A Riemann-Lagrange geometrization for metrical Multi-Time Lagrange Spaces

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Dedicated to the memory of Radu Rosca (1908-2005)

Abstract. In this paper we construct a geometrization on the 1-jet fiber bundle $J^1(T,M)$ for the multi-time quadratic Lagrangian function

$$L = h^{\alpha\beta}(t)g_{ij}(t,x)x^i_\alpha x^j_\beta + U^{(\alpha)}_{(i)}(t,x)x^i_\alpha + F(t,x).$$

Our geometrization includes a nonlinear connection Γ, a generalized Cartan canonical Γ-linear connection $C\Gamma$ together with its d-torsions and d-curvatures, naturally provided by the given multi-time quadratic Lagrangian function L.

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1 Metrical multi-time Lagrange spaces

It is important to note that quadratic multi-time Lagrangians dominates most scientific domains. We can remind only the theory of elasticity [19], the dynamics of ideal fluids, the magnetohydrodynamics [6], [7], the theory of bosonic strings [5] or the multi-time evolution (p-flow) of some physical or economical phenomena [21, 22, 23, 24, 25, 26]. This fact emphasizes the importance of the geometrization of quadratic multi-time Lagrangians. In conclusion, a Riemann-Lagrange geometry on 1-jet spaces was required. Such a geometry was initially developed by Saunders [20] and Asanov [2], and continued, in a Miron's approach [10], by Udrişte ([21]-[26]) and Neagu ([13], [16]).

In the sequel, let us fix $h = (h_{\alpha\beta}(t^\gamma))$ a semi-Riemannian metric on the temporal manifold T and let $g = (g_{ij}(t^\gamma, x^k, x^k_\gamma))$ be a symmetric d-tensor on $E = J^1(T,M)$, of rank n and having a constant signature.

Generally, a smooth multi-time Lagrangian function

$$L : E \to \mathbb{R},\ E \ni (t^\alpha, x^i, x^i_\alpha) \to L(t^\alpha, x^i, x^i_\alpha) \in \mathbb{R}, \tag{1.1.1}$$

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produces a *fundamental vertical metrical d-tensor*

$$G^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2}\frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta}, \tag{1.1.2}$$

where $i, j = 1, ..., n$ and $\alpha, \beta = 1, ..., p$.

Definition 1.1. *A multi-time Lagrangian function $L : E \to \mathbb{R}$, having the fundamental vertical metrical d-tensor of the form*

$$G^{(\alpha)(\beta)}_{(i)(j)}(t^\gamma, x^k, x^k_\gamma) = h^{\alpha\beta}(t^\gamma) g_{ij}(t^\gamma, x^k, x^k_\gamma), \tag{1.1.3}$$

is called a **Kronecker h-regular multi-time Lagrangian function**.

In this context, we can introduce the following important concept:

Definition 1.2. *A pair $ML^n_p = (J^1(T, M), L)$, $p = \dim T$, $n = \dim M$, consisting of the 1-jet fibre bundle and a Kronecker h-regular multi-time Lagrangian function $L : J^1(T, M) \to \mathbb{R}$, is called a* **multi-time Lagrange space**.

Remark 1.3. *i) In the particular case $(T, h) = (\mathbb{R}, \delta)$, a multi-time Lagrange space is called a* **relativistic rheonomic Lagrange space** *and is denoted by*

$$RL^n = (J^1(\mathbb{R}, M), L).$$

For more details about the relativistic rheonomic Lagrangian geometry, the reader may consult [15].

ii) If the temporal manifold T is 1-dimensional one, then, via a temporal reparametrization, we have

$$J^1(T, M) \equiv J^1(\mathbb{R}, M).$$

In other words, a multi-time Lagrange space, having $\dim T = 1$, is a reparametrized relativistic rheonomic Lagrange space.

