Symmetries of second order potential differential systems

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Abstract

We characterize the family of second order potential differential systems, with n degrees of freedom, via their symmetries.

Firstly, we calculate explicitly the equivalence Lie algebra and the weak equivalence Lie algebra. It is shown that the equivalence Lie algebra has the dimension $n + 4 + \frac{n(n-1)}{2}$ whereas the weak equivalence Lie algebra is infinite-dimensional. The later contains strictly the former.

Secondly, we investigate the Lie-point symmetry structure. We start with a quadratic potential, and we provide an analysis relying on the spectral theorem. In the case of non-quadratic potentials, we establish the conditions for the existence of additional symmetries, deriving the classifying conditions. These conditions are greatly simplified under the action of the equivalence group.

Finally we show how the Lie point symmetries can be obtained using (weak) equivalence transformations and we give an example where the existence of symmetries can be used to prove integrability.

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1 Introduction

The second order potential differential system

$$\ddot{x}^i = -\frac{\partial V}{\partial x^i}, \quad i = 1, ..., n, \tag{1.1}$$

associated to a potential $V = V(x^1, \ldots, x^n)$, models conservative phenomena amongst which we can cite motion in molecular systems and multibody mechanical systems. These are Euler-Lagrange equations associated to the Lagrangian $L = \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j - V(x)$ (Einstein's summation convention is assumed).

Particularly, the least squares Lagrangian

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$$L = \frac{1}{2}\delta_{ij}(\dot{x}^i - X^i(x))(\dot{x}^j - X^j(x))$$

(see Geometric Dynamics [27]), associated to the irrotational vector field $X = X^i \frac{\partial}{\partial x^i}$, produces the Euler-Lagrange equations

$$\ddot{x}^i = \frac{\partial f}{\partial x^i}, \quad i = 1, ..., n, \tag{1.2}$$

where $f(x) = \frac{1}{2}\delta_{ij}X^i(x)X^j(x)$ is a density of energy.

Of course, it is well known that the Hamiltonian $H = \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j + V(x)$ is a first integral. Sometimes, the Hamiltonian H can be used like Lyapunov function attached to the Hamiltonian vector field, to decide the stability or unstability of the (first order) Hamiltonian differential system [27].

The differential system (1) is called *integrable* if it admits n first integrals (integrals of motion depending on velocity). The systematic investigation of the integrability of differential systems (1) took off in the nineteenth century with the pioneering works of Poisson, Hamilton, Ostrogradskii and Liouville and was consolidated in the early twentieth century with the classical work of Whittaker [26] and later with the work of Arnold [1].

Whittaker [26] looked for differential systems of type (1), with two degrees of freedom, admitting a second integral of motion quadratic in the velocity. He found only one potential of the two possible classes which possess second-order invariants. The second potential was obtained by Sen [24]. Hietarinta [11, 12, 13, 14] combines ad-hoc methods and computer algebra in his search for 2-dimensional integrable differentiable systems of type (1), with polynomial potentials. Grammaticos *et al* [10] showed how to extend some integrable 2-dimensional systems to integrable n-dimensional systems. Also, Dorizzi *et al* [7] investigated integrable three-dimensional Hamiltonian systems with quartic potentials. Calogero [3, 4, 5] investigated integrable non-relativistic n-body problem.

Seifert [29], Gordon [33], Weinstein [31], Mawhin-Willem [30] proved the existence of trajectories of the system (1) with arbitrary given energy which join two fixed points and have arbitrary topological type.

The common feature of available integrable potential differential systems is that they possess symmetries (translation, scaling, rotation,...). Therefore it makes sense to look for integrable systems amongst systems admitting symmetries. This approach initiated by Sophus Lie, has been used recently for general potential differential systems with two degrees of freedom by Sen [23]. The work of Sen was rediscovered by Damianou *et al* [6]. Some authors have rather focus on specific potential systems: Sahadevan and Lakshmanan [22] performed a symmetry analysis of Hénon-Heiles and two coupled quartic harmonic oscillator systems. Duarte *et al* [8] work out the symmetry Lie algebra of the n-dimensional harmonic oscillator. Bryant [32] refers to the general approaches of Lie symmetries.

