Cooperative guards in art galleries with one hole

Paweł Żyliński

Abstract

The coopeartive art gallery problem asks for the minimum number of cooperative point guards that can collectively monitor a simple polygon with n vertices. A guard set is *cooperative* if its visibility graph is connected. For simple polygons without holes, a tight bound of $\lfloor \frac{n-2}{2} \rfloor$ was shown by Ahlfeld and Hecker and, independently, by Hernández-Peñalver. For galleries with holes, up to know, the best upper bound has been $\lfloor \frac{n+2h}{2} \rfloor - 1$ (due to Ahlfeld and Hecker), where h denotes the number of holes. In this paper, we improve this bound for $h = 1$ by asserting that $\lfloor \frac{n+1}{2} \rfloor - 1$ is a tight bound.

Mathematics Subject Classification: 52C99, 05C90.
Key words: art gallery theorem, cooperative guards, polygons with holes.

1 Introduction

The original *art gallery problem* raised by Victor Klee asks how many guards are sufficient to watch every point of the interior of an n-vertex simple polygon. The guard is a stationary point that can see any point which can be connected to it with a line segment within the polygon. In 1975, Chvátal [3] proved that $\lfloor \frac{n}{3} \rfloor$ guards are occasionally necessary and always sufficient to cover a polygon with n vertices. Since then many different variations of this problem have arisen; see [8], [11] for more details.

Herein we analyze the concept of *cooperative guards* that was proposed by Liaw, Huang and Lee in [6]. The guard set S is said to be *cooperative* provided that its visibility graph is connected. The idea behind this concept is that if something goes wrong with one guard, all the others can be informed. It is worth pointing out that a variation of the cooperative guards problem had been already raised before [6]: in 1992, Ahlfeld and Hecker had studied the problem of determining the minimum link-number for polygons, see [1] for more details.

Liaw *et al.* [6] established that the *minimum cooperative guards problem* for simple polygons is NP-hard, but for spiral and 2-spiral polygons this problem can be solved in linear time. For k-spiral polygons, the minimum number of cooperative guards is at most N_k, the total number of reflex vertices in the k-spiral polygon [4]. The

Balkan Journal of Geometry and Its Applications, Vol.10, No.2, 2005, pp. 142-153.

cooperative guards problem for general simple polygons has been completely settled by Ahlfeld and Hecker [1] and, independently, by Hernández-Peñalver [5], who proved that $\lfloor \frac{n}{2} \rfloor - 1$ cooperative guards are always sufficient and occasionally necessary to guard a polygon with n vertices.

As far as galleries with holes are considered, then up to now, the best upper bound for cooperative guards in galleries with holes has been $\lfloor \frac{n+2h}{2} \rfloor - 1$ (Ahlfeld and Hecker [1]). In this paper, we improve this bound for the case $h = 1$: we show that $\lfloor \frac{n+1}{2} \rfloor - 1$ cooperative guards are always sufficient and occasionally necessary to guard a one-hole polygon with n vertices, even if guards are restricted to be located at the vertices. Our approach is similar to that of Shermer's for arbitrary guards: in 1983, Shermer [9, 10] established that $\lfloor \frac{n+1}{3} \rfloor$ is a tight bound for arbitrary vertex guards in any polygon with exactly one hole. To show the sufficiency of this bound, he uses an arbitrary triangulation of the polygon. This triangulation must contain a cycle of triangles, i.e., the cycle of triangles corresponding to the cycle in the dual graph surrounding the hole. Shermer first shows that to prove the sufficiency of $\lfloor \frac{n+1}{3} \rfloor$ vertex guards for any triangulation, it is enough to provide a proof for a *reduced triangulation* consisting of a cycle of triangles perhaps with at most single triangles attached. In some of these triangulations, it is not possible to pick $\lfloor \frac{n+1}{3} \rfloor$ vertex guards so that every triangle has a guard at one of its vertices. Shermer calls these configurations *tough triangulations*, and makes a case study to show that in each situation $\lfloor \frac{n+1}{3} \rfloor$ vertex guards are still sufficient.

The organization of our paper is as follows. Section 2 is devoted to notation, terminology, and some preliminary results. The sufficiency proof for cooperative vertex guards will be presented in Section 3. Finally, galleries with arbitrary number of holes are discussed.

