Some modified connection structures associated with the Finslerian gravitational field

Satoshi Ikeda and Mihai Anastasiei

Dedicated to the memory of Radu Rosca (1908-2005)

Abstract. Some modified connection structures of the Finslerian gravitational field are considered by modifying the Finslerian independent variables (x^i, y^i) $(i, j = 1, 2, 3, 4)$ in the following forms: (x^i, y^i, p_j) (p_j: co-vector dual to y^i); (x^i, y^j, z^k) (z^k: another vector chosen at one more microscopic level than the $y-$level); (x, ψ) (ψ: spinor).

Mathematics Subject Classification: 53FB40, 83D05.
Key words: Finslerian gravitational fields, connection structures.

1 Introduction

The independent variables of the Finslerian gravitational field are chosen as (x^i, y^j), where the vector y $(= y^j; j = 1, 2, 3, 4)$ is attached to each point x $(= x^i; i = 1, 2, 3, 4)$ as the independent internal variable. Therefore, the Finslerian gravitational field is regarded as a unified field between the external $(x)-$field spanned by points $\{x\}$ and the internal $(y)-$field spanned by vectors $\{y\}$ (cf.[1]).

From another viewpoint, the Finslerian gravitational field is considered the unified field over the total space of the vector bundle whose fibre at each point x is the $(y)-$field and base manifold is the $(x)-$field. This vector bundle is not necessarily the tangent bundle TM over the base manifold M. It can be any vector bundle case in which one applies the geometry of vector bundles as it is described in [2]. But the theory is simpler for the tangent bundle. Thus we consider the tangent bundle as starting point and we will replace it with certain vector bundles in the next sections. Then, the adapted frame in the total space is set as follows ([2]):

$$\begin{aligned} &dX^A = (dx^i, \delta y^i = dy^i + N^i_j dx^j) \\ &\frac{\partial}{\partial X^A} = \left(\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}\right), \end{aligned} \tag{1.1}$$

where $X^A = (x^i, y^j)$ $(A = 1, 2, ..., 8)$, and N^i_j denotes the nonlinear connection representing physically the interaction between the $(x)-$ and $(y)-$fields.

Balkan Journal of Geometry and Its Applications, Vol.11, No.1, 2006, pp. 81-86.

We say that this interaction is holonomic if $d(\delta y^i) = 0$ (modulo δy^i). A direct calculation gives $d(\delta y^i) + \omega^i_j \wedge \delta y^j = \Omega^i$, where $\omega^i_j = \dfrac{\partial N^i_k}{\partial y^j} dx^k$ and $2\Omega^k = (\dfrac{\delta N^k_j}{\delta x^i} - \dfrac{\delta N^k_i}{\delta x^j}) dx^i \wedge dx^j$. Thus the said interaction is holonomic if and only if $\Omega^k = 0$. The local vector fields $(\dfrac{\partial}{\partial y^i})$ span a distribution on TM called vertical and the local vector fields $(\dfrac{\delta}{\delta x^i})$ span a distribution that is supplementary to it, called the horizontal distribution. From the equation $[\dfrac{\delta}{\delta x^j}, \dfrac{\delta}{\delta x^i}] = \Omega^k_{ij} \dfrac{\partial}{\partial y^k}$ it comes out that the interaction between the $(x)-$ and $(y)-$ fields is holonomic if and only if the horizontal distribution is integrable.

On the basis of (1.1), the Finslerian connection structure is stipulated as

$$\begin{aligned} &\nabla_{\frac{\partial}{\partial X^C}} \frac{\partial}{\partial X^B} = \Gamma^A_{BC} \frac{\partial}{\partial X^A}; \\ &\Gamma^A_{BC} \equiv (L^i_{jk}, C^i_{jk}). \end{aligned} \tag{1.2}$$

Namely, two kinds of connection coefficients L^i_{jk} and C^i_{jk} appear and the covariant derivatives of an arbitrary vector V^i can be defined by

$$\begin{aligned} V^i_{|k} &= \frac{\delta V^i}{\delta x^k} + L^i_{jk} V^j, \\ V^i|_k &= \frac{\partial V^i}{\partial y^k} + C^i_{jk} V^j. \end{aligned} \tag{1.3}$$

This is the most simplified connection structure and geometrically it corresponds to the Finslerian structure. It preserves by parallelism the vertical and the horizontal distributions and makes covariant constant the almost tangent structure on TM.