In this work we obtain general results about the symmetry analysis of the family of n-dimensional potential differential systems. For that Section 2 contains some remarks about the case where the potential V is a quadratic form. This case arise naturally during the symmetry analysis of the general case. In Section 3 we calculate the equivalence and the weak equivalence group which is common for all potential systems of type (1.1). In Section 4 we obtain a classifying condition for (1.1) and we use the equivalence group to simplify it. In Section 5, we show how we can use the (weak) equivalence group to obtain the symmetries of (1.1). Section 6 examines an example with two first integrals of motion. In the final section, we summarize our results.

In the sequel we assume that the reader is familiar with basic notions of Lie's symmetry theory [2, 19, 20, 25] so that routine calculations are omitted.

2 Potential differential systems with quadratic potentials

Here we assume that the potential V is a quadratic form, i.e.,

$$V = \frac{1}{2}a_{ij}x^i x^j + b_i x^i + c, \tag{2.3}$$

where $a_{ij} = a_{ji}$, b_i and c are constants. By making the change of dependent variable $x' = x - x_p$, where x_p is a particular solution of (1.1), we can assume without loss of generality that $b_i = 0$. Also we can let $c = 0$ since it does not appear explicitly in (1.1). We are then left with

$$V = \frac{1}{2}a_{ij}x^i x^j.$$

To sum up at this point, the differential system of type (1), with quadratic potential, can be written

$$\ddot{x} = Ax, \tag{2.4}$$

where A is a symmetric matrix. Now, we know from the spectral theorem that symmetric matrices are diagonalizable. Thus there exists an invertible matrix P such that

$$A = P^{-1}DP,$$

where D is a diagonal matrix. Next, by making the change of variables

$$x' = Px,$$

equation (2.4) becomes

$$\ddot{x}' = Dx'.$$

Thus in the study of symmetries of potential systems with quadratic potential, we may assume without loss of generality that the potential is given by

$$V = \frac{1}{2}\sum_{i=1}^{n}\lambda_i (x^i)^2. \tag{2.5}$$

Using Lie algorithm, it can be shown that the symmetries of the potential system with the potential (2.5) depend on the multplicity of the eigenvalues of the matrix diag $[\lambda_1, \lambda_2, \ldots, \lambda_n]$ (see [9], section 3).

Remarks.

1) The second order ODEs system (1) can be linearized around a point or along a trajectory.

2) In time that a linearization about a point returns a time invariant linear dynamics, the linearization about a trajectory returns a linear time-varying differential system.

3) In order to linearize the second order ODEs system (1) in the neighborhood of the critical point $x = x_0$, or by translation, $x = 0$, it is enough to replace V by the quadratic part $\sum_{i,j=1}^{n} \frac{\partial^2 V}{\partial x^i \partial x^j}(0) x^i x^j$. The motions described by the Lagrangian $L = \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j - \sum_{i,j=1}^{n} \frac{\partial^2 V}{\partial x^i \partial x^j}(0) x^i x^j$ are called *small oscillations* in a neighborhood of the equilibrium point $x = 0$.

3 Equivalence and weak equivalence group of family of potential systems

Equivalence transformations of a given differential system are invertible transformations of independent and dependent variables that preserve the structure of the system. Amongst equivalence transformations, those depending continuously on a parameter and forming a group can be calculated algorithmically [20]. An equivalence transformation (1.1) will in general map a system into another of the same form, but with a different potential V, even if the potential continues to depend on the same arguments. A weak equivalence transformation can also affect the arguments of the transformed potential, i.e., $V(x)$ can be transformed to $\bar{V}(\bar{t}, \bar{x})$.