2 Definitions and preliminaries

An *art gallery* is a simple polygon P, i.e., a region bounded by a simple polyline $\bar{P}$ (together with $\bar{P}$). A *guard* g is any point of P. A point $x \in P$ is said to be *seen* by a guard g if the line segment with endpoints x and g is a subset of P. A collection of guards S is said to *cover* polygon P if every point $x \in P$ can be seen by some guard $g \in S$. For a guard set S, we define the *visibility graph* $VG(S)$ as follows: the vertex set is S and two vertices v_1, v_2 are adjacent if they see each other. The guard set S is said to be *cooperative* if its visibility graph $VG(S)$ is connected. Finally, a *gallery with holes* is a simple polygon P enclosing other simple polygons $H_1, \ldots, H_h$, known as the holes; the boundaries of $P, H_1, \ldots, H_h$ are mutually disjoint.

For an art gallery P with one hole, let us define $CG_1(P)$ to be the minimum cardinality of a cooperative guards set for P. Next, let us define $cg_1(n)$ to be the maximum value of $CG_1(P)$ over all one-hole polygons with n vertices, counting vertices on the hole as well as on the outer boundary. The function $cg_1(n)$ represents the maximum number of cooperative guards that are ever needed for an n-gon with one hole – $cg_1(n)$ cooperative guards always suffice and $cg(n)$ cooperative guards are necessary for at least one n-vertex polygon with a hole.

2.1 Reduction to combinatorial guards

A *triangulation* T of a polygon P is a partitioning of P into a set of triangles with pair-wise disjoint interiors in such a way that the edges of those triangles are either edges or diagonals of P joining pairs of vertices. As we know, such a triangulation always exists ([8], Theorem 1.2).

A *triangulation graph* G_T of an n-vertex polygon P is a graph whose vertices correspond to n vertices of P and whose edges correspond to the edges of P and diagonals of a triangulation T. A *vertex guard* in a triangulation graph G_T is a single vertex of G_T. A set of guards S is said to *dominate* G_T if every triangular face of G_T, except for the external face and the faces corresponding to the holes, has at least one of its vertices assigned as a guard. The collection of guards S is said to be *cooperative* if the subgraph of G_T induced by set S is connected. Guards in a graph are called *combinatorial* to distinguish them from *geometric* guards introduced earlier. The reason for introducing triangulation graphs is that a proof of sufficiency of a certain number of combinatorial guards establishes the sufficiency of the same number of geometric guards.

Lemma 2.1. [5] *Let P be an n-vertex polygon with one hole, and G_T be one of its triangulation graphs. If G_T can be dominated by $f(n)$ cooperative guards, then P can be covered by $f(n)$ geometric cooperative vertex guards.*

Thus in general, the idea of the sufficiency proof is to solve the cooperative guards problem on triangulation graphs, and then to extend this result to polygons. Before commencing the proof, let us recall two facts.

Theorem 2.2. [5] *A triangulation graph of a hole-free polygon with $n \geq 4$ vertices can always be dominated by $\lfloor \frac{n}{2} \rfloor - 1$ cooperative guards.*

Lemma 2.3. [12] *Let G_T be a triangulation graph of a hole-free polygon P with $n \geq 3$ vertices, and let $e = \{v_1, v_2\}$ be an edge of P. Then:*

a) if n is odd, then $\lfloor \frac{n-1}{2} \rfloor$ cooperative guards with one guard placed at any endpoint of e suffice to dominate G_T;

b) otherwise, $\lfloor \frac{n-1}{2} \rfloor$ cooperative guards with one guard placed either at v_1 or at v_2 suffice to dominate G_T.

3 Triangulation graphs of galleries with one hole

In this section, we will prove that $cg_1(n) = \lfloor \frac{n-1}{2} \rfloor$, even for vertex guards. The idea of the proof of the sufficiency of $\lfloor \frac{n-1}{2} \rfloor$ cooperative vertex guards for galleries with one hole follows the main outlines of Shermer's proof for vertex (arbitrary) guards in [9, 10].