Example 1.4. *Let us suppose that the spatial manifold M is also endowed with a semi-Riemannian metric $g = (g_{ij}(x))$. Then, the multi-time Lagrangian function*

$$L_1 : E \to \mathbb{R}, \quad L_1 = h^{\alpha\beta}(t) g_{ij}(x) x^i_\alpha x^j_\beta \tag{1.1.4}$$

is a Kronecker h-regular one. It follows that the pair

$$\mathcal{BS}ML^n_p = (J^1(T, M), L_1)$$

is a multi-time Lagrange space. It is important to note that the multi-time Lagrangian $\mathcal{L}_1 = L_1\sqrt{|h|}$ is exactly the "energy" Lagrangian, whose extremals are ultra-harmonic maps between the semi-Riemannian manifolds (T, h) and (M, g) [4]. At the same time, the multi-time Lagrangian that governs the physical theory of bosonic strings is of type $\mathcal{L}_1$ [6].

Example 1.5. (**geometric dynamics** *[21, 22, 23, 24, 25, 26]) Let us start with a p-flow described by the completely integrable PDES system*

$$\frac{\partial x^i}{\partial t^\alpha} = X^i_\alpha(t, x(t)),\ i = 1, ..., n;\ \alpha = 1, ..., p.$$

This system and the semi-Riemannian metrics $h^{\alpha\beta}(t)$ and $g_{ij}(t, x)$ determine the quadratic Lagrangian function

$$L_2 : E \to \mathbb{R}, \quad L_2 = h^{\alpha\beta}(t)g_{ij}(t, x)(x^i_\alpha - X^i_\alpha(t, x(t))(x^j_\beta - X^j_\beta(t, x(t)), \tag{1.1.5}$$

which is Kronecker h-regular. If the metrics h and g are positive definite, then this is the least squares Lagrangian. Also, we remark that any PDEs system can be replaced with a p-flow (multi-time evolution), and consequently it produces a Lgrange-Hamilton problem via any quadratic Lagrange function of preceding form.

Example 1.6. *In the above notations, taking $U^{(\alpha)}_{(i)}(t, x)$ a d-tensor field on E and $F : T \times M \to \mathbb{R}$ a smooth function, the quadratic multi-time Lagrangian function*

$$L_2 : E \to \mathbb{R}, \quad L_2 = h^{\alpha\beta}(t)g_{ij}(x)x^i_\alpha x^j_\beta + U^{(\alpha)}_{(i)}(t, x)x^i_\alpha + F(t, x), \tag{1.1.6}$$

is also a Kronecker h-regular one. The multi-time Lagrange space

$$\mathcal{EDML}^n_p = (J^1(T, M), L_2)$$

is called the **autonomous multi-time Lagrange space of electrodynamics**. *This is because, in the particular case $(T, h) = (\mathbb{R}, \delta)$, the space $\mathcal{EDML}^n_1$ naturally generalizes the classical Lagrange space of electrodynamics, that governs the movement law of a particle placed concomitently into a gravitational field and an electromagnetic one. From physical point of view, the semi-Riemannian metric $h_{\alpha\beta}(t)$ (resp. $g_{ij}(x)$) represents the* **gravitational potentials** *of the manifold T (resp. M), the d-tensor $U^{(\alpha)}_{(i)}(t, x)$ plays the role of the* **electromagnetic potentials** *which produce a gyroscopic force, and F is a* **potential function**. *The non-dynamical character of the spatial gravitational potentials $g_{ij}(x)$ motivates us to use the term* "autonomous".

Example 1.7. *More general, if we consider the symmetrical d-tensor $g_{ij}(t, x)$ on E, of rank n and having a constant signature on E, we can define the Kronecker h-regular multi-time Lagrangian function*

$$L_3 : E \to \mathbb{R}, \quad L_3 = h^{\alpha\beta}(t)g_{ij}(t, x)x^i_\alpha x^j_\beta + U^{(\alpha)}_{(i)}(t, x)x^i_\alpha + F(t, x). \tag{1.1.7}$$

The multi-time Lagrange space

$$\mathcal{NEDML}^n_p = (J^1(T, M), L_3)$$

is called the **non-autonomous multi-time Lagrange space of electrodynamics**. *We use the term "non-autonomous", in order to emphasize the dynamical character of spatial gravitational potentials $g_{ij}(t, x)$, i.e., their dependence of the temporal coordinates t^γ.*

An important role and, at the same time, an obstruction in the subsequent development of the theory of the multi-time Lagrange spaces, is played by the following theorem, proved in [12]:

Theorem 1.8. *(of characterization of multi-time Lagrange spaces)*
If $p = \dim T \geq 2$, then the following statements are equivalent:
i) L is a Kronecker h-regular Lagrangian function on $J^1(T, M)$.
ii) The multi-time Lagrangian function L reduces to a multi-time Lagrangian function of non-autonomous electrodynamic type, that is

$$L = h^{\alpha\beta}(t)g_{ij}(t,x)x^i_\alpha x^j_\beta + U^{(\alpha)}_{(i)}(t,x)x^i_\alpha + F(t,x).$$

Corollary 1.9. *The fundamental vertical metrical d-tensor of an arbitrary Kronecker h-regular multi-time Lagrangian function L is of the form*

$$G^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2}\frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta} = \begin{cases} h^{11}(t)g_{ij}(t,x^k,y^k), & p = \dim T = 1 \\ h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma,x^k), & p = \dim T \geq 2. \end{cases} \tag{1.1.8}$$

Remark 1.10. *i) It is obvious that the preceding theorem is an obstruction in the development of a fertile geometrical theory for the multi-time Lagrange spaces. This obstruction was surpassed in the paper [13], by introducing the more general notion of a* **generalized multi-time Lagrange space**. *The generalized multi-time Riemann-Lagrange geometry on $J^1(T, M)$ will be constructed using only a Kronecker h-regular vertical metrical d-tensor $G^{(\alpha)(\beta)}_{(i)(j)}$ and a nonlinear connection Γ,* "a priori" *given on the 1-jet space $J^1(T, M)$.*

ii) In the case $p = \dim T \geq 2$, the preceding theorem obliges us to continue our geometrical study of the multi-time Lagrange spaces, directing our attention upon the non-autonomous multi-time Lagrange spaces of electrodynamics.

Let $ML^n_p = (J^1(T, M), L)$, where $\dim T = p$, $\dim M = n$, be a multi-time Lagrange space whose fundamental vertical metrical d-tensor metric is

$$G^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2}\frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta} = \begin{cases} h^{11}(t)g_{ij}(t,x^k,y^k), & p = 1 \\ h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma,x^k), & p \geq 2. \end{cases}$$

Supposing that the semi-Riemannian temporal manifold (T, h) is compact and orientable, by integration on the manifold T, we can define the *energy functional* associated to the multi-time Lagrange function L, taking

$$\mathcal{E}_L : C^\infty(T, M) \to \mathbb{R}, \quad \mathcal{E}_L(f) = \int_T L(t^\alpha, x^i, x^i_\alpha)\sqrt{|h|}\, dt^1 \wedge dt^2 \wedge \ldots \wedge dt^p,$$

where the smooth map f is locally expressed by $(t^\alpha) \to (x^i(t^\alpha))$ and $x^i_\alpha = \dfrac{\partial x^i}{\partial t^\alpha}$.

The extremals of the energy functional $\mathcal{E}_L$ verify the Euler-Lagrange PDEs

$$2G^{(\alpha)(\beta)}_{(i)(j)}x^j_{\alpha\beta} + \frac{\partial^2 L}{\partial x^j \partial x^i_\alpha}x^j_\alpha - \frac{\partial L}{\partial x^i} + \frac{\partial^2 L}{\partial t^\alpha \partial x^i_\alpha} + \frac{\partial L}{\partial x^i_\alpha}H^\gamma_{\alpha\gamma} = 0, \tag{1.1.9}$$

where $x^j_{\alpha\beta} = \dfrac{\partial^2 x^j}{\partial t^\alpha \partial t^\beta}$ and $H^\gamma_{\alpha\beta}$ are the Christoffel symbols of the semi-Riemannian temporal metric $h_{\alpha\beta}$.

Taking into account the Kronecker h-regularity of the Lagrangian function L, it is possible to rearrange the Euler-Lagrange equations of the Lagrangian $\mathcal{L} = L\sqrt{|h|}$ in the following *generalized Poisson form (ultra-hyperbolic partial differential equations):*

$$\text{Œ}_h x^k + 2\mathcal{G}^k(t^\mu, x^m, x^m_\mu) = 0, \tag{1.1.10}$$

where

$$\text{Œ}_h x^k = h^{\alpha\beta}\{x^k_{\alpha\beta} - H^\gamma_{\alpha\beta} x^k_\gamma\},$$

$$2\mathcal{G}^k = \frac{g^{ki}}{2}\left\{\frac{\partial^2 L}{\partial x^j \partial x^i_\alpha} x^j_\alpha - \frac{\partial L}{\partial x^i} + \frac{\partial^2 L}{\partial t^\alpha \partial x^i_\alpha} + \frac{\partial L}{\partial x^i_\alpha} H^\gamma_{\alpha\gamma} + 2 g_{ij} h^{\alpha\beta} H^\gamma_{\alpha\beta} x^j_\gamma\right\}.$$