3.1 *Equivalence transformations*

Following the infinitesimal method described in [20], we look for the equivalence generator Γ in the form

$$\Gamma = \tau(t,x)\frac{\partial}{\partial t} + \xi^i(t,x)\frac{\partial}{\partial x^i} + \omega(t,x,V)\frac{\partial}{\partial V}. \tag{3.6}$$

We need the second extension $\Gamma^{[2]}$ of Γ given by

$$\tilde{\Gamma} = \Gamma + \dot{\xi}^i\frac{\partial}{\partial \dot{x}^i} + \ddot{\xi}^i\frac{\partial}{\partial \ddot{x}^i} + \omega_t\frac{\partial}{\partial V_{,t}} + \omega_i\frac{\partial}{\partial V_{,i}}, \tag{3.7}$$

where

$$\begin{aligned}
\dot{\xi}^i &= D\xi^i - \dot{x}^i D\tau, \quad \ddot{\xi}^i = D\dot{\xi}^i - \ddot{x}^i D\tau, \quad D = \frac{\partial}{\partial t} + \dot{x}^i\frac{\partial}{\partial x^i} + \cdots,\\
\omega_i &= \tilde{D}_i\omega - V_{,t}\tilde{D}_i\tau - V_{,j}\tilde{D}_i\xi^j, \quad \tilde{D}_i = \frac{\partial}{\partial x^i} + V_{,i}\frac{\partial}{\partial V} + \cdots,\\
\omega_t &= \tilde{D}_t\omega - V_{,t}\tilde{D}_t\tau - V_{,i}\tilde{D}_t\xi^i, \quad \tilde{D}_t = \frac{\partial}{\partial t} + V_{,t}\frac{\partial}{\partial V} + \cdots.
\end{aligned}$$

The invariant conditions

$$\tilde{\Gamma}(V_{,t})|_{V_{,t}=0} = 0, \quad \tilde{\Gamma}(\ddot{x}^i + V_{,i})|_{V_{,t}=0,\,(1.1)} = 0, \tag{3.8}$$

yield, after expansion and separation, the following equations

$$\omega_{,t} = 0, \quad \xi^i_{,t} = 0, \quad \tau_{,ij} = 0 \tag{3.9}$$

$$\xi^i_{,jk} - \delta^i{}_k \tau_{,tj} - \delta^i{}_j \tau_{,tk} = 0 \tag{3.10}$$

$$\delta^i{}_j \tau_{,tt} - \delta^i{}_j \tau_{,k} V_{,k} - 2V_{,i}\tau_{,j} = 0 \tag{3.11}$$

$$V_{,j}\xi^i_{,j} - 2V_{,i}\tau_{,t} = \omega_{,i} + V_{,i}\omega_{,V} - V_{,j}\xi^j_{,i}. \tag{3.12}$$

The PDEs system consisting of the last two equations in (9) and the equation (10) give

$$\tau = (A_i t + B_i)x^i + C(t), \ \xi^i = (\delta^i_k A_j + \delta^i_j A_k)x^j x^k + D^i_j x^j.$$

Substituting τ into the PDEs system (11), we obtain

$$(\delta^i_j(A_k t + B_k) + 2\delta^k_i(A_j t + B_j))V_{,k} = \delta^i_j C''.$$

Let us look this system as a linear nonhomogeneous PDEs system of n^2 equations with the unknown function V.

Case 1. Let $A_i t + B_i \neq 0$. Then

$$rank\left(\delta^i_j(A_k t + B_k) + 2\delta^k_i(A_j t + B_j)\right) = n,$$

and consequently the previous PDEs system has no solution.

Case 2. Let $A_i = 0, B_i = 0$. Then $C'' = 0$ and V rests arbitrary. In this case the relations (9)-(12) reduce to the PDEs

$$\tau_{,i} = 0, \quad \tau_{,tt} = 0$$

$$\omega_{,t} = 0, \quad \omega_{,i} = 0$$

$$\xi^i_{,t} = 0, \quad \xi^i_{,jk} = 0$$

$$\xi^i_{,l} + \xi^l_{,i} - 2\tau_{,t}\delta^l_i - \omega_{,V}\delta^l_i = 0.$$

The integration of (3.9)-(3.12) yields after some calculations

$$\tau = \alpha t + \beta, \quad \xi^i = A^i{}_j x^j + B^i, \quad A^i{}_j = -A^j{}_i, \ i \neq j, \ A^i{}_i = \lambda, \tag{3.13}$$

$$\omega = \gamma - 2(\alpha - \lambda)V, \tag{3.14}$$

where $\alpha, \beta, \gamma, \lambda$, $A^i{}_j$ and B^i are arbitrary constants. Consequently, we determined the equivalence transformations for each V-ODEs system (1).