We say that a triangulation of a one-hole polygon is *simple* if the dual graph of the triangulation is a cycle (surrounding the hole). Let us recall that the (weak) dual graph of a triangulation of a one-hole polygon, with a vertex for each triangle and an edge connecting two vertices whose triangles share a diagonal, is a single cycle with some number of attached trees, and each vertex of the dual graph is of degree at most 3.

a) b)

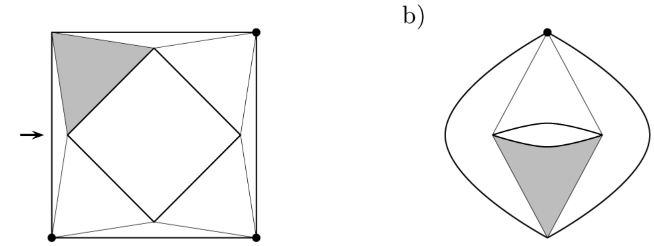

Fig. 1. (a) A triangulation graph of 8 vertices with string $(t/)^8$ that requires 4 combinatorial cooperative guards: the 3 shown (dots) do not cover the shaded triangle. (b) A triangulation graph of 4 vertices with string $(t/)^4$ that requires 2 combinatorial cooperative guards: the 1 shown (dot) do not cover the shaded triangle.

Let P be a one-hole polygon, and let T be one of its triangulation. Suppose T to be simple. A cycle triangle of T is *based on the inner boundary* if it has exactly one vertex, its apex, on the outer boundary, and *based on the outer boundary* if just its apex vertex is on the inner boundary. Let us label a cycle triangle "t". Then T is represented as a string of characters over the alphabet {"t","/"}, formed by concatenating all the labels of the cycle triangles, and inserting a "/" between labels t_1 and t_2 if the triangle t_1 is based on the inner boundary and the triangle t_2 is based on the outer boundary, or vice versa. Thus each "/" records a switch in basing. This string of characters will be called the *string* associated with T.

Fig. 1(a) shows an example. Starting at the indicated leftmost triangle and proceeding counterclockwise, we obtain the string $t/t/t/t/t/t/t/t/$. We employ standard regular expression notation to condense the strings: s^k for k repetitions of string s, and $s^\star$ for zero or more repetitions of s. Thus the above string is equivalent to $(t/)^8$. We consider two strings equivalent if one is a cyclic shift of the other, or a cyclic shift of the reverse of another. Finally, note that the strings make no distinction between the inner and outer boundaries.

The main difficulty in the sufficiency proof is the existence of triangulation graphs that require as many as $\lfloor \frac{n}{2} \rfloor$ combinatorial cooperative guards for complete domination.

Lemma 3.1. *The triangulation graph of a triangulation T of a one-hole polygon requires $\lfloor \frac{n}{2} \rfloor$ combinatorial cooperative guards for complete domination iff the string for T has the form $(t/)^{2k+4}$, $k \geq 0$.*

We will call a string that is an instance of $(t/)^{2k+4}$ *tough*. Fig. 1(a) satisfies the lemma: $n = 8$ and it requires $\lfloor \frac{8}{2} \rfloor = 4$ combinatorial cooperative guards; an attempted cover with three guards is shown in the figure. Note that even triangulations whose strings are tough but do not correspond to any non-degenerate polygon require $\lfloor \frac{n}{2} \rfloor$ combinatorial cooperative guards. Fig. 1(b) shows the smallest possible instance, $(t/)^4$, where $n = 4$ and it requires $\lfloor \frac{4}{2} \rfloor = 2$ combinatorial cooperative guards.

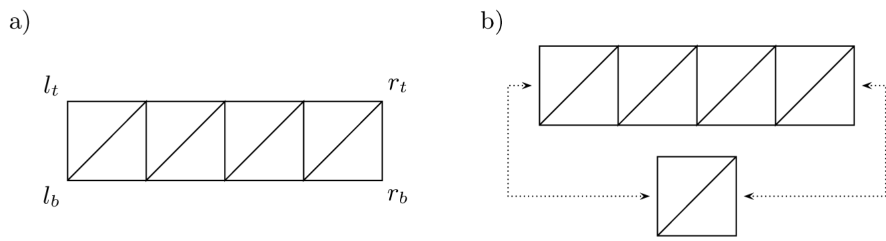

Fig. 2. A tough triangulation requires $\lfloor \frac{n}{2} \rfloor$ combinatorial cooperative guards.

*Proof.*We will first prove that a triangulation graph G_T with a tough string requires $\lfloor \frac{n}{2} \rfloor$ combinatorial cooperative guards. Let us consider a sequence s of triangles that is shown in Fig. 2(a). It is easy to see that it has the following properties:

1) It requires $\lfloor \frac{m}{2} \rfloor$ combinatorial cooperative guards, where m is the number of triangles.
2) If a guard is required either at vertex l_t or r_b, then the sequence requires $\lfloor \frac{m+2}{2} \rfloor$ combinatorial cooperative guards, where m is the number of triangles.
3) If guards are required at both vertices l_b and r_t, then the sequence requires $\lfloor \frac{m+2}{2} \rfloor$ combinatorial cooperative guards, where m is the number of triangles.