Proposition 1.11. *i) The geometrical object $\mathcal{G} = (\mathcal{G}^r)$ is a multi-time dependent spatial h-*spray.

ii) Moreover, the spatial h-spray $\mathcal{G} = (\mathcal{G}^l)$ is the h-trace of a multi-time dependent spatial spray $G = (G^{(i)}_{(\alpha)\beta})$, that is $\mathcal{G}^l = h^{\alpha\beta} G^{(l)}_{(\alpha)\beta}$.

Proof. The proof of this proposition is given in [12]. □

Following previous reasonings and the preceding result, we can regard the equations (1.1.10) as being the equations of the ultra-harmonic maps of a multi-time dependent spray.

Theorem 1.12. *The extremals of the energy functional $\mathcal{E}_L$ attached to the Kronecker h-regular Lagrangian function L are ultra-harmonic maps on $J^1(T, M)$ of the multi-time dependent spray (H, G) defined by the temporal components*

$$H^{(i)}_{(\alpha)\beta} = \begin{cases} -\dfrac{1}{2} H^1_{11}(t) y^i, & p = 1 \\ -\dfrac{1}{2} H^\gamma_{\alpha\beta} x^i_\gamma, & p \geq 2 \end{cases}$$

and the local spatial components $G^{(i)}_{(\alpha)\beta} =$

$$= \begin{cases} \dfrac{h_{11} g^{ik}}{4}\left[\dfrac{\partial^2 L}{\partial x^j \partial y^k} y^j - \dfrac{\partial L}{\partial x^k} + \dfrac{\partial^2 L}{\partial t \partial y^k} + \dfrac{\partial L}{\partial x^k} H^1_{11} + 2h^{11} H^1_{11} g_{kl} y^l\right], & p = 1 \\ \dfrac{1}{2}\Gamma^i_{jk} x^j_\alpha x^k_\beta + T^{(i)}_{(\alpha)\beta}, & p \geq 2, \end{cases}$$

where $p = \dim T$.

Definition 1.13. *The multi-time dependent spray (H, G) constructed in the preceding Theorem is called the* **canonical multi-time spray attached to the multi-time Lagrange space** ML^n_p.

In the sequel, by local computations, the canonical multi-time spray (H, G) of the multi-time Lagrange space ML^n_p induces naturally a nonlinear connection Γ on $J^1(T, M)$.

Theorem 1.14. The canonical nonlinear connection

$$\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$$

of the multi-time Lagrange space ML^n_p *is defined by the temporal components*

$$M^{(i)}_{(\alpha)\beta} = 2H^{(i)}_{(\alpha)\beta} = \begin{cases} -H^1_{11}y^i, & p = 1 \\ -H^\gamma_{\alpha\beta}x^i_\gamma, & p \geq 2, \end{cases} \tag{1.1.11}$$

and the spatial components

$$N^{(i)}_{(\alpha)j} = \frac{\partial \mathcal{G}^i}{\partial x^j_\gamma}h_{\alpha\gamma} = \begin{cases} h_{11}\dfrac{\partial \mathcal{G}^i}{\partial y^j}, & p = 1 \\ \Gamma^i_{jk}x^k_\alpha + \dfrac{g^{ik}}{2}\dfrac{\partial g_{jk}}{\partial t^\alpha} + \dfrac{g^{ik}}{4}h_{\alpha\gamma}U^{(\gamma)}_{(k)j}, & p \geq 2, \end{cases} \tag{1.1.12}$$

where $\mathcal{G}^i = h^{\alpha\beta}G^{(i)}_{(\alpha)\beta}$.

