we have*

(i) $\overline{R}\left(X,X'\right)=0$ *for all* $X,X'\in\Gamma L$*,* $\overline{R}\left(Y,Y'\right)=0$ *for all* $Y,Y'\in\Gamma Q$*;*

(ii) $\overline{T}\left(V,W\right)=-2\Phi\left(V,W\right)\bar{\xi}$.

Proof. The flatness of $\mathcal{F}$ and of $\mathcal{G}$ implies $\widetilde{R}^L\equiv 0$ and $\widetilde{R}^Q\equiv 0$. Then, since also the Bott connection is flat, it follows that ∇' and ∇'' are flat connections along L and Q, respectively, so the curvature of $\overline{\nabla}$ vanishes along the leaves of the foliations $\mathcal{F}$ and $\mathcal{G}$ and (i) is proved. Now we prove (ii). Take $X\in\Gamma L$ and $Y\in\Gamma Q$. Then, with the above notations,

$$\begin{aligned}\overline{T}(X,Y) &= \nabla'_X Y - \nabla''_Y X - [X,Y] = \nabla^{L^\perp}_X Y - \nabla^{Q^\perp}_Y X - [X,Y]\\ &= p_{L^\perp}([X,Y]) - p_{Q^\perp}([Y,X]) - [X,Y]\\ &= [X,Y]_Q + [X,Y]_E - [Y,X]_L - [Y,X]_E - [X,Y]\\ &= \sum_{i=1}^r \eta_i([X,Y])\xi_i = -\sum_{i=1}^r 2d\eta_i(X,Y)\xi_i = -2\Phi(X,Y)\overline{\xi}.\end{aligned}$$

The same formula holds in the other cases. Indeed first of all note that, for all $V \in \Gamma TM$, $\overline{\nabla}_V \xi_i = p_{L^\perp}([V_L,\xi_i]) + p_{Q^\perp}([V_Q,\xi_i]) = 0$, as L and Q are strongly flat. Hence

$$\begin{aligned}\overline{T}(V,\xi_i) &= \overline{\nabla}_V\xi_i - \overline{\nabla}_{\xi_i}V - [V,\xi_i] = -\nabla'''_{\xi_i}V - [V,\xi_i] = -[\xi_i,V_L]_L - [\xi_i,V_Q]_Q\\ &- \sum_{j=1}^r \xi_i(\eta_j(V))\xi_j - [V_L,\xi_i] - [V_Q,\xi_i] - \sum_{j=1}^r [\eta_j(V)\xi_j,\xi_i] = 0\end{aligned}$$

and also $-2\Phi(V,\xi_i)\overline{\xi} = 0$. Finally, if $X, X' \in \Gamma L$ then $\overline{T}(X,X') = \widetilde{\nabla}^L_X X' - \widetilde{\nabla}^L_{X'} X - [X,X'] = \widetilde{T}(X,X') = 0$, because $\widetilde{\nabla}^L$ is torsion free. Note that also $\Phi(X,X') = 0$, so $\overline{T}(X,X') = -2\Phi(X,X')\overline{\xi}$. Similarly, for any $Y,Y' \in \Gamma Q$, $\overline{T}(Y,Y') = -2\Phi(Y,Y')\overline{\xi} = 0$. □

Corollary 7.7. *Let $\mathcal{F}$ a strongly flat Legendrian foliation on an almost $\mathcal{S}$-manifold (M,η_i,ξ_i,ϕ,g) such that each ξ_i is a Killing vector field, and suppose that $Q = \phi(L)$ is integrable. Then the connection $\overline{\nabla}$ adapted to the bi-Legendrian structure (L,Q) satisfies properties (i) and (ii) of Theorem 7.6.*

Remark 7.8. *From the proof of Theorem 7.6 it follows that for any bi-Legendrian structure, without the hypothesis of strongly flatness, the relations $\overline{T}(X,Y) = -2\Phi(X,Y)\overline{\xi}$, $\overline{T}(X,X') = 0$ and $T(Y,Y') = 0$, for all $X,X' \in \Gamma L$, $Y,Y' \in \Gamma Q$, are still true.*

Example 7.9. Consider $\mathbb{R}^{2n+r}$ with coordinates $x_1,\dots,x_r,y_1,\dots,y_n,z_1,\dots,z_r$ and its standard almost $\mathcal{S}$-structure (ϕ,η_i,ξ_i,g) where

$$\eta_i = dz_i - \sum_{j=1}^n y_j dx_j,\ \xi_i = \frac{\partial}{\partial z_i}$$

$$g = \sum_{i=1}^r \eta_i \otimes \eta_i + \frac{1}{2}\sum_{j=1}^n \left((dx_j)^2 + (dy_j)^2\right)$$

and ϕ is represented by the matrix

$$\begin{pmatrix} 0 & I_n & 0\\ -I_n & 0 & 0\\ 0 & Y & 0\end{pmatrix}$$

where Y is the $(r \times n)$-matrix given by

$$\begin{pmatrix} y_1 & \cdots & y_n \\ \vdots & \ddots & \vdots \\ y_1 & \cdots & y_n \end{pmatrix}$$

It is not difficult to check that $\Phi = d\eta_1 = \cdots = d\eta_r = \sum_{i=1}^{n} dx_i \wedge dy_i$. Set, for each $k \in \{1, \ldots, n\}$, $X_k := \frac{\partial}{\partial y_k}$ and $Y_k := \frac{\partial}{\partial x_k} + y_k \sum_{\alpha=1}^{r} \frac{\partial}{\partial z_\alpha}$. Note that $\phi(X_k) = Y_k$. Now define $L := span\{X_1, \ldots, X_n\}$, $Q := span\{Y_1, \ldots, Y_n\}$, obtaining in this way two strongly flat Legendrian foliations on $\mathbb{R}^{2n+r}$ such that $\phi(L) = Q$. Let $\overline{\nabla}$ denote the connection associated to (L, Q) as in (7.18). Then the curvature tensor of $\overline{\nabla}$ vanishes identically. Indeed by a straightforward computation one has $\overline{\nabla}_{\frac{\partial}{\partial y_i}} \frac{\partial}{\partial y_j} = \widetilde{\nabla}^L_{\frac{\partial}{\partial y_i}} \frac{\partial}{\partial y_j} = 0$, $\overline{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$, $\overline{\nabla}_{\frac{\partial}{\partial y_i}} \frac{\partial}{\partial x_j} = 0$, $\overline{\nabla}_{\frac{\partial}{\partial z_\alpha}} \frac{\partial}{\partial y_i} = \left[\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial y_i}\right]_L = 0$ and $\overline{\nabla}_{\frac{\partial}{\partial z_\alpha}} \frac{\partial}{\partial x_i} = 0$, from which $\overline{R} \equiv 0$.

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