Next, if we enclose the sequence s with two triangles, see Fig. 2(b), we will get a tough string. The above properties ensure us that at least $\lfloor \frac{m+2}{2} \rfloor$ combinatorial cooperative guards are required for complete domination of the string. As $m + 2 = n$, we are done.

Now we will prove the lemma in the other direction, in the contrapositive form: if a triangulation T is not an instance of a tough string, then fewer than $\lfloor \frac{n}{2} \rfloor$ combinatorial cooperative guards suffice for domination. Each t in a tough triangulation must be followed by $/$. Thus any non-tough triangulation must contain a fragment of the form tt with the apex at some vertex v.

Without loss of generality, we can assume that the sequence $t/$ is followed by tt. Otherwise, we can remove triangles ttt from the triangulation graph G_T and split vertex v into two. The resulted hole-free triangulation graph can be dominated by $\lfloor \frac{n-1}{2} \rfloor - 1$ combinatorial cooperative guards. The same guard placement in G_T with one additional guard at v yields a domination of G_T by $\lfloor \frac{n-1}{2} \rfloor$ combinatorial cooperative guards.

Therefore let triangles from the tt/t-fragment be labeled t_1, t_2 and t_3, respectively, and let $\{x, v\}$ be the diagonal shared between triangles t_1 and t_2. Removing triangles t_2 and t_3 from graph G_T results in a triangulation graph G'_T of a hole-free polygon with n vertices, see Fig. 3. By Lemma 2.3, graph G'_T can be dominated by $\lfloor \frac{n-1}{2} \rfloor$ combinatorial cooperative guards with one guard located either at x or at v. As t_1 and t_2 form the tt-fragment in G_T, without loss of generality, we can assume v to be assigned as a guard. Then the same guard placement in G_T yields a domination of G_T by $\lfloor \frac{n-1}{2} \rfloor$ combinatorial cooperative guards, as triangles t_2 and t_3 will be dominated by the guard at v. $\square$

Fig. 3. An existence of a tt-fragment leads to a domination by $\lfloor \frac{n-1}{2} \rfloor$ combinatorial cooperative guards.

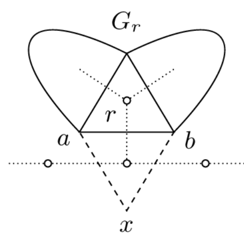

Fig. 4. A graph G_r attached at diagonal $\{a, b\}$ to a cycle triangle.

Now, let P be one-hole polygon with n vertices, and let T be one of its triangulations. The next lemma shows that the only triangulations "difficult" to be dominated by $\lfloor \frac{n-1}{2} \rfloor$ combinatorial cooperative guards are the tough ones.

Lemma 3.2. *Let P be a one-hole polygon with n vertices, and suppose that there exists a triangulation T of P that is not tough, that is, P has either a triangulation whose dual is a cycle with at most one tree attached to the cycle or P has a non-tough simple triangulation. Then the triangulation graph of T can be dominated by $\lfloor \frac{n-1}{2} \rfloor$ cooperative guards.*

*Proof.*The proof is by induction on the number of trees attached to a cycle of the dual graph of a triangulation T. The basis is established by Lemma 3.1: $\lfloor \frac{n-1}{2} \rfloor$ combinatorial cooperative guards are sufficient to dominate a non-tough simple triangulation graph, which by definition has no attached trees in the dual. For the general step, assume that $\lfloor \frac{n-1}{2} \rfloor$ combinatorial cooperative guards suffice for any non-tough triangulation with the dual of $s' < s$ trees.

Let G_T be the triangulation graph of T, and let G_r be the triangulation graph whose dual graph corresponds to a tree detachable from G_T by the removal of one arc r. This situation is illustrated in Fig. 4. Let a and b be the endpoints of the diagonal whose dual is r. Let m be the number of vertices in G_r, *not including* a and b. The proof proceeds in three cases, depending on the value (n mod 2) and (m mod 2). The easiest cases are considered first.

