Determination of metrics by boundary energy

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Dedicated to the memory of Radu Rosca (1908-2005)

Abstract. This paper reformulates a problem of Sharafutdinov [4] and extend the new variant from the single-time context to the multi-time context.

Section 1 is dedicated to the single-time case. It starts with well-known facts of describing geodesics as extremals. Then it is formulated and studied the problem of determination of a metric by the boundary energy. The linearization of this problem leads to the ray transform of a tensor field and to moment problem.

Section 2 extend the single-time case to the multi-time case. It begins with well-known facts about harmonic maps and continues with determining a pair of metrics from boundary energy. Using the linearization, we extend the idea to multi-ray transform of a distinguished tensor field (moment problem).

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1 Single-time Case

1.1 Geodesics

Let (M, g) be a Riemannian manifold, $\dim M = n$. Consider $(x^1, \ldots, x^n)$ the local coordinates and Γ^i_{jk} the Christoffel symbols of the second type.

Definition 1.1 *Let $x\colon [0,1] \to M$, $x(t) = (x^1(t), \ldots, x^n(t))$ be a curve on M joining the points $x(0) = p$ and $x(1) = q$ of M. The integral*

$$E_g(x) = \frac{1}{2}\int_0^1 ||\dot{x}(t)||^2 dt = \frac{1}{2}\int_0^1 g_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t)dt$$

is called the energy of the curve x.

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Proposition 1.1 *A minimum point of the energy functional E_g, with the boundary conditions $x(0) = p$ and $x(1) = q$, necessarily verifies the boundary value problem:*

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = 0, \quad i = \overline{1,n}, \qquad (1.1)$$
$$x(0) = p, \qquad x(1) = q,$$

where $L(x^i, \dot{x}^i) = \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j$ is the Lagrangian (kinetic energy) determining the functional.

Explicitly,

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0, \quad i = \overline{1,n},$$
$$x(0) = p, \qquad x(1) = q.$$

Proof. Let us refer to the second part of the Proposition. We have

$$\frac{\partial L}{\partial x^i} = \frac{1}{2}\frac{\partial}{\partial x^i}\left(g_{jk}\dot{x}^j\dot{x}^k\right) = \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\dot{x}^j\dot{x}^k$$
$$\frac{\partial L}{\partial \dot{x}^i} = \frac{1}{2}\frac{\partial}{\partial \dot{x}^i}\left(g_{jk}\dot{x}^j\dot{x}^k\right) = g_{ij}\dot{x}^j$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{d}{dt}\left(g_{ij}\dot{x}^j\right) = \frac{\partial g_{ij}}{\partial x^k}\dot{x}^k\dot{x}^j + g_{ij}\ddot{x}^j.$$

Hence, the Euler-Lagrange equations become

$$\frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\dot{x}^j\dot{x}^k - \frac{\partial g_{ij}}{\partial x^k}\dot{x}^k\dot{x}^j - g_{ij}\ddot{x}^j = 0 \Leftrightarrow \qquad (1.2)$$
$$g_{ij}\ddot{x}^j + \frac{1}{2}\left(2\frac{\partial g_{ij}}{\partial x^k}\dot{x}^k\dot{x}^j - \frac{\partial g_{jk}}{\partial x^i}\dot{x}^j\dot{x}^k\right) = 0 \Leftrightarrow$$
$$\ddot{x}^p + \frac{1}{2}g^{ip}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right)\dot{x}^j\dot{x}^k = 0 \Leftrightarrow$$
$$\ddot{x}^p + \Gamma^p_{jk}\dot{x}^j\dot{x}^k = 0, \quad p = \overline{1,n}.$$

Remark 1.1 a) *We suppose that the problem* (1.1) *stated in the previous proposition has a unique solution.*

b) *The equations*

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0, \quad i = \overline{1,n}$$

of the extremals of the energy functional E_g coincide with the equations of the geodesics of (M, g).

1.2 Determination of a Metric by Boundary Energy

Definition 1.2 *Let (M, g) be a compact Riemannian manifold and ∂M be the boundary of M. The Riemannian metric g is called simple if any two points $p, q \in \partial M$ can be joined by a unique geodesic*

$$x_{pq} \colon [0,1] \to \overline{M}, \quad \overline{M} = M \cup \partial M,$$
$$x(0) = p, \qquad x(1) = q, \qquad x(0,1) \subset M.$$

Definition 1.3 *Let (M, g) be a simple Riemannian manifold. The function*

$$E_g \colon \partial M \times \partial M \to \mathbf{R}, \quad E_g(p,q) = E(x_{pq}),$$

where x_{pq} is the geodesic joining the points p, q, $(x_{pq} \setminus \{p,q\}) \subset M \setminus \partial M$, is called the boundary energy produced by the metric g.

The problem of existence of a simple metric g with the property that a given function $E \colon \partial M \times \partial M \to \mathbf{R}$ represents the boundary energy attached to g cannot have a unique solution. To justify this statement, let $\varphi \colon M \to M$ be a diffeomorphism of M such that $\varphi|_{\partial M} = \text{id}$ and $g^1 = \varphi^* g^0$. The diffeomorphism φ transforms the simple metric g^0 into the simple metric g^1. The relation

$$g^1(x)(\xi, \eta) = g^0((d_x\varphi)\xi, (d_x\varphi)\eta)_{\varphi(x)},$$

where $d_x\varphi \colon T_xM \to T_{\varphi(x)}M$ is the differential of φ, implies that g^0 and g^1 have different families of geodesics, but the energy is the same.