Remark 1.15. *In the particular case* $(T, h) = (\mathbb{R}, \delta)$*, the canonical nonlinear connection* $\Gamma = (0, N^{(i)}_{(1)j})$ *of the relativistic rheonomic Lagrange space* $RL^n = (J^1(\mathbb{R}, M), L)$ *generalizes naturally the canonical nonlinear connection of the classical rheonomic Lagrange space* $L^n = (\mathbb{R} \times \mathbf{T}M, L)$ *[10].*

2 Generalized Cartan canonical connection $C\Gamma$ of a metrical multi-time Lagrange space

Now, let us consider that $ML^n_p = (J^1(T, M), L)$ is a multi-time Lagrange space, whose fundamental vertical metrical d-tensor is

$$G^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2}\frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta} = \begin{cases} h^{11}(t)g_{ij}(t, x^k, y^k), & p = 1 \\ h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma, x^k), & p \geq 2. \end{cases}$$

Let $\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$ be the canonical nonlinear connection of the multi-time Lagrange space ML^n_p.

The main result of this Section is the Theorem of existence and uniqueness of the *generalized Cartan canonical connection* $C\Gamma$, which allowed us to develop in the paper [14] the *multi-time Riemann-Lagrange geometry of physical fields*, theory that represents a natural generalization of the classical field theories (the *Finslerian theory* [1], [2] and the *ordinary Lagrangian theory* [10]).

Theorem 2.1. *(the generalized Cartan canonical connection)*
On the multi-time Lagrange space $ML^n_p = (J^1(T, M), L)$*, endowed with the canonical nonlinear connection* Γ*, there is a unique h-normal* Γ*-linear connection*

$$C\Gamma=(H^{\gamma}_{\alpha\beta},G^{k}_{j\gamma},L^{i}_{jk},C^{i(\gamma)}_{j(k)}),$$

having the metrical properties:

i) $g_{ij|k}=0,\quad g_{ij}|^{(\gamma)}_{(k)}=0,$

ii) $G^{k}_{j\gamma}=\dfrac{g^{ki}}{2}\dfrac{\delta g_{ij}}{\delta t^{\gamma}}\ ,\quad L^{k}_{ij}=L^{k}_{ji}\ ,\quad C^{i(\gamma)}_{j(k)}=C^{i(\gamma)}_{k(j)}\ ,$

where "$_{|\alpha}$", "$_{|i}$" *and* "$|^{(\alpha)}_{(i)}$" *are the local covariant derivatives of the h-normal Γ-linear connection $C\Gamma$.*

Proof. Let $C\Gamma=(\bar{G}^{\gamma}_{\alpha\beta},G^{k}_{j\gamma},L^{i}_{jk},C^{i(\gamma)}_{j(k)})$ be an h-normal Γ-linear connection, whose local coefficients are defined by the relations $\bar{G}^{\gamma}_{\alpha\beta}=H^{\gamma}_{\alpha\beta}$, $G^{k}_{j\gamma}=\dfrac{g^{ki}}{2}\dfrac{\delta g_{ij}}{\delta t^{\gamma}}$ and

$$L^{i}_{jk}=\frac{g^{im}}{2}\left(\frac{\delta g_{jm}}{\delta x^{k}}+\frac{\delta g_{km}}{\delta x^{j}}-\frac{\delta g_{jk}}{\delta x^{m}}\right),$$

$$C^{i(\gamma)}_{j(k)}=\frac{g^{im}}{2}\left(\frac{\partial g_{jm}}{\partial x^{k}_{\gamma}}+\frac{\partial g_{km}}{\partial x^{j}_{\gamma}}-\frac{\partial g_{jk}}{\partial x^{m}_{\gamma}}\right). \tag{2.2.1}$$

Taking into account the local expressions of the local covariant derivatives induced by the connection Γ, by a local calculation, we deduce that $C\Gamma$ satisfies the conditions *i) and ii)*.