By the way, the metrical structure is introduced by

$$G \equiv G_{AB} dX^A dX^B = g_{ij}(x,y) dx^i \otimes dx^j + g_{ij}(x,y) \delta y^i \otimes \delta y^j. \tag{1.4}$$

Here $g_{ij}(x,y)$ are the gravitational potentials depending also on the $(y)-$field. The condition that ∇ is metrical, that is, $\nabla G = 0$ is equivalent with $g_{ij|k} = 0$ and $g_{ij}|_k = 0$. In the other words the tensor field is $h-$ covariant constant and $v-$ covariant constant.One can impose only one of these conditions. The most important is $g_{ij|k} = 0$ as it can be seen from [2]. Given the connection ∇ we may consider in particular $y^i_{|k} = L^i_{jk} y^j - N^i_k$ called the $h-$ deflection tensor and $y^i|_k = \delta^i_k + C^i_{jk} y^j$ called the $v-$ deflection tensor. If we set $D_{ij} = g_{ik} y^k_{|j}$ and $d_{ij} = g_{ik} y^k|_j$ we can associate to g_{ij} two tensor fields $F_{ij} = \dfrac{1}{2}(D_{ij} - D_{ji}), f_{ij} = \dfrac{1}{2}(d_{ij} - d_{ji})$ that may be regarded as $h-$ and $v-$ electromagnetic tensors associated to g_{ij}. They verify the identities that generalize the Maxwell equations (see [2]).

As is understood from (1.2) or (1.3), the connection structure depends essentially on the adapted frame (1.1), which is also constructed by the independent variables (x^i, y^j). Therefore, if the independent variables are modified, then the adapted frame is also modified accordingly and then, the Finslerian connection structure (1.2) is modified correspondingly.

Along this line, in the following, we shall take up the following cases: the independent variables (x^i, y^j) are generalized to (x^i, y^j, p_j), p_j being a covector dual to y^j; (x^i, y^j, z^k), z^k being a vector chosen at one more microscopic level than the $y-$level; (x, ψ), ψ being a spinor.

2 Modified Connection Structures - I

First, we shall consider the case where the independent variables become (x^i, y^j, p_j), p_j being a covector dual to y_j.In fact we replace the tangent bundle TM with the vector bundle $TM \times T^*M$ over M, where T^*M is the cotangent bundle of M. Then, the adapted frame is set as follows (cf. [5])):

$$\begin{aligned} &dX^A \equiv (dx^i, \delta y^i = dy^i + N^i_j dx^j, \delta p_i = dp_i - M_{ji} dx^j), \\ &\frac{\partial}{\partial X^A} \equiv \left(\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j} + M_{ij} \frac{\partial}{\partial p_j}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial p_i} \right), \end{aligned} \tag{2.1}$$

where two kinds of nonlinear connections N^i_j and M_{ij} appear.

The interaction between the $(x)-(y)-$ and $(p)-$ fields is holonomic if $d(\delta y^i = 0$ (modulo δy^i) and $d(\delta p_i) = 0$(modulo δp_i). We have $d(\delta p_i) = -\dfrac{\partial M_{ji}}{\partial p_k}\delta p_k - \Theta_{jki} dx^k \wedge dx^j$ for

$$2\Theta_{jki} = \frac{\partial M_{ji}}{\partial x^k} - \frac{\partial M_{ki}}{\partial x^j} + M_{jh}\frac{\partial M_{ki}}{\partial p_h} - M_{kh}\frac{\partial M_{ji}}{\partial p_h}.$$

Thus the said interaction is holonomic if and only if $\Omega^i_{jk} = 0, \Theta_{jki} = 0$. The last equation is equivalent with the integrability of the distribution spanned by (δp_i).

Now, the connection structure is given by

$$\Gamma^A_{BC} \equiv (L^i_{jk}, C^i_{jk}, H^{ik}_j), \tag{2.2}$$

by which the covariant derivatives can be defined as follows: e.g.,

$$\begin{aligned} V^i_{|k} &= \frac{\delta V^i}{\delta x^k} + L^i_{jk} V^j, \\ V^i|_k &= \frac{\partial V^i}{\partial y^k} + C^i_{jk} V^j, \\ V^i\|^k &= \frac{\partial V^i}{\partial p_k} + H^{ik}_j V^j, \end{aligned} \tag{2.3}$$

In particular, we have

$$\begin{aligned} &p_{i|j} = Mij - L^k_{ij} p_k \\ &p_i|_j = \delta ij - C^k_{ij}, \\ &p_i\|^k = \delta^k_i + H^{jk}_i p_j, \end{aligned} \tag{2.3'}$$

Thus new types of electromagnetic tensors could be considered.

The metrical structure is in this case introduced by

$$G \equiv g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j + g^{ij}\delta p_i \otimes \delta p_j. \tag{2.4}$$

A new condition of metrizability of ∇ appears in the form $g^{ij}\|^k = 0$.

Next, if the independent variables are chosen as (x^i, y^j, z^k), z^k being a vector chosen at one more microscopic level than the $y-$level, then the adapted frame is set as follows:

$$\begin{aligned} &dX^A \equiv (dx^i, \delta y^i = dy^i + N^i_j dx^j, \delta z^k = dz^k + A^k_j dy^j + B^k_j dx^j), \\ &\frac{\partial}{\partial X^A} \equiv \left(\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j} - B^j_i \frac{\partial}{\partial z^j}, \frac{\delta}{\delta y^i} = \frac{\partial}{\partial y^i} - A^j_i \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^i}\right), \end{aligned} \tag{2.5}$$

where three kinds of nonlinear connections N^i_j, A^i_j and B^i_j appear. In fact here we replace the bundle TM with the product $TM \times TM$ over M. The meaning of the assertion that z^k is a vector chosen at one more microscopic level than the $y-$level is provided by the expression $\dfrac{\delta}{\delta y^i} = \dfrac{\partial}{\partial y^i} - A^j_i \dfrac{\partial}{\partial z^j}$. In some sense the variable (y^i) become more external then the variables (z^k).

Then, the connection structures is prescribed by

$$\Gamma^A_{BC} \equiv (L^i_{jk}, C^i_{jk}, E^i_{jk}), \tag{2.6}$$

by which the covariant derivatives can be defined as follows: e.g.,

$$\begin{aligned} V^i_{|k} &= \frac{\delta V^i}{\delta x^k} + L^i_{jk}V^j, \\ V^i|_k &= \frac{\delta V^i}{\delta y^k} + C^i_{jk}V^j, \\ V^i\|^k &= \frac{\partial V^i}{\partial z_k} + E^i_{jk}V^j, \end{aligned} \tag{2.7}$$

etc. The metrical structure is given by

$$G \equiv g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j + g_{ij}\delta z^i \otimes \delta z^j. \tag{2.8}$$

3 Modified Connection Structures - II

In this section, we shall consider the case where the independent variables become (x, ψ), ψ being a spinor. Then, if we put the adapted frame in the form (cf. [4])

$$\begin{aligned} &dX^A \equiv (dx^i, \overline{\delta}\psi = d\psi + \Theta_i \psi dx^i + \Xi\psi d\psi \equiv P\delta), \\ &\frac{\partial}{\partial X^A} \equiv \left(\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i \frac{\partial}{\partial \psi}, \frac{\overline{\partial}}{\partial \psi} = P^{-1}\frac{\partial}{\partial \psi}\right), \end{aligned} \tag{3.1}$$

where Θ_i and Ξ denote the spin gauge fields and $\delta\psi = d\psi + N_i dx^i$, $P = I + \Xi\psi$ and $PN_i \equiv \Theta_i \psi$, then the connection structure can be introduced by

$$\Gamma^A_{BC} \equiv (L^i_{jk}, L_k, C^i_j, C), \tag{3.2}$$

by which the following covariant derivatives can be defined: e.g.,

$$\begin{aligned} V^i_{|k} &= \frac{\delta V^i}{\delta x^k} + L^i_{jk} V^j, \\ V^i| &= \frac{\overline{\partial} V^i}{\partial \psi} + C^i_j V^j, \end{aligned} \tag{3.3}$$

etc. Further, for a spinorial field $\Phi(x, \psi)$, its covariant derivatives are given by

$$\begin{aligned} &D\Phi = d\Phi - \Theta_i \Theta dx^i - \Xi \Phi d\psi = (\Phi_{|k}) dx^k + (\Phi|)\overline{\delta}\psi, \\ &\Phi_{|k} = \frac{\delta \Phi}{\delta x^k} - \overline{\Theta}_k \Phi, \\ &\Phi| = \frac{\overline{\partial}\Phi}{\partial \psi} - \overline{\Xi}\Phi, \end{aligned} \tag{3.4}$$

where $\overline{\Theta}_k = \Theta_k - \Xi N_k$ and $\overline{\Xi} = P^{-1}\Xi$.

4 Other Comments

In this Section, we shall show some other modified connection structures.

First, if the adapted frame (1.1) is changed to

$$\begin{aligned} &dX^A \equiv (dx^i, \delta y^i = P^i_j dy^j + Q^i_j dx^j), \\ &\frac{\partial}{\partial X^A} \equiv \left(\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}, \frac{\delta}{\delta y^i} = (P^{-1})^j_i \frac{\partial}{\partial y^j} \right), \end{aligned} \tag{4.1}$$

where $N^i_j = (P^{-1})^i_l Q^l_j$, then the connection structure is given by

$$\Gamma^A_{BC} \equiv (L^i_{jk} - N^l_k C^i_{jl},\ (P^{-1})^l_j C^i_{lk}). \tag{4.2}$$

Second, if the independent variables are chosen as (x^i, p_j) and the adapted frame in this case is given by (cf. [5])

$$\begin{aligned} &dX^A \equiv (\delta x^i = dx^i + \Pi^{ji} \delta p_j, \delta p_i = dp_i - M_{ji} dx^j), \\ &\frac{\partial}{\partial X^A} \equiv \left(\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} + M_{ij} \frac{\partial}{\partial p_j}, \frac{\delta}{\delta p_i} = \frac{\partial}{\partial p_j} - \Pi^{ij} \frac{\delta}{\delta x^j} \right), \end{aligned} \tag{4.3}$$

then the connection structure is given by (see (2.2))

$$\Gamma^A_{BC} \equiv (L^i_{jk}, H^{ik}_j - \Pi^{kl} L^i_{jl}). \tag{4.4}$$

Finally, if the nonlinear connection N^i_j in (1.1) is changed as $\overline{N}^i_j = N^i_j - F^i_j$, then the connection structure becomes (cf.[2])

$$\Gamma^A_{BC} \equiv (\overline{L}^i_{jk} = L^i_{jk} + C^i_{jl} F^l_k,\ \overline{C}^i_{jk} = C^i_{jk}). \tag{4.5}$$

Thus, we can consider many interesting modified connection structures by modifying the independent variables or the adapted frame in various ways.

References

[1] S. Ikeda, *Advanced Studies in Applied Geometry*, Seizansha, Sagamihara, 1995.

[2] R. Miron, M. Anastasiei, *Vector Bundles and Lagrange Space with Applications to Relativity*, Geometry Balkan Press, Bucharest, 1997.

[3] R. Miron, *The Geometry of Higher Order Hamilton Spaces*, Kluwer Acad. Publ., Dordrecht, 2003.

[4] S. Ikeda, *Some structural considerations on the Finslerian gravitational field*, Tensor, N.S., 59 (1998), 1-5.

[5] R. Miron, D. Hrimiuc, H. Shimada, S.V. Sabau, *The Geometry of Hamilton and Lagrange Spaces*, Kluwer Acad. Publ., Dordrecht, 2001.

Authors' addresses:

Satoshi Ikeda
Department of Mechanical Engineering, Faculty of Science and Technology
Tokyo University of Science, Noda, Chiba 278-8510, Japan.

Mihai Anastasiei
Faculty of Mathematics, University "AL.I.Cuza" Iaşi, 700506, Iaşi, Romania

Mathematics Institute "O.Mayer" Iaşi, Iaşi Branch of the Romanian Academy, Iaşi, Romania.
email: anastas@uaic.ro