Is the nonuniqueness of the proposed problem settled by the above-mentioned cons-truction?

Problem 1. Let g^0 and g^1 be simple metrics on the manifold M, with the boundary ∂M, such that $E_{g^0} = E_{g^1}$. Is there a diffeomorphism $\varphi \colon M \to M$, such that $\varphi|_{\partial M} = \text{id}$ and $g^1 = \varphi^* g^0$? (the problem of determination of a metric by its boundary energy).

1.3 Linearization of The Problem of Determination of a Metric by its Boundary Energy

Let us linearize the above-mentioned problem.

Let (g^τ) be a family of simple metrics on M, depending smoothly on the parameter $\tau \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$. Let $x^\tau \colon [0,1] \to M$ be a geodesic joining the points $p = x^\tau(0)$ and $q = x^\tau(1)$. Consider $x^\tau(t) = (x^1(t,\tau), \ldots, x^n(t,\tau))$ as the representation of x^τ in a local coordinate system. Suppose that $g^\tau = (g^\tau_{ij})$ and $x^i(t,\tau)$, $i,j = \overline{1,n}$, are C^∞ functions.

We start with the boundary energy

$$E_{g^\tau}(p,q) = \frac{1}{2}\int_0^1 g^\tau_{ij}(x^\tau(t))\dot{x}^i(t,\tau)\dot{x}^j(t,\tau)dt.$$

Differentiating with respect to τ and then considering $\tau = 0$, we have

$$\begin{aligned}\left.\frac{\partial}{\partial\tau}\right|_{\tau=0} E_{g^\tau}(p,q) \;=\; & \int_0^1 f_{ij}(x^0(t))\dot{x}^i(t,0)\dot{x}^j(t,0)dt \\ & +\frac{1}{2}\int_0^1 \left[\frac{\partial g^0_{ij}}{\partial x^k}(x^0(t))\dot{x}^i(t,0)\dot{x}^j(t,0)\frac{\partial x^k}{\partial\tau}(t,0)\right. \\ & \left. +2g^0_{ij}(x^0(t))\dot{x}^i(t,0)\ \frac{\partial \dot{x}^j}{\partial\tau}(t,0)\right] dt,\end{aligned}$$

where

$$f_{ij} = \frac{1}{2}\frac{\partial}{\partial \tau}\Big|_{\tau=0} g^{\tau}_{ij}. \tag{1.3}$$

Using an integration by parts, having in mind that $\frac{\partial x^i}{\partial \tau}(0,0) = \frac{\partial x^i}{\partial \tau}(1,0) = 0$ and x^0 is extremal of energy functional, we obtain

$$\begin{aligned}\int_0^1 g^0_{ij}(x^0(t))\dot{x}^i(t,0)\frac{\partial \dot{x}^j}{\partial \tau}(t,0)dt &= \int_0^1 g^0_{ij}(x^0(t))\dot{x}^i(t,0)\frac{\partial}{\partial t}\left(\frac{\partial x^j}{\partial \tau}\right)(t,0)dt\\ &= g^0_{ij}(x^0(t))\dot{x}^i(t,0)\frac{\partial x^j}{\partial \tau}(t,0)\Big|_0^1 - \int_0^1 \frac{\partial}{\partial t}\left[g^0_{ij}(x^0(t))\dot{x}^i(t,0)\right]\frac{\partial x^j}{\partial \tau}(t,0)dt\\ &= -\int_0^1 \left[\frac{\partial g^0_{ij}}{\partial x^k}(x^0(t))\dot{x}^k(t,0)\dot{x}^i(t,0) + g^0_{ij}(x^0(t))\ddot{x}^i(t,0)\right]\frac{\partial x^j}{\partial \tau}(t,0)dt.\end{aligned} \tag{1.4}$$

The equations (1.2) can be written in the form

$$\frac{\partial g^0_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j = 2\left(\frac{\partial g^0_{jk}}{\partial x^i}\dot{x}^i\dot{x}^j + g_{jk}\ddot{x}^k\right). \tag{1.5}$$

By replacing the relations (1.4) and (1.5) in the equality (1.3), we find

$$\frac{\partial}{\partial \tau}\Big|_{\tau=0} E_{g\tau}(p,q) = \int_0^1 f_{ij}(x^0(t))\dot{x}^i(t,0)\dot{x}^j(t,0)dt.$$

If we denote

$$I_f(x_{pq}) = \int_0^1 f_{ij}(x^0(t))\dot{x}^i(t,0)\dot{x}^j(t,0)dt,$$

the previous relation becomes

$$\frac{\partial}{\partial \tau}\Big|_{\tau=0} E_{g\tau}(p,q) = I_f(x_{pq}), \tag{1.6}$$

x_{pq} being a geodesic of the metric g^0.

In the particular case when the energy $E_{g\tau}$ does not depend on τ, the left side of the equality (1.6) is null.

From $\varphi|_{\partial M} = \mathrm{id}$ we obtain $v|_{\partial M} = 0$, so we have the following linearization of the problem 1: to what extent the family of integrals

$$I_f(x_{pq}) = \int_{x_{pq}} f_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t)dt,$$

counted after $p, q \in \partial M$, determine the tensor $f = (f_{ij})$ over a Riemannian manifold (M, g^0)?