Case 1: $n = 2k + 1$. The sufficiency of $\lfloor \frac{n-1}{2} \rfloor$-bound follows immediately from Theorem 2.2. The idea is to cut graph G_T along an internal diagonal in order to remove the hole by connecting it to the exterior face. The resulted graph G'_T has $n + 2$ vertices,

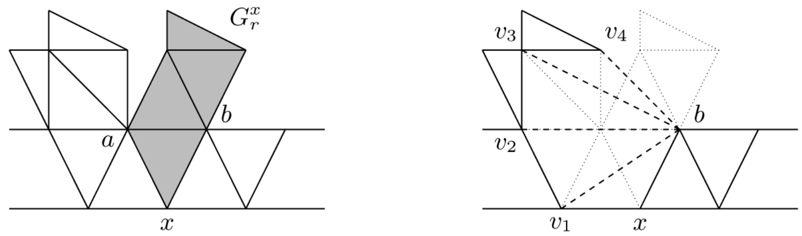

Fig. 5. Case 3: $n = 2t$ and $m = 2l + 1$.

as two new vertices are introduced, and it can be dominated by $\lfloor \frac{n+2}{2} \rfloor - 1 = \lfloor \frac{n-1}{2} \rfloor$ cooperative guards, as n is odd. The same guard placement will dominate all of G_T.

Case 2: $n = 2k$ and $m = 2l$. Augment G_r to G_r^x by adding a triangle on the other side of $\{a, b\}$, whose apex is x. Next, cutting G_T along diagonals $\{x, a\}$ and $\{x, b\}$ results in two triangulation graphs G'_T and G_r^x, each of $n - m + 1$ and $m + 3$ vertices, respectively. By Theorem 2.2, G'_T can be dominated by $\lfloor \frac{n-m-1}{2} \rfloor$ combinatorial cooperative guards. With the same arguments, G_r^x can be dominated by $\lfloor \frac{m+3}{2} \rfloor - 1$ combinatorial cooperative guards with a guard at one of vertices a, b or x. As m is even, $\lfloor \frac{m+3}{2} \rfloor - 1 = \lfloor \frac{m}{2} \rfloor$. The same guard placement in G_T yields a domination by $\lfloor \frac{n-m-1}{2} \rfloor + \lfloor \frac{m}{2} \rfloor \leq \lfloor \frac{n-1}{2} \rfloor$ combinatorial guards, and the guard set is cooperative, as one of vertices a, b or x was chosen for the location of a guard.

Case 3: $n = 2k$ and $m = 2l + 1$. Again augment G_r to G_r^x by adding a triangle on the other side of $\{a, b\}$, whose apex is x. By Lemma 2.3, G_r^x can be dominated by $\lfloor \frac{m+2}{2} \rfloor$ combinatorial cooperative guards with one guard located either at a or at x. If x is assigned as a guard, it may be removed to a. Thus we can assume vertex a to be assigned as a guard in G_r^x. Let G'_T be the result of removing all triangles of G_r^x and all triangles incident to a. G'_T has $n - m - 1$ vertices, since it is missing m vertices of G_D^x and vertex a. Note that G'_T is not necessarily a triangulation graph of a polygon, as pieces may be attached at vertices only. But now connect each vertex of G'_T that was adjacent to a in G_T to b. In Fig. 5, vertices $v_1, \ldots, v_4$ are connected. These connections are not always geometrically possible, but for this case we are only concerned with the combinatorial structure of the graph. The reconnections do not increase the number of vertices, but they restore G'_T to be a triangulation graph of a polygon with one hole, with smaller number of trees attached to the cycle of the dual graph of G'_T.

If G'_T is non-tough, then by the induction hypothesis, G'_T can be dominated by $\lfloor \frac{n-m-2}{2} \rfloor = \lfloor \frac{n-m-3}{2} \rfloor$ combinatorial cooperative guards, as $n - m$ is odd. And it is easy to see that placing guards at vertices of G_T assigned as guards either in G'_T or in G_r^x yields a combinatorial cooperative domination of G_T by at most $\lfloor \frac{n-1}{2} \rfloor$ guards. Otherwise, the toughness of G'_T implies the following properties of G_T:

1) The dual graph of G_T is a single cycle with one attached tree (corresponding to graph G_r^x).
2) Since connecting b to vertices to which a was adjacent results in a tough

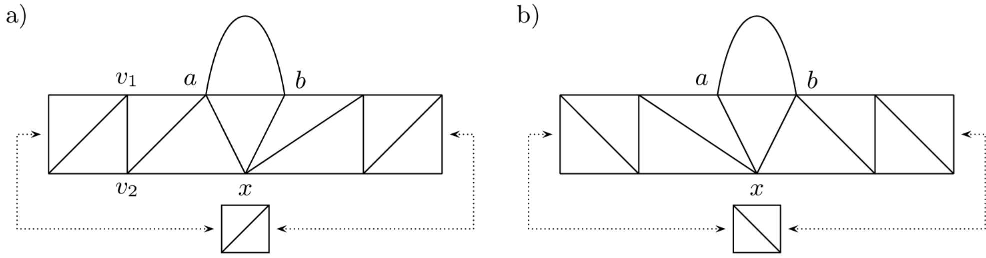

Fig. 6. In the case of toughness of G'_T, one of the above configurations must have occurred.