Case 3. Suppose that $A = (A_i t + B_i)$ is a non-zero vector. If at least one component of this vector is zero , then

$$rank\left(\delta^i_j(A_k t + B_k) + 2\delta^k_i(A_j t + B_j)\right)$$

is smaller than n. If

$$rank\left(\delta^i_j(A_k t + B_k) + 2\delta^k_i(A_j t + B_j), \ \ \delta^i_j C''\right) < n,$$

then the PDEs system has solutions in V. In this case we can have completely new symmetries. PLEASE TRY TO GIVE DETAILS!

Theorem. *The equivalence Lie algebra of all potential systems of type (1.1) is generated by* $n+4+\dfrac{n(n-1)}{2}$ *vector fields:*

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_{1+i} = \frac{\partial}{\partial x^i}, \quad \Gamma_{n+2} = \frac{\partial}{\partial V}, \tag{3.15}$$

$$\Gamma_{n+3} = t\frac{\partial}{\partial t} - 2V\frac{\partial}{\partial V}, \quad \Gamma_{n+4} = x^i\frac{\partial}{\partial x^i} + 2V\frac{\partial}{\partial V}, \tag{3.16}$$

$$\Gamma_{ij} = x^i\frac{\partial}{\partial x^j} - x^j\frac{\partial}{\partial x^i}, \quad i<j, \tag{3.17}$$

where i and j vary from 1 *to n.*

The operators or vector fields (3.15)-(3.17) respectively describe translations, scalings and rotations.

3.2 *Weak equivalence transformations*

In order to compute weak equivalence transformations, which are common to all potential systems, we follow almost the same path as for equivalence transformations. The only change is that we do not impose the invariance of the equation $V_t = 0$ this time. After applying the invariance conditions and solving the resulting determining equation we obtain

$$\tau = \tau(t),\ \xi^i = \frac{1}{2}\dot\tau x^i + C^i_{\ j}x^j + B^i(t),\ C^i_{\ j} = -C^j_{\ i},\ i\neq j; C^i_i = C, \tag{3.18}$$

$$\omega = \frac{1}{4}\tau^{(3)}\delta_{ij}x^ix^j - \delta_{ij}\ddot B^i x^j - (\dot\tau - 2C)V + D(t), \tag{3.19}$$

where C and $C^i_{\ j}$ are arbitrary constants, $\tau(t)$, $D(t)$ and $B^i(t)$ are sufficiently smooth arbitrary functions.

Theorem. *The weak equivalence group is generated by the vector fields*

$$\Gamma_D = D(t)\frac{\partial}{\partial V}, \quad \Gamma_B = B^i(t)\frac{\partial}{\partial x^i} - \delta_{ij}\ddot B^i x^j\frac{\partial}{\partial V}, \tag{3.20}$$

$$\begin{aligned}\Gamma_\tau &= 4\tau(t)\frac{\partial}{\partial t} + 2\dot\tau(t)x^i\frac{\partial}{\partial x^i}\\ &+ \left(\tau^{(3)}(t)\delta_{ij}x^ix^j - 4\dot\tau(t)V\right)\frac{\partial}{\partial V}\end{aligned} \tag{3.21}$$

$$\Gamma_{n+4} = x^i\frac{\partial}{\partial x^i} + 2V\frac{\partial}{\partial V}, \quad \Gamma_{jk} = x^j\frac{\partial}{\partial x^k} - x^k\frac{\partial}{\partial x^j}, \quad j<k. \tag{3.22}$$

It is straightforward to verify that the equivalence Lie algebra of (1.1) is a proper subalgebra of its weak equivalence Lie algebra.