The existence of the solutions of the stated problem for the family (g^τ) implies the existence of an one-parameter group of diffeomorphisms $\varphi\colon M \to M$, such that $\varphi^\tau|_{\partial M} = \mathrm{id}$ and $g^\tau = (\varphi^\tau)^* g^0$, that is

$$g^\tau_{ij} = (g^0_{k\ell} \circ \varphi^\tau)\frac{\partial x'^k}{\partial x^i}(x,\tau)\frac{\partial x'^\ell}{\partial x^j}(x,\tau), \tag{1.7}$$

where $\varphi^\tau(x) = (\varphi^1(x,\tau), \dots, \varphi^n(x,\tau))$ and $x' = \varphi^\tau(x)$.

If we differentiate with respect to τ and we make $\tau = 0$, we obtain

Theorem 1.1 *The relation* (1.7) *implies*

$$f_{ij} = \frac{1}{2}(v_{i;j} + v_{j;i}), \tag{1.8}$$

$v = (v^i)$ being the covariant vector field that generates the one-parameter group (φ^τ) and $v_{i;j}$ is the covariant derivative of the covariant vector field $(v_i = g^0_{ij} v^j)$ in the metric g^0.

Proof. The covariant derivative of v in the metric g° is

$$v_{i;j} = \frac{\partial v_i}{\partial x^j} - \Gamma^k_{ij} v_k,$$

where

$$\begin{aligned} v_i &= g^0_{ij} v^j, \\ \Gamma^k_{ij} &= \frac{1}{2} g^{kp} \left(\frac{\partial g^0_{jp}}{\partial x^i} + \frac{\partial g^0_{ip}}{\partial x^j} - \frac{\partial g^0_{ij}}{\partial x^p} \right), \\ v^k(x) &= \left.\frac{\partial}{\partial \tau}\right|_{\tau=0} \varphi^k(x,\tau), \quad i,j,k = \overline{1,n}. \end{aligned}$$

Differentiating the relation (1.7) with respect to τ and then considering $\tau = 0$, we find

$$\begin{aligned} 2f_{ij} &= \left.\frac{\partial}{\partial \tau}\right|_{\tau=0} g^\tau_{ij} = \frac{\partial g^0_{k\ell}}{\partial \varphi^m} \left(\left.\frac{\partial}{\partial \tau}\right|_{\tau=0} \varphi^m \right) \frac{\partial x'^k}{\partial x^i}(x,0) \frac{\partial x'^\ell}{\partial x^j}(x,0) \\ &+ (g^0_{k\ell} \circ \varphi^0) \frac{\partial}{\partial \tau} \left(\frac{\partial x'^k}{\partial x^i} \right)(x,0) \frac{\partial x'^\ell}{\partial x^j}(x,0) \\ &+ (g^0_{k\ell} \circ \varphi^0) \frac{\partial x'^k}{\partial x^i}(x,0) \frac{\partial}{\partial \tau} \left(\frac{\partial x'^\ell}{\partial x^j} \right)(x,0) = \frac{\partial g^0_{k\ell}}{\partial x^m} v^m \frac{\partial x'^k}{\partial x^i} \cdot \frac{\partial x'^\ell}{\partial x^j} \\ &+ g^0_{k\ell} \frac{\partial}{\partial x^i} \left(\left.\frac{\partial}{\partial \tau}\right|_{\tau=0} x'^k \right) \frac{\partial x'^\ell}{\partial x^j} \\ &+ g^0_{k\ell} \frac{\partial x'^k}{\partial x^i} \cdot \frac{\partial}{\partial x^j} \left(\left.\frac{\partial}{\partial \tau}\right|_{\tau=0} x'^\ell \right) = \frac{\partial g^0_{k\ell}}{\partial x^m} v^m \delta^k_i \delta^\ell_j + g^0_{k\ell} \frac{\partial v^k}{\partial x^i} \delta^\ell_j \\ &+ g^0_{k\ell} \delta^k_i \frac{\partial v^\ell}{\partial x^j} = \frac{\partial g^0_{ij}}{\partial x^p} v^p + g^0_{jp} \frac{\partial v^p}{\partial x^i} + g^0_{ip} \frac{\partial v^p}{\partial x^j}. \end{aligned}$$

On the other hand

$$\begin{aligned} v_{i;j} + v_{j;i} &= \frac{\partial v_i}{\partial x^j} - \Gamma^m_{ij} v_m + \frac{\partial v_j}{\partial x^i} - \Gamma^m_{ji} v_m = \frac{\partial g^0_{im}}{\partial x^j} v^m + g^0_{im} \frac{\partial v^m}{\partial x^j} + \frac{\partial g^0_{jm}}{\partial x^i} v^m \\ &+ g^0_{jm} \frac{\partial v^m}{\partial x^i} - g^{mp}_0 \left(\frac{\partial g^0_{jp}}{\partial x^i} + \frac{\partial g^0_{ip}}{\partial x^j} - \frac{\partial g^0_{ij}}{\partial x^p} \right) g^0_{ms} v^s = \left(\frac{\partial g^0_{im}}{\partial x^j} + \frac{\partial g^0_{jm}}{\partial x^i} \right) v^m \end{aligned}$$

$$+\left(g^0_{im}\frac{\partial v^m}{\partial x^j}+g^0_{jm}\frac{\partial v^m}{\partial x^i}\right)-\left(\frac{\partial g^0_{jp}}{\partial x^i}+\frac{\partial g^0_{ip}}{\partial x^j}-\frac{\partial g^0_{ij}}{\partial x^p}\right)v^p$$
$$=\ \frac{\partial g^0_{ij}}{\partial x^p}v^p+g^0_{jp}\frac{\partial v^p}{\partial x^i}+g^0_{ip}\frac{\partial v^p}{\partial x^j}.$$

Hence, we obtained the equality

$$f_{ij}=\frac{1}{2}(v_{i;j}+v_{j;i}).$$

1.4 Ray Transform of a Tensor field

Let us generalize this problem from covariant tensors of second order to covariant tensors of superior order.