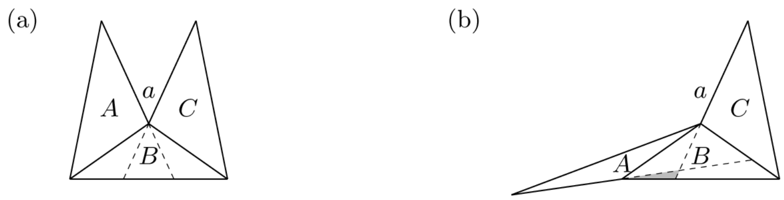

Fig. 7. A c-triplet is covered if A and C are covered (a), but B would not be necessarily covered if the triangles do not form a c-triplet.

string $(t/)^{2k+4}$, by simple enumeration of cases, cycle triangles of G_T (without triangles of G_r) must have been of the form $tt/t/(t/)^{2k+2}$, and either vertex a or b must has belong only to triangles from the tt-fragment. Therefore without loss of generality, either the configuration shown in Fig. 6(a) or Fig. 6(b) must hold.

Subcase 3.a: G_T is of the form shown in Fig. 6(a). Then instead of connecting b to v_2, let us connect v_1 with x, that is, we have to flip a diagonal in a quadrilateral (b, v_1, v_2, x) in G'_T. Again let us note that this is only a combinatorial "procedure". This flipping results in a non-tough triangulation, and thus we can proceed in a way similar to that in the case of non-toughness of G'_T.

Subcase 3.b: G_T is of the form shown in Fig. 6(b). Then instead of considering vertex a in the first step of the proof, we have to consider vertex b. It is easy to see that this will lead to Subcase 3.a. □

The next step is to involve the geometry on the triangulation and use geometric guards in the case of a tough triangulation. In particular, if a tough triangulation contains either "c-pair" or a "c-triplet", then $\lfloor \frac{n-1}{2} \rfloor$ vertex cooperative guards suffice. The final step is to show that every tough triangulation contains one of these two structures. The next section follows exactly the same reasoning as in [9, 10].

3.1 c-pairs and c-triplets

A *c-pair* is a pair of adjacent cycle triangles that together form a convex quadrilateral.

Lemma 3.3. *A polygon with a tough triangulation containing a c-pair may be covered with* $\lfloor \frac{n-1}{2} \rfloor$ *vertex cooperative guards.*

*Proof.*Flipping the diagonal of the c-pair will change the structure of the triangulation to non-tough. By Lemma 3.2, the resulted triangulation graph can be dominated by $\lfloor \frac{n-1}{2} \rfloor$ combinatorial cooperative guards, and hence all of P can be covered with $\lfloor \frac{n-1}{2} \rfloor$ vertex cooperative guards. □

A *c-triplet* is a triple (A, B, C) of consecutive triangles such that the union of three triangles may be partitioned into two convex pieces. By cutting of a polygon along diagonals shared by triangles A, B and B, C of any triple (A, B, C), we get the following lemma (compare [9, 10]).

Lemma 3.4. *A polygon with a tough triangulation containing a c-triplet may be covered with* $\lfloor \frac{n-1}{2} \rfloor$ *vertex cooperative guards.*

*Proof.*Let a be a vertex common to the c-triplet triangles A, B and C, as shown in Fig 7(a). Delete B and split vertex a into two. The result is a hole-free polygon with $n + 1$ vertices, which may therefore be covered with $\lfloor \frac{n-1}{2} \rfloor$ vertex guards by [1, 5]. In particular, both A and C must have a guard in one of its corners (Theorem 2.2). Now put back B. Because the three triangles form a c-triplet, B is also covered by the guards covering A and C. Note that if the triangles did not form a c-triplet, as in Fig. 7(b), B would not be necessarily covered. □

The final property we need is that every tough triangulation contains one of these structures. And this is guaranted by Shermer's lemma[1].