Conversely, let us consider an h-normal Γ-linear connection

$$\tilde{C}\Gamma=(\tilde{\bar{G}}^{\gamma}_{\alpha\beta},\tilde{G}^{k}_{j\gamma},\tilde{L}^{i}_{jk},\tilde{C}^{i(\gamma)}_{j(k)})$$

which satisfies the metrical conditions *i)* and *ii)*. In this context, we have

$$\tilde{\bar{G}}^{\gamma}_{\alpha\beta}=H^{\gamma}_{\alpha\beta},\quad \tilde{G}^{k}_{j\gamma}=\frac{g^{ki}}{2}\frac{\delta g_{ij}}{\delta t^{\gamma}}.$$

Moreover, the condition $g_{ij|k}=0$ is equivalent to

$$\frac{\delta g_{ij}}{\delta x^{k}}=g_{mj}\tilde{L}^{m}_{ik}+g_{im}\tilde{L}^{m}_{jk}.$$

Applying now a Christoffel process to the indices $\{i,j,k\}$, we find

$$\tilde{L}^{i}_{jk}=\frac{g^{im}}{2}\left(\frac{\delta g_{jm}}{\delta x^{k}}+\frac{\delta g_{km}}{\delta x^{j}}-\frac{\delta g_{jk}}{\delta x^{m}}\right).$$

By analogy, using the relations $C^{i(\gamma)}_{j(k)}=C^{i(\gamma)}_{k(j)}$ and $g_{ij}|^{(\gamma)}_{(k)}=0$ and using also a Christoffel process applied to the indices $\{i,j,k\}$, we obtain

$$\tilde{C}^{i(\gamma)}_{j(k)}=\frac{g^{im}}{2}\left(\frac{\partial g_{jm}}{\partial x^{k}_{\gamma}}+\frac{\partial g_{km}}{\partial x^{j}_{\gamma}}-\frac{\partial g_{jk}}{\partial x^{m}_{\gamma}}\right).$$

In conclusion, the uniqueness of the generalized Cartan canonical connection $C\Gamma$ is clear. □

Remark 2.2. *i) Replacing the canonical nonlinear connection Γ with an arbitrary nonlinear connection, the preceding Theorem holds good.*

ii) In the particular case $(T,h)=(\mathbb{R},\delta)$, the generalized δ-normal Γ-linear Cartan connection associated to the relativistic rheonomic Lagrange space

$$RL^n=(J^1(\mathbb{R},M),L)$$

generalizes naturally the canonical Cartan connection of a classical rheonomic Lagrange space $L^n=(\mathbb{R}\times \mathbf{T}M,L)$, constructed in [10].

iii) The generalized Cartan canonical connection of the multi-time Lagrange space ML^n_p verifies also the metrical properties

$$h_{\alpha\beta/\gamma}=h_{\alpha\beta|k}=h_{\alpha\beta}|^{(\gamma)}_{(k)}=0,\quad g_{ij/\gamma}=0.$$

iv) In the case $p=\dim T\geq 2$, the coefficients of the generalized Cartan canonical connection of the multi-time Lagrange space ML^n_p reduce to

$$\bar{G}^{\gamma}_{\alpha\beta}=H^{\gamma}_{\alpha\beta}\ ,\ G^k_{j\gamma}=\frac{g^{ki}}{2}\frac{\partial g_{ij}}{\partial t^\gamma}\ ,\ L^i_{jk}=\Gamma^i_{jk}\ ,\ C^{i(\gamma)}_{j(k)}=0.$$

3 Local d-torsions and d-curvatures of $C\Gamma$

Applying the formulas that determine the local d-torsions and d-curvatures of an h-normal Γ-linear connection $\nabla\Gamma$ (see [16]) to the generalized Cartan canonical connection $C\Gamma$, we obtain the following results:

Theorem 3.1. *The torsion d-tensor $\mathbf{T}$ of the generalized Cartan canonical connection $C\Gamma$ of the multi-time Lagrange space ML^n_p is determined by the local components*

(3.3.1)

	h_T		h_M		v	
	$p=1$	$p\geq 2$	$p=1$	$p\geq 2$	$p=1$	$p\geq 2$
h_Th_T	0	0	0	0	0	$R^{(m)}_{(\mu)\alpha\beta}$
h_Mh_T	0	0	T^m_{1j}	$T^m_{\alpha j}$	$R^{(m)}_{(1)1j}$	$R^{(m)}_{(\mu)\alpha j}$
h_Mh_M	0	0	0	0	$R^{(m)}_{(1)ij}$	$R^{(m)}_{(\mu)ij}$
vh_T	0	0	0	0	$P^{(m)\ (1)}_{(1)1(j)}$	$P^{(m)\ (\beta)}_{(\mu)\alpha(j)}$
vh_M	0	0	$P^{m(1)}_{i(j)}$	0	$P^{(m)\ (1)}_{(1)i(j)}$	0
vv	0	0	0	0	0	0