Let (M,g) be a simple Riemannian manifold and $\tau_M=(TM,p,M)$, $\tau'_M=(T'M,p',M)$ the tangent and the cotangent bundle of M, respectively. Let $S^m\tau'_M$ be the set of the symmetric tensor fields on τ'_M and $C^\infty(S^m\tau'_M)$ the space of the sections of this bundle. Consider ∇ the covariant derivative and σ the symmetrization.

Problem 2. Let (M,g) be a simple Riemannian manifold. Do integrals

$$I_f(x_{pq})=\int_{x_{pq}} f_{i_1\ldots i_m}(x(t))\dot{x}^{i_1}(t)\cdots\dot{x}^{i_m}(t)dt, \tag{1.9}$$

$p,q\in\partial M$, determine a symmetric tensor field $f\in C^\infty(S^m\tau'_M)$ (x_{pq} is the geodesic joining the endpoints p, q and dt is the geodesic arc length)? Particulary, the equality $I_f(x_{pq})=0$ allows us to state the existence of a field $v\in C^\infty(S^{m-1}\tau'_M)$, such that $v|_{\partial M}=0$ and $\sigma(\nabla v)=f$?

The function I_f, determined by the equality (1.9) on the set of the geodesics joining the points situated on the boundary of M, is called *single-ray transform of the tensor field f*.

Remark 1.2 *According[4],there are some known results on problem* 1.

R. Michel obtained a posi-tive answer to problem 1 in the two-dimensional case when g^0 has constant Gauss curvature.

R. G. Mukhometov, J. W. Bernstein and M. L. Gerver found a solution to the linear problem 2 for simple metrics, in the case $m=0$. When $m=1$, Yu. E. Anikonov and V. G. Romanov solved problem 2. R. G. Mukhometov generalised these results to metrics whose geodesics form a typical caustics.

2 Multi-time Case

2.1 Harmonic maps

Let (M,g) be a compact Riemannian manifold of dimension $n>1$, with the boundary ∂M. Let $(x^1,\ldots,x^n)$ be the local coordinates, $\Gamma_{ij,k}$, respectively Γ^i_{jk} the Christoffel symbols of the first type, respectively the second type.

Definition 2.1 *The pair of metrics (h,g) is called simple if for any $\sigma \in \partial M$ there is a unique minimal submanifold N represented by $x\colon T \cup \partial T \to M \cup \partial M$, $x\Big|_{\partial T} = \sigma$, T parallelipiped, $x(T \setminus \partial T) \subset M$, such that x depends smoothly on σ.*

Let (N,h) be a minimal Riemannian submanifold of M, $\dim N = p$, $2 \le p \le n$, fixed by a closed border σ of dimension $p-1$, included in ∂M. Suppose that ∂M is foliated by submanifolds of type σ. Let $(t^1,\dots,t^p)$ be the local coordinates in N.

Definition 2.2 *Let $x\colon T \to M$, $x(t) = (x^1(t),\dots,x^n(t))$, $t=(t^1,\dots,t^p)$, $x\Big|_{\partial T} = \sigma$, $x \in C^\infty(T,M)$.*

The integral

$$E_{(h,g)}(x) = \frac{1}{2}\int_T h^{\alpha\beta}(t) g_{ij}(x(t)) x^i_\alpha(x(t)) x^j_\beta(x(t)) dv_h,$$

where $(h^{\alpha\beta}) = (h_{\alpha\beta})^{-1}$, $h_{\alpha\beta} = h\left(\frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta}\right)$, $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ and $x^i_\alpha = \frac{\partial x^i}{\partial t^\alpha}$, is called the energy of the application x.

Proposition 2.1 *A minimum of the energy functional $E_{(h,g)}$, with the boundary condition $x\Big|_{\partial T} = \sigma$, necessarily verifies the boundary value problem*

$$\frac{\partial L}{\partial x^i} - \frac{\partial}{\partial t^\alpha}\left(\frac{\partial L}{\partial x^i_\alpha}\right) = 0, \quad i = \overline{1,n},$$

$$x\Big|_{\partial T} = \sigma,$$

where $L(x^i, x^i_\alpha) = \frac{1}{2}\sqrt{h} h^{\alpha\beta} g_{ij} x^i_\alpha x^j_\beta$ is the Lagrangian of the functional, $h = \det(h_{\alpha\beta})$.

Explicitly, $\tau(x) = 0$, $x\Big|_{\partial T} = \sigma$, where

$$\tau(x) = h^{\alpha\beta}\left\{\frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta} - \Gamma^\gamma_{\alpha\beta} x^i_\gamma + \Gamma^i_{jk} x^j_\alpha x^k_\beta\right\}\frac{\partial}{\partial x^i}$$

is the tension field of the application x.