4 Symmetry analysis of a potential system

According to Lie's infinitesimal invariance criteria [2, 19, 20, 25], the operator

$$X = \tau(t,x)\frac{\partial}{\partial t} + \xi^i(t,x)\frac{\partial}{\partial x^i}$$

is a Lie point symmetry of (1.1) if and only if the following determining equations are satisfied

$$\xi^i_{,tt} + V_{,ij}\xi^j + 2\tau_{,t}V_{,i} - V_{,j}\xi^i_{,j} = 0, \tag{4.23}$$

$$2\xi^i_{,tj} - \delta^i{}_j\tau_{,tt} - \delta^i{}_jV_{,k}\tau_{,k} + 2V_{,i}\tau_{,j} = 0, \tag{4.24}$$

$$\xi^i_{,jk} - \delta^i{}_k\tau_{,tj} - \delta^i{}_j\tau_{,tk} = 0, \tag{4.25}$$

$$\tau_{,jk} = 0. \tag{4.26}$$

There are $\frac{1}{2}(n+1)(n^2+3n)$ equations for $n+1$ unknowns τ and ξ^i. The general solution of the PDE (26) is $\tau = A_i(t)x^i + B(t)$. Then the PDEs system (25) is reduced to $\xi^i_{,jk} = \delta^i_k\dot{A}_j + \delta^i_j\dot{A}_k$, with the general solution

$$\xi^i = (\delta^i_k\dot{A}_j + \delta^i_j\dot{A}_k)x^jx^k + C^i_j(t) + D^i(t).$$

Substituting τ and ξ^i into the PDEs system (24), we obtain

$$(\delta^i_jA_k - 2\delta^k_iA_j)V_{,k} = 2(\delta^i_k\ddot{A}_j + \delta^i_j\ddot{A}_k)x^k + 2\dot{C}^i_j - \delta^i_j(\ddot{A}_kx^k + \ddot{B}).$$

This is a linear nonhomogeneous PDEs system of n^2 equations with the unknown function V.

Case 1. Let $A_i \neq 0$. Then $\operatorname{rank}\big(\delta^i_jA_k - 2\delta^k_iA_j\big) = n$, and consequently the PDEs system has no solution.

Case 2. Let $A_i = 0$. Then V rests arbitrary and $2\dot{C}^i_j = \delta^i_j\ddot{B}$, i.e., $C^i_j = \frac{1}{2}\delta^i_j\dot{B} + e^i_j$, where e^i_j are constants.

Substituting

$$\tau = B(t), \tag{4.27}$$

$$\xi^i = \left(\frac{1}{2}\delta^i{}_j\dot{B} + e^i{}_j\right)x^j + D^i(t), \tag{4.28}$$

into (4.23) we obtain the PDEs system

$$\ddot{D}^i + \frac{1}{2}B^{(3)}x^i + V_{,ij}\left[\left(\frac{1}{2}\delta^j{}_k\dot{B} + e^j{}_k\right)x^k + D^j\right] + \frac{3}{2}\dot{B}V_{,i} - V_{,j}e^i{}_j = 0, \tag{4.29}$$

with the unknown V. Simple calculations show that the compatibility conditions for the system (4.29) are

$$(e^j{}_k + e^k{}_j)V_{,ik} - (e^i{}_k + e^k{}_i)V_{,jk} = 0. \tag{4.30}$$

The system (4.30) forms the so-called classifying conditions. In oder to simplify the analysis of (4.30), we introduce the notation

$$E^p_q = \frac{1}{2}(e^p{}_q + e^q{}_p).$$

Thus (4.30) becomes

$$E^i_{\ k} V_{,jk} - E^j_{\ k} V_{,ik} = 0. \tag{4.31}$$

Since two equivalent equations have the same symmetry structure, we can use the equivalence Lie algebra to simplify the classifying equation (4.31). In particular by using the equivalence subgroup of rotation (generated by the Γ_{ij}s) on (4.31), we are lead to $E^p_{\ q} = 0$ when $p \neq q$. That is, we do not affect the symmetry structure by assuming $E^p_{\ q} = 0$ when $p \neq q$. Thus (4.31) reduces to

$$(E^i_{\ i} - E^j_{\ j}) V_{,ij} = 0 \text{ (no summations)} . \tag{4.32}$$

The symmetry classification will rely on the solution of (4.32) which depends on the muliplicity of the eingenvalues of E.