Lemma 3.5. [9, 10] *Any tough triangulation of a polygon contains either a c-pair or a c-triplet.*

With all preceding lemmas we are just led to the following theorem.

Theorem 3.6. $\lfloor \frac{n-1}{2} \rfloor$ *vertex cooperative guards suffice to cover any* n*-vertex polygon with one hole.*

*Proof.*Lemma 3.2 established that if there exists a non-tough triangulation, then $\lfloor \frac{n-1}{2} \rfloor$ vertex cooperative guards suffice. So we only need to consider polygons with tough triangulations. Lemmas 3.3 and 3.4 show that if a tough triangulation contains c-pair or c-triplet, then $\lfloor \frac{n-1}{2} \rfloor$ vertex cooperative guards suffice. And Lemma 3.5 shows that every tough triangulation contains one of these structures, so there are no further possibilities. □

Theorem 3.7. *For all* $n \geq 6$, $cg_1(n) = \lfloor \frac{n-1}{2} \rfloor$, *even for vertex guards.*

[1] Although our definition of the tough string differs a little from that introduced by Shermer, exactly the same proof provides the required property for both definitions, see [9, 10] for more details.

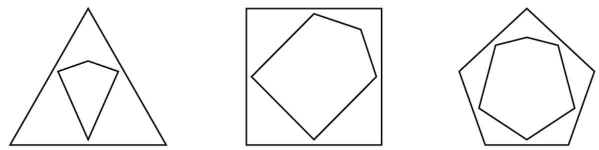

Fig. 8. A one-hole polygon can require as many as $\lfloor\frac{n-1}{2}\rfloor$ cooperative guards.

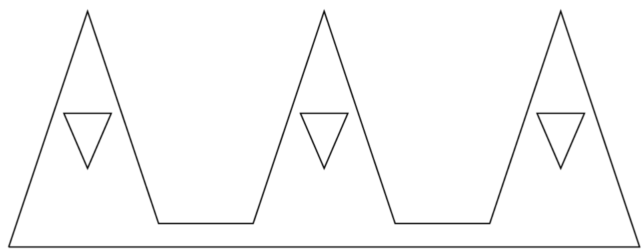

Fig. 9. A polygon with holes can require as many as $\lfloor\frac{n}{2}\rfloor$ cooperative vertex guards.

*Proof.*The necessity is established by a class of polygons shown in Fig. 8. It is easy to see that they require as many as $\lfloor\frac{n-1}{2}\rfloor$ cooperative guards. Therefore $cg_1(n) \geq \lfloor\frac{n-1}{2}\rfloor$. For sufficiency, apply Theorem 3.6. □

4 Final remarks

It does not seem easy to extend the proof of Theorem 3.6 to more than one hole. Let us recall that an *art gallery with holes* is a polygon P enclosing other polygons $H_1, \ldots, H_h$, known as the holes ($H_1, \ldots, H_h$, and the boundary of P are mutually disjoint. Similarly to one-hole art galleries, let us define $cg_h(n)$ to be the maximum value of $CG(P)$ over all polygons with h holes and n vertices in total, i.e., counting vertices on the holes as well as on the outer boundary. The function $cg_h(n)$ represents the maximum number of cooperative guards that are ever needed for an art gallery with n vertices and h holes. The cooperative guards problem for polygons with an arbitrary number of holes generally remains unsolved. The best upper bound for vertex guards, due to Ahlfeld and Hecker [1], is $\lfloor\frac{n}{2}\rfloor + h - 1$.

Theorem 4.1. [1] *A triangulation graph of an n-vertex simple polygon with h holes can always be dominated by $\lfloor\frac{n}{2}\rfloor + h - 1$ cooperative guards.*

But this theorem is weak in a sense that so far no one has found examples of polygons that require so many guards. Fig. 9 shows an example from the class of n-vertex polygons with h holes that require $\lfloor\frac{n}{2}\rfloor$ vertex cooperative guards. We conjecture that this bound is tight.

Conjecture 4.2. $\lfloor \frac{n}{2} \rfloor$ *cooperative vertex guards always suffice to cover an n-vertex polygon with h holes.*

Although the cooperative guards problem for arbitrary polygons with an arbitrary number of holes remains unsolved, by using the result of Bjorling-Sachs and Souvaine [2], we can easily provide a new upper bound for cooperative point guards.

In 1995, Bjorling-Sachs and Souvaine [2] showed that $\lfloor \frac{n+h}{3} \rfloor$ arbitrary guards are sufficient in any polygon with n vertices and h holes. Their approach is to connect each hole to the exterior by cutting away a quadrilateral *channel* c_i, $i = 1, \ldots, h$, such that one vertex is introduced for each channel, and there is a triangle T_i in the remaining polygon such that any point in it sees all of the channel c_i, $i = 1, \ldots, h$. This triangle is then forced to be in a triangulation of the hole-free version of the polygon. A guard assignment based on 3-coloring will cover the hole-free polygon and all the channels as well. Since the new polygon has $n + h$ vertices, the number of guards is $\lfloor \frac{n+h}{3} \rfloor$. These guards are vertex guards in the hole-free polygon, but point guards in the original polygon, since new vertices were added during the channel constructions. The main result of Bjorling-Sachs and Souvaine's paper is the following theorem.

Theorem 4.3. [2] *In any polygon P with n vertices and h holes, all channels c_i, $i = 1, \ldots, h$, can be removed in such a way that the remaining polygon has*:

1) *$n + h$ vertices.*
2) *No holes.*
3) *A triangulation T with* (*disjoint*) *triangles t_i, $i = 1, \ldots, h$, as leaves in the dual graph of T from whose vertices the areas of the removed channels are visible in P.*

Let P be a polygon with n vertices and h holes, and let T be a triangulation whose existence is guaranteed by the above theorem. By Theorem 2.2 a triangulation graph G_T can be dominated by $\lfloor \frac{n+h}{2} \rfloor - 1$ combinatorial cooperative guards, and thus all of the hole-free polygon can be covered by the same number of cooperative guards. Since each of the triangles t_i of T, $i = 1, \ldots, h$, has a guard at one of its vertices, by Theorem 4.3 these guards see all of the channels as well. Thus all of P is covered by $\lfloor \frac{n+h}{2} \rfloor - 1$ cooperative guards. Note that the guards are point guards, since the hole-free polygon has vertices not present in P. This proves the following corollary.

Corollary 4.4. *For all $h \geq 0$ and $n \geq 3+3h$, $cg_h(n) \leq \lfloor \frac{n+h}{2} \rfloor - 1$, that is, $\lfloor \frac{n+h}{2} \rfloor - 1$ cooperative guards are sufficient to cover the interior of an art gallery with n vertices and h holes.*

References

[1] U. Ahlfeld, H.-D. Hecker, The computational complexity of some guard sets in polygons, *J. Inform. Process. Cyber.* EIK 28(6) (1992), 331-342.

[2] I. Bjorling-Sachs, D.L. Souvaine, An efficient algorithm for guard placement in polygons with holes. *Discrete Comput. Geom.* 13 (1995), 77-109.

[3] V. Chvátal, *A combinatorial theorem in plane geometry*, J. Combin. Theory Ser. B 18 (1975), 39-41.

[4] J.S. Deogun, S.T. Sarasamma, *On the minimum cooperative guard problem*, J. Combin. Math. Combin. Comput. 22 (1996), 161-182.

[5] G. Hernández-Peñalver, *Controlling guards*, Proc. of Sixth Canadian Conference on Computational Geometry (1994), 387-392.

[6] B-C. Liaw, N.F. Huang, R.C.T. Lee, *The minimum cooperative guards problem on k-spiral polygons*, Proc. of Fifth Canadian Conference on Computational Geometry (1993), 97-101.

[7] G.H. Meisters, *Polygons have ears*, Amer. Math. Monthly 82 (1975), 648-651.

[8] J. O'Rourke, *Art Gallery Theorems and Algorithms*, Oxford University Press (1987).

[9] T. Shermer, Triangulation graphs that require extra guards. NYIT Computer Graphics Technical Report 3D-13 (1984).

[10] T. Shermer, Polygon guarding II: efficient reduction of triangulation fragments, NYIT Computer Graphics Technical Report 3D-16 (1985).

[11] J. Urrutia. *Art Gallery and Illumination Problems*, in: Handbook on Computational Geometry, Elsevier Science, Amsterdam (2000).

[12] P. Żyliński, *Cooperative guards in the fortress problem*, Balkan Journal of Geometry and Its Applications 9(2) (2004), 103-119.

Paweł Żyliński
Institute of Mathematics, Gdańsk University
57 Wita Stwosza, 80952 Gdańsk, Poland
e-mail: impz@univ.gda.pl