Proof. We have

$$\begin{aligned}
\frac{\partial L}{\partial x^i} &= \frac{1}{2}\sqrt{h} h^{\beta\gamma}\frac{\partial}{\partial x^i}(g_{jk} x^j_\beta x^k_\gamma) = \frac{1}{2}\sqrt{h} h^{\beta\gamma}\frac{\partial g_{jk}}{\partial x^i} x^j_\beta x^k_\gamma \\
&= \frac{1}{2}\sqrt{h} h^{\beta\gamma}(\Gamma_{ij,k} + \Gamma_{ik,j}) x^j_\beta x^k_\gamma = \frac{1}{2}\sqrt{h} h^{\beta\gamma}\Gamma_{ij,k} x^j_\beta x^k_\gamma + \frac{1}{2}\sqrt{h} h^{\beta\gamma}\Gamma_{ik,j} x^j_\beta x^k_\gamma \\
&= \frac{1}{2}\sqrt{h} h^{\beta\gamma}\Gamma_{ij,k} x^j_\beta x^k_\gamma + \frac{1}{2}\sqrt{h} h^{\beta\gamma}\Gamma_{ij,k} x^j_\beta x^k_\gamma = \sqrt{h} h^{\beta\gamma}\Gamma_{ij,k} x^j_\beta x^k_\gamma; \\
\frac{\partial L}{\partial x^i_\alpha} &= \frac{1}{2}\sqrt{h} h^{\beta\gamma}\frac{\partial}{\partial x^i_\alpha}(g_{jk} x^j_\beta x^k_\gamma) = \sqrt{h} h^{\alpha\beta} g_{ij} x^j_\beta; \\
\frac{\partial}{\partial t^\alpha}\left(\frac{\partial L}{\partial x^i_\alpha}\right) &= \frac{\partial}{\partial t^\alpha}\left(\sqrt{h} h^{\alpha\beta} g_{ij} x^j_\beta\right) = \frac{1}{2\sqrt{h}}\frac{\partial h}{\partial h_{\gamma\mu}}\frac{\partial h_{\gamma\mu}}{\partial t^\alpha} h^{\alpha\beta} g_{ij} x^j_\beta + \sqrt{h}\frac{\partial h^{\alpha\beta}}{\partial t^\alpha} g_{ij} x^j_\beta
\end{aligned}$$

$$
\begin{aligned}
&+\sqrt{h}h^{\alpha\beta}\frac{\partial g_{ij}}{\partial x^k}x^j_\beta x^k_\alpha+\sqrt{h}h^{\alpha\beta}g_{ij}\frac{\partial^2 x^j}{\partial t^\alpha\partial t^\beta}\\
=\;&\frac{1}{2}\sqrt{h}h^{\gamma\mu}\left(h_{\gamma\nu}\Gamma^\nu_{\mu\alpha}+h_{\mu\nu}\Gamma^\nu_{\gamma\alpha}\right)h^{\alpha\beta}g_{ij}x^j_\beta-\sqrt{h}\left(h^{\alpha\nu}\Gamma^\beta_{\alpha\nu}+h^{\beta\nu}\Gamma^\alpha_{\nu\alpha}\right)g_{ij}x^j_\beta\\
&+\sqrt{h}h^{\alpha\beta}\frac{\partial g_{ij}}{\partial x^k}x^j_\beta x^k_\alpha+\sqrt{h}h^{\alpha\beta}g_{ij}\frac{\partial^2 x^i}{\partial t^\alpha\partial t^\beta}\\
=\;&\frac{1}{2}\sqrt{h}\left(\delta^\mu_\nu\Gamma^\nu_{\mu\alpha}+\delta^\gamma_\nu\Gamma^\nu_{\gamma\alpha}\right)h^{\alpha\beta}g_{ij}x^j_\beta-\sqrt{h}\left(h^{\alpha\nu}\Gamma^\beta_{\alpha\nu}+h^{\beta\nu}\Gamma^\alpha_{\nu\alpha}\right)g_{ij}x^j_\beta\\
&+\sqrt{h}h^{\alpha\beta}\left(\Gamma_{ik,j}+\Gamma_{jk,i}\right)x^j_\beta x^k_\alpha+\sqrt{h}h^{\alpha\beta}g_{ij}\frac{\partial^2 x^j}{\partial t^\alpha\partial t^\beta}\\
=\;&\frac{1}{2}\sqrt{h}\left(\Gamma^\mu_{\mu\alpha}+\Gamma^\gamma_{\gamma\alpha}\right)h^{\alpha\beta}g_{ij}x^j_\beta-\sqrt{h}\left(h^{\alpha\nu}\Gamma^\beta_{\alpha\nu}+h^{\beta\nu}\Gamma^\alpha_{\nu\alpha}\right)g_{ij}x^j_\beta\\
&+\sqrt{h}h^{\alpha\beta}\left(\Gamma_{ik,j}+\Gamma_{jk,i}\right)x^j_\beta x^k_\alpha+\sqrt{h}h^{\alpha\beta}g_{ij}\frac{\partial^2 x^j}{\partial t^\alpha\partial t^\beta}\\
=\;&\sqrt{h}\Gamma^\mu_{\mu\alpha}h^{\alpha\beta}g_{ij}x^j_\beta-\sqrt{h}h^{\alpha\nu}\Gamma^\beta_{\alpha\nu}g_{ij}x^j_\beta-\sqrt{h}h^{\beta\nu}\Gamma^\alpha_{\nu\alpha}g_{ij}x^j_\beta\\
&+\sqrt{h}h^{\alpha\beta}\left(\Gamma_{ik,j}+\Gamma_{jk,i}\right)x^j_\beta x^k_\alpha+\sqrt{h}h^{\alpha\beta}g_{ij}\frac{\partial^2 x^j}{\partial t^\alpha\partial t^\beta}\\
=\;&\sqrt{h}\Gamma^\mu_{\mu\alpha}h^{\alpha\beta}g_{ij}x^j_\beta-\sqrt{h}h^{\alpha\nu}\Gamma^\beta_{\alpha\nu}g_{ij}x^j_\beta-\sqrt{h}\Gamma^\mu_{\mu\alpha}h^{\alpha\beta}g_{ij}x^j_\beta\\
&+\sqrt{h}h^{\alpha\beta}\left(\Gamma_{ik,j}+\Gamma_{jk,i}\right)x^j_\beta x^k_\alpha+\sqrt{h}h^{\alpha\beta}g_{ij}\frac{\partial^2 x^j}{\partial t^\alpha\partial t^\beta}\\
=\;&\sqrt{h}\left[h^{\alpha\beta}g_{ij}\frac{\partial^2 x^j}{\partial t^\alpha\partial t^\beta}-h^{\alpha\nu}\Gamma^\beta_{\alpha\nu}g_{ij}x^j_\beta+h^{\alpha\beta}\left(\Gamma_{ik,j}+\Gamma_{jk,i}\right)x^j_\beta x^k_\alpha\right].
\end{aligned}
$$