Case 3. Suppose that $A = (A_i)$ is a non-zero vector. If at least one component of this vector is zero, then $\operatorname{rank}\big(\delta^i_j A_k - 2\delta^k_i A_j\big)$ is smaller than n. If

$$rank\left(\delta^i_j A_k - 2\delta^k_i A_j,\;\; 2(\delta^i_k \ddot{A}_j + \delta^i_j \ddot{A}_k)x^k + 2\dot{C}^i_j - \delta^i_j(\ddot{A}_k x^k + \ddot{B})\right) < n,$$

then the PDEs system in the unknown V has solutions and between them we have also a unique quadratic solution. In this case we can have completely new symmetries. PLEASE LOOK FOR DETAILS!

5 Symmetries of potential system using the equivalence group

The use of equivalence transformations to obtain partial informations on the symmetry structure of differential equations is well documented in the literature (see for instance [15, 16, 18, 20, 21] and references therein). The use of equivalence transformations to obtain symmetries is particularly important when directly symmetry classsification is too complicated.

The only common symmetry of all potential systems of type (1.1), indexed by V, is time-translation. However for some specifications of V, there might be more symmetries. In order to get optimally some of these specifications, nonsimilar subalgebras of an appropriate projection of the equivalence group can be calculated. This is a very difficult exercise that can be performed manually only for lower-dimensional algebras.

The equivalence operators of (1.1) when projected on the space of independent and dependent variables t and x^i will be a symmetry of (1.1) provided they leave invariant the equation $V - V(x) = 0$. Below we use this important property to get some extensions of the symmetry Lie algebra of (1.1). We insist on the fact that other linear combinations of the equivalence operators are also possible.

5.1 *Extensions using* Γ_B

The operator Γ_B leaves the equation $V - V(x) = 0$ invariant provided $B^i = F(t)$, $i = 1, 2, \ldots, n$, and $F(t)$ satisfies the ordinary differential equation

$$\ddot{F} - \lambda F = 0,$$

where λ is a constant. Therefore, the following subcases must be considered.

i) $\lambda = 0$: in this case $F = C_1 t + C_2$, where C_1 and C_2 are arbitrary constants and $V(x) = \psi(x^1 - x^2, x^2 - x^3, \ldots, x^{n-1} - x^n)$, where ψ is an arbitrary smooth function. The symmetries are then

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \sum_{i=1}^{n} \frac{\partial}{\partial x^i}, \quad X_3 = tX_2.$$

ii) $\lambda \neq 0$: $V(x) = \dfrac{\lambda}{2}\left(\displaystyle\sum_{i=1}^{n} x^i\right)^2 + \psi(x^1 - x^2, x^2 - x^3, \ldots, x^{n-1} - x^n)$ and the symmetries are

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = F_1(t)\sum_{i=1}^{n} \frac{\partial}{\partial x^i}, \quad X_3 = F_2(t)\sum_{i=1}^{n} \frac{\partial}{\partial x^i},$$

where

$$F_1(t) = \begin{cases} \cosh(\sqrt{\lambda} t) & \text{if } \lambda > 0 \\ \cos(\sqrt{-\lambda} t) & \text{if } \lambda < 0, \end{cases}$$

$$F_2(t) = \begin{cases} \sinh(\sqrt{\lambda} t) & \text{if } \lambda > 0 \\ \sin(\sqrt{-\lambda} t) & \text{if } \lambda < 0. \end{cases}$$

5.2 *Extensions using* Γ_τ

The projection of Γ_τ on t and x leaves $V - V(x) = 0$ unchanged provided $\tau^{(3)} = 4k\tau$, where k is a constant. We distinguish the following subcases.

a) $k = 0$: $V(x) = \dfrac{1}{r^2}\psi(x_2/x_1, x_3/x_2, \ldots, x^n/x^{n-1})$, where $r^2 = \delta_{ij} x^i x^j$ and the symmetries are

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = 2t\frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i}, \quad X_3 = t^2 \frac{\partial}{\partial t} + t x^i \frac{\partial}{\partial x^i}.$$

b) $k \neq 0$: $V(x) = \dfrac{k}{2} r^2 + \dfrac{1}{r^2}\psi(x_2/x_1, x_3/x_2, \ldots, x^n/x^{n-1})$ and the symmetries are

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \cosh(2t\sqrt{k})\frac{\partial}{\partial t} + \sqrt{k}\sinh(2t\sqrt{k}) x^i \frac{\partial}{\partial x^i},$$

$$X_3 = \sinh(2t\sqrt{k})\frac{\partial}{\partial t} + \sqrt{k}\cosh(2t\sqrt{k}) x^i \frac{\partial}{\partial x^i}, \text{ if } k > 0$$

and

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \cos(2t\sqrt{-k})\frac{\partial}{\partial t} - \sqrt{-k}\sin(2t\sqrt{-k}) x^i \frac{\partial}{\partial x^i},$$

$$X_3 = \sin(2t\sqrt{-k})\frac{\partial}{\partial t} + \sqrt{-k}\cos(2t\sqrt{-k}) x^i \frac{\partial}{\partial x^i}, \text{ if } k < 0.$$

Remark. Note that we can further extend the symmetry Lie algebras obtained in a) and b) using the equivalence operators Γ_{ij}. The potential assume the forms $V = h/r^2$ and $V = kr^2/2 + h/r^2$ (h and k are constants) respectively.

6 Example of potential system with two first integrals of motion

Consider the 2-dimensional potential system with

$$V(x,y)=\alpha x+\Phi(y-kx), \tag{6.33}$$

where α and k are real constants and Φ is an arbitrary smooth function. This case arises in the symmetry classification of 2-dimensional potential systems presented in [17, 23]. Its symmetries are

$$X_1=\frac{\partial}{\partial t},\quad X_2=\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\cdot \tag{6.34}$$

These two point symmetries turn out to be Noether symmetries for the natural Lagrangian $L=\frac{1}{2}(\dot{x}^2+\dot{y}^2)-\alpha x-\Phi(y-kx)$. Using Noether's theorem [2, 19, 20, 25], we find that the corresponding integrals of motion are respectively

$$H_1 = \frac{1}{2}(\dot{x}^2+\dot{y}^2)+\alpha x+\Phi(y-kx), \tag{6.35}$$

$$H_2 = \alpha t+\dot{x}+k\dot{y}. \tag{6.36}$$

We recognise (6.35) as being the Hamiltonian of the system. Using X_2, we reduce the potential system with potential (6.33) to

$$\dot{x}+k\dot{y}+\alpha t=c, \tag{6.37}$$

$$\ddot{y}+\Phi'(y-kx)=0, \tag{6.38}$$

where c is an arbitrary constant. By noticing that X_2 is a symmetry of (6.37)-(6.38), we introduce the change of coordinate $X=y-kx$, $Y=x$, which transforms X_2 to the translation in Y, i.e., $\partial/\partial Y$. In the new variables, the system (6.37)-(6.38) read

$$\alpha t+(1+k^2)\dot{Y}+k\dot{X} = c \tag{6.39}$$

$$k\ddot{Y}+\ddot{X}=-\Phi'(X). \tag{6.40}$$

Solve equation (6.39) for $\dot{Y}$ and substitute the result into (6.40) to obtain

$$\ddot{X}=-(1+k^2)\Phi'(X)+k\alpha, \tag{6.41}$$

This last equation has the symmetry X_1. Integrating (6.41) once we find

$$\dot{X}^2=-2(1+k^2)\Phi(X)+2k\alpha X+c_1, \tag{6.42}$$

where c_1 is a constant of integration. Equation (6.42) has separable variables. Thus the potential system with potential (6.33) is solvable by quadratures.

7 Conclusion

We have determined the equivalence Lie algebra and the weak equivalence Lie algebra of an n-dimensional potential differential system. We used the equivalence Lie algebra to simplify the classifying conditions. Also, we showed how to use equivalence operators to construct symmetries of an n-dimensional second order potential differential system. Finally we have given an example in which the existence of point symmetries confirm the integrability of the underlying systems.

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