where,

i) for $p=\dim T=1$ we have

$$T^m_{1j}=-G^m_{j1}\ ,\ P^{m(1)}_{i(j)}=C^{m(1)}_{i(j)}\ ,\ P^{(m)\ (1)}_{(1)1(j)}=-G^m_{j1}\ ,$$

$$P^{(m)\ (1)}_{(1)i(j)}=\frac{\partial N^{(m)}_{(1)i}}{\partial y^j}-L^m_{ji}\ ,\quad R^{(m)}_{(1)ij}=\frac{\delta N^{(m)}_{(1)i}}{\delta x^j}-\frac{\delta N^{(m)}_{(1)j}}{\delta x^i}\ ,$$

$$R^{(m)}_{(1)1j} = -\frac{\partial N^{(m)}_{(1)j}}{\partial t} + H^1_{11}\left[N^{(m)}_{(1)j} - \frac{\partial N^{(m)}_{(1)j}}{\partial y^k} y^k\right];$$

ii) for $p = \dim T \geq 2,$ *denoting*

$$F^m_{i(\mu)} = \frac{g^{mp}}{2}\left[\frac{\partial g_{pi}}{\partial t^\mu} + \frac{1}{2} h_{\mu\beta} U^{(\beta)}_{(p)i}\right],$$

$$H^\gamma_{\mu\alpha\beta} = \frac{\partial H^\gamma_{\mu\alpha}}{\partial t^\beta} - \frac{\partial H^\gamma_{\mu\beta}}{\partial t^\alpha} + H^\eta_{\mu\alpha} H^\gamma_{\eta\beta} - H^\eta_{\mu\beta} H^\gamma_{\eta\alpha},$$

$$r^m_{pij} = \frac{\partial \Gamma^m_{pi}}{\partial x^j} - \frac{\partial \Gamma^m_{pj}}{\partial x^i} + \Gamma^k_{pi}\Gamma^m_{kj} - \Gamma^k_{pj}\Gamma^m_{ki},$$

we have

$$T^m_{\alpha j} = -G^m_{j\alpha},\ P^{m\ (\beta)}_{(\mu)\alpha(j)} = -\delta^\beta_\gamma G^m_{j\alpha},\ R^{(m)}_{(\mu)\alpha(j)} = -H^\gamma_{\mu\alpha\beta} x^m_\gamma,$$

$$R^{(m)}_{(\mu)\alpha j} = -\frac{\partial N^{(m)}_{(\mu)j}}{\partial t^\alpha} + \frac{g^{mk}}{2} H^\beta_{\mu\alpha}\left[\frac{\partial g_{jk}}{\partial t^\beta} + \frac{h_{\beta\gamma}}{2} U^{(\gamma)}_{(k)j}\right],$$

$$R^{(m)}_{(\mu)ij} = r^m_{ijk} x^k_\mu + \left[F^m_{i(\mu)|j} - F^m_{j(\mu)|i}\right];$$

Theorem 3.2. *The curvature d-tensor* $\mathbf{R}$ *of the generalized Cartan canonical connection* $C\Gamma$ *of the multi-time Lagrange space* ML^n_p *is determined by the local components*

	h_T		h_M		v	
	$p=1$	$p\geq 2$	$p=1$	$p\geq 2$	$p=1$	$p\geq 2$
$h_T h_T$	0	$H^\alpha_{\eta\beta\gamma}$	0	$R^l_{i\beta\gamma}$	0	$R^{(l)(\alpha)}_{(\eta)(i)\beta\gamma}$
$h_M h_T$	0	0	R^l_{i1k}	$R^l_{i\beta k}$	$R^{(l)(1)}_{(1)(i)1k} = R^l_{i1k}$	$R^{(l)(\alpha)}_{(\eta)(i)\beta k}$
$h_M h_M$	0	0	R^l_{ijk}	R^l_{ijk}	$R^{(l)(1)}_{(1)(i)jk} = R^l_{ijk}$	$R^{(l)(\alpha)}_{(\eta)(i)jk}$
$v h_T$	0	0	$P^{(l)\ (1)}_{i1(k)}$	0	$P^{(l)(1)\ (1)}_{(1)(i)1(k)} = P^{(l)\ (1)}_{i1(k)}$	0
$v h_M$	0	0	$P^{l\ (1)}_{ij(k)}$	0	$P^{(l)(1)\ (1)}_{(1)(i)j(k)} = P^{l\ (1)}_{ij(k)}$	0
vv	0	0	$S^{l(1)(1)}_{i(j)(k)}$	0	$S^{(l)(1)(1)(1)}_{(1)(i)(j)(k)} = S^{l(1)(1)}_{i(j)(k)}$	0

where $R^{(l)(\alpha)}_{(\eta)(i)\beta\gamma} = \delta^\alpha_\eta R^l_{i\beta\gamma} + \delta^l_i H^\alpha_{\eta\beta\gamma}$, $R^{(l)(\alpha)}_{(\eta)(i)\beta k} = \delta^\alpha_\eta R^l_{i\beta k}$, $R^{(l)(\alpha)}_{(\eta)(i)jk} = \delta^\alpha_\eta R^l_{ijk}$ *and*

i) for $p = \dim T = 1$ *we have*

$$R^l_{i1k} = \frac{\delta G^l_{i1}}{\delta x^k} - \frac{\delta L^l_{ik}}{\delta t} + G^m_{i1} L^l_{mk} - L^m_{ik} G^l_{m1} + C^{l(1)}_{i(m)} R^{(m)}_{(1)1k},$$

$$R^l_{ijk} = \frac{\delta L^l_{ij}}{\delta x^k} - \frac{\delta L^l_{ik}}{\delta x^j} + L^m_{ij} L^l_{mk} - L^m_{ik} L^l_{mj} + C^{l(1)}_{i(m)} R^{(m)}_{(1)jk},$$

$$P^{l\ (1)}_{i1(k)} = \frac{\partial G^l_{i1}}{\partial y^k} - C^{l(1)}_{i(k)/1} + C^{l(1)}_{i(m)} P^{(m)\ (1)}_{(1)1(k)},$$

$$P^{l\ (1)}_{ij(k)} = \frac{\partial L^l_{ij}}{\partial y^k} - C^{l(1)}_{i(k)|j} + C^{l(1)}_{i(m)} P^{(m)\ (1)}_{(1)j(k)},$$

$$S^{l(1)(1)}_{i(j)(k)} = \frac{\partial C^{l(1)}_{i(j)}}{\partial y^k} - \frac{\partial C^{l(1)}_{i(k)}}{\partial y^j} + C^{m(1)}_{i(j)} C^{l(1)}_{m(k)} - C^{m(1)}_{i(k)} C^{l(1)}_{m(j)} \ ;$$

ii) for $p = \dim T \geq 2$ *we have*

$$H^{\alpha}_{\eta\beta\gamma} = \frac{\partial H^{\alpha}_{\eta\beta}}{\partial t^{\gamma}} - \frac{\partial H^{\alpha}_{\eta\gamma}}{\partial t^{\beta}} + H^{\mu}_{\eta\beta} H^{\alpha}_{\mu\gamma} - H^{\mu}_{\eta\gamma} H^{\alpha}_{\mu\beta} \ ,$$

$$R^{l}_{i\beta\gamma} = \frac{\delta G^{l}_{i\beta}}{\delta t^{\gamma}} - \frac{\delta G^{l}_{i\gamma}}{\delta t^{\beta}} + G^{m}_{i\beta} G^{l}_{m\gamma} - G^{m}_{i\gamma} G^{l}_{m\beta} \ ,$$

$$R^{l}_{i\beta k} = \frac{\delta G^{l}_{i\beta}}{\delta x^{k}} - \frac{\delta \Gamma^{l}_{ik}}{\delta t^{\beta}} + G^{m}_{i\beta} \Gamma^{l}_{mk} - \Gamma^{m}_{ik} G^{l}_{m\beta} \ ,$$

$$R^{l}_{ijk} = r^{l}_{ijk} = \frac{\partial \Gamma^{l}_{ij}}{\partial x^{k}} - \frac{\partial \Gamma^{l}_{ik}}{\partial x^{j}} + \Gamma^{m}_{ij} \Gamma^{l}_{mk} - \Gamma^{m}_{ik} \Gamma^{l}_{mj} \ .$$

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