Finally, we obtain

$$\frac{\partial L}{\partial x^\alpha}-\frac{\partial}{\partial t^i}\left(\frac{\partial L}{\partial x^\alpha_i}\right)=0\Leftrightarrow$$

$$h^{\alpha\beta}\left[\Gamma_{ij,k}x^j_\alpha x^k_\beta-g_{ij}\frac{\partial^2 x^j}{\partial t^\alpha\partial t^\beta}+\Gamma^\gamma_{\alpha\beta}g_{ij}x^j_\gamma-\left(\Gamma_{ik,j}+\Gamma_{jk,i}\right)x^k_\alpha x^j_\beta\right]=0\Leftrightarrow$$

$$h^{\alpha\beta}g_{i\ell}\left(\frac{\partial^2 x^\ell}{\partial t^\alpha\partial t^\beta}-\Gamma^\gamma_{\alpha\beta}x^i_\gamma+\Gamma^i_{jk}x^j_\alpha x^k_\beta\right)=0.$$

We suppose that the problem stated in the previous proposition has a unique solution.

Remark 2.1 a) *The mapping $x\in C^\infty(T,M)$ for which $\tau(x)=0$ is called harmonic mapping.*

b) *If the mapping $x\in C^\infty(T,M)$ is a Riemannian immersion, x is harmonic if and only if x is minimal.*

2.2 Determining a Pair of Metrics by Boundary Energy

Let us consider (h,g) a simple pair of metrics.

Definition 2.3 *Let $\sigma\in\partial M$ and $E_{(h,g)}(\sigma)$ the energy of the submanifold N that corres-ponds to the border σ. The function $E_{(h,g)}\colon\partial M\to\mathbf{R}$ generated by the correspondence $\sigma\to E_{(h,g)}(\sigma)$ is called the boundary energy.*

Given an energy function E, is there a pair of simple metrics (h, g) that realize that energy? How can these metrics be found?

Let us show that the existence problem of the metrics with the property that $E: \partial M \to \mathbf{R}$ represents the boundary energy cannot have a unique solution.

Let $\Phi: T \times M \to T \times M$, $\Phi(t^1, \ldots, t^p; x^1, \ldots, x^n) = (\psi(t), \varphi(x))$ be a diffeomorphism with the properties $\psi|_{\partial T} = \mathrm{id}$, $\varphi|_{\partial M} = \mathrm{id}$. The diffeomorphism transforms the simple metrics h^0, g^0 into the simple metrics $h^1 = \psi^* h^0$ and $g^1 = \varphi^* g^0$, because we have

$$h^1(t)(\mu, \nu) = h^0((d_t\psi)\mu, (d_t\psi)\nu)_{\psi(t)},$$

where $d_t\psi: T_tT \to T_{\psi(t)}T$ is the differential of ψ, and

$$g^1(x)(\xi, \eta) = g^0((d_x\varphi)\xi, (d_x\varphi)\eta)_{\varphi(x)},$$

$d_x\varphi: T_xM \to T_{\varphi(x)}M$ is the differential of φ.

(h^0, g^0) and (h^1, g^1) give different families of minimal submanifolds with the same boundary energy E.

Problem $1'$. Let (h^0, g^0) and (h^1, g^1) be pairs of simple metrics, h^0, h^1 on T, respectively g^0, g^1 on M. The equality $E_{(h^0,g^0)} = E_{(h^1,g^1)}$ implies the existence of a diffeomorphism $\Phi: T \times M \to T \times M$, $\Phi = (\psi, \varphi)$, $\psi|_{\partial T} = \mathrm{id}$, $\varphi|_{\partial M} = \mathrm{id}$, $h^1 = \psi^* h^0$ and $g^1 = \varphi^* g^1$ (the problem of finding a pair metrics by the boundary energy)?

2.3 Linearization of the Problem of Determining Metrics by the Boundary Energy

Let us linearize the problem $1'$. Let (g^τ) be a family of simple metrics on M which depends smoothly on $\tau \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$. Let $\sigma \in \partial M$ and $a = E(\sigma)$, $E: \partial M \to \mathbf{R}$ the given frontier energy. Consider $x^\tau: T \to M$ a minimal submanifold of the metric g^τ, for which $x^\tau\Big|_{\partial T} = \sigma$, $x^i = x^i(t^\alpha)$, $\alpha = \overline{1,p}$, $i = \overline{1,n}$. Let $T = [0, a]^p$ with the induced Riemannian metric $(h^\tau_{\alpha\beta})$, $t = (t^1, \ldots, t^p)$.

Let $x^\tau(t) = (x^1(t, \tau), \ldots, x^n(t, \tau))$ be the representation of x^τ in a coordinate system and $g^\tau = (g^\tau_{ij})$. The energy of the deformation x^τ is

$$E_{(h^\tau, g^\tau)}(\sigma) = \frac{1}{2} \int_T h^{\alpha\beta}_\tau(t) g^\tau_{ij}(x^\tau(t)) x^i_\alpha(t, \tau) x^j_\beta(t, \tau) dv_h.$$

Differentiating with respect to τ, we obtain

$$\begin{aligned}
\frac{\partial}{\partial\tau}\Big|_{\tau=0} E_{(h^\tau, g^\tau)}(\sigma) &= \int_T \left[h^{\alpha\beta}_0(t) f_{ij}(x^0(t)) + k^{\alpha\beta}(t) g^0_{ij}(x^0(t))\right] x^i_\alpha(t, 0) x^j_\beta(t, 0) dv_h \\
&+ \frac{1}{2} \int_T \left[h^{\alpha\beta}_0(t) \frac{\partial g^0_{ij}}{\partial x^k}(x^0(t)) x^i_\alpha(t, 0) x^j_\beta(t, 0) \frac{\partial x^k}{\partial\tau}(t, 0)\right. \\
&\left. + 2h^{\alpha\beta}_0(t) g^0_{ij}(x^0(t)) x^i_\alpha(t, 0) \frac{\partial x^j_\beta}{\partial\tau}(t, 0)\right] dv_h,
\end{aligned}$$

(2.1)

where $f_{ij} = \frac{1}{2}\frac{\partial}{\partial\tau}\Big|_{\tau=0} g^{\tau}_{ij}$ and $k^{\alpha\beta} = \frac{1}{2}\frac{\partial}{\partial\tau}\Big|_{\tau=0} h^{\alpha\beta}_{\tau}$.

Integrating by parts and using the fact that $\frac{\partial x^i}{\partial\tau}\Big|_{\partial T} = 0$, we have

$$\int_T h_0^{\alpha\beta}(t)g^0_{ij}(x^0(t))x^i_\alpha(t,\tau)\frac{\partial x^j_\beta}{\partial\tau}(t,0)dv_h = \int_T h_0^{\alpha\beta}g^0_{ij}x^i_\alpha\frac{\partial}{\partial t^\beta}\left(\frac{\partial x^j}{\partial\tau}\right)\sqrt{h}dt^1\wedge\cdots\wedge dt^p$$

$$= -\int_T\left[\frac{\partial h_0^{\alpha\beta}}{\partial t^\beta}g^0_{ij}x^i_\alpha\sqrt{h} + h_0^{\alpha\beta}\frac{\partial g^0_{ij}}{\partial x^k}x^k_\beta x^i_\alpha\sqrt{h} + h_0^{\alpha\beta}g^0_{ij}\frac{\partial^2 x^i}{\partial t^\alpha\partial t^\beta}\sqrt{h}\right.$$
$$\left.+h_0^{\alpha\beta}g^0_{ij}x^i_\alpha\frac{1}{2\sqrt{h}}\frac{\partial h}{\partial h_{\gamma\delta}}\frac{\partial h_{\gamma\delta}}{\partial t^\beta}\right]\frac{\partial x^j}{\partial\tau}\frac{1}{\sqrt{h}}dv_h$$
$$= -\int_T\left[\frac{\partial h_0^{\alpha\beta}}{\partial t^\beta}g^0_{ij}x^i_\alpha + h_0^{\alpha\beta}\frac{\partial g^0_{ij}}{\partial x^k}x^k_\beta x^i_\alpha + h_0^{\alpha\beta}g^0_{ij}\frac{\partial^2 x^i}{\partial t^\alpha\partial t^\beta}\right.$$
$$\left.+\frac{1}{2}h_0^{\alpha\beta}h_0^{\gamma\delta}g^0_{ij}x^i_\alpha\frac{\partial h_{\gamma\delta}}{\partial t^\beta}\right]\frac{\partial x^j}{\partial\tau}dv_h.$$

The second integral of (2.1) becomes

$$\int_T\left[h_0^{\alpha\beta}\frac{\partial g^0_{i\ell}}{\partial x^j}x^i_\alpha x^\ell_\beta - 2\frac{\partial h_0^{\alpha\beta}}{\partial t^\beta}g^0_{ij}x^i_\alpha - 2h_0^{\alpha\beta}\frac{\partial g^0_{ij}}{\partial x^\ell}x^\ell_\beta x^i_\alpha - 2h_0^{\alpha\beta}g^0_{ij}\frac{\partial^2 x^i}{\partial t^\alpha\partial t^\beta}\right.$$
$$\left.-h_0^{\alpha\beta}h_0^{\gamma\delta}g^0_{ij}x^i_\alpha\frac{\partial h^0_{\gamma\delta}}{\partial t^\beta}\right]\frac{\partial x^j}{\partial\tau}dv_h$$
$$= \int_T\left\{h_0^{\alpha\beta}\left[\frac{\partial g^0_{i\ell}}{\partial x^j} - 2\frac{\partial g^0_{ij}}{\partial x^\ell}\right]x^i_\alpha x^\ell_\beta - 2\frac{\partial h_0^{\alpha\beta}}{\partial t^\beta}g^0_{ij}x^i_\alpha - 2h_0^{\alpha\beta}g^0_{ij}\frac{\partial^2 x^i}{\partial t^\alpha\partial t^\beta}\right.$$
$$\left.-h_0^{\alpha\beta}h_0^{\gamma\delta}g^0_{ij}x^i_\alpha\frac{\partial h^0_{\gamma\delta}}{\partial t^\beta}\right\}\frac{\partial x^j}{\partial\tau}dv_h$$
$$= \int_T\left[-2h_0^{\alpha\beta}\Gamma_{i\ell,j}x^i_\alpha x^\ell_\beta - 2\frac{\partial h_0^{\alpha\beta}}{\partial t^\beta}g^0_{ij}x^i_\alpha - 2h_0^{\alpha\beta}g^0_{ij}\frac{\partial^2 x^i}{\partial t^\alpha\partial t^\beta}\right.$$
$$\left.-h_0^{\alpha\beta}h_0^{\gamma\delta}g^0_{ij}x^i_\alpha\frac{\partial h^0_{\gamma\delta}}{\partial t^\beta}\right]\frac{\partial x^j}{\partial\tau}dv_h$$
$$= \int_T\left\{-2h_0^{\alpha\beta}\Gamma_{i\ell,j}x^i_\alpha x^\ell_\beta - 2\frac{\partial h_0^{\alpha\beta}}{\partial t^\beta}g^0_{ij}x^i_\alpha - 2h_0^{\alpha\beta}g^0_{ij}\frac{\partial^2 x^i}{\partial t^\alpha\partial x^\beta}\right.$$
$$\left.-h_0^{\alpha\beta}g^0_{ij}x^i_\alpha h_0^{\gamma\delta}\left[\Gamma_{\gamma\beta,\delta} + \Gamma_{\delta\beta,\gamma}\right]\right\}\frac{\partial x^j}{\partial\tau}dv_h$$
$$= \int_T\left\{-2h_0^{\alpha\beta}g^0_{jp}\Gamma^p_{i\ell}x^i_\alpha x^\ell_\beta - 2\frac{\partial h_0^{\alpha\beta}}{\partial t^\beta}g^0_{ij}x^i_\alpha - 2h_0^{\alpha\beta}g^0_{ij}\frac{\partial^2 x^i}{\partial t^\alpha\partial t^\beta}\right.$$
$$\left.-h_0^{\alpha\beta}g^0_{ij}x^i_\alpha\left(\Gamma^\gamma_{\gamma\beta} + \Gamma^\delta_{\delta\beta}\right)\right\}\frac{\partial x^j}{\partial\tau}dv_h$$
$$= \int_T\left\{2h_0^{\alpha\beta}g^0_{jp}\left(-\Gamma^p_{i\ell}x^i_\alpha x^\ell_\beta - \frac{\partial^2 x^p}{\partial t^\alpha\partial t^\beta}\right) - 2\frac{\partial h_0^{\alpha\beta}}{\partial t^\beta}g^0_{ij}x^i_\alpha\right.$$

$$-2h_0^{\alpha\beta}g^0_{ij}x^i_\alpha\Gamma^\gamma_{\gamma\beta}\Big\}\frac{\partial x^j}{\partial\tau}dv_h.$$

Since x_0 is an extremal of the energy, we have

$$h_0^{\alpha\beta}g^0_{jp}\left(-\Gamma^p_{i\ell}x^i_\alpha x^\ell_\beta-\frac{\partial^2x^p}{\partial t^\alpha\partial t^\beta}\right)=-2g^0_{jp}h^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}x^p_\gamma$$

and the previous integral becomes

$$\int_T\left[-2g^0_{jp}h_0^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}x^p_\gamma-2\frac{\partial h_0^{\alpha\beta}}{\partial t^\beta}g^0_{ij}x^i_\alpha-2h_0^{\alpha\beta}g^0_{ij}x^i_\alpha\Gamma^\gamma_{\gamma\beta}\right]\frac{\partial x^j}{\partial\tau}dv_h$$
$$=-2\int_T g^0_{ij}x^i_\mu\left(h_0^{\alpha\beta}\Gamma^\mu_{\alpha\beta}+\frac{\partial h_0^{\mu\beta}}{\partial t^\beta}+h_0^{\mu\beta}\Gamma^\gamma_{\mu\beta}\right)\frac{\partial x^j}{\partial\tau}dv_h$$
$$=-2\int_T g^0_{ij}x^i_\mu\left(h_0^{\alpha\beta}\Gamma^\mu_{\alpha\beta}-h_0^{\mu\nu}\Gamma^\beta_{\beta\gamma}-h^{\gamma\beta}\Gamma^\mu_{\beta\nu}+h_0^{\mu\beta}\Gamma^\gamma_{\gamma\beta}\right)\frac{\partial x^j}{\partial\tau}dv_h=0.$$

Denoting $F^{\alpha\beta}_{ij}=h_0^{\alpha\beta}f_{ij}+k^{\alpha\beta}g^0_{ij}$, we have the equality

$$\frac{\partial}{\partial\tau}\bigg|_{\tau=0}E_{(h^\tau,g^\tau)}(\sigma)=\int_T F^{\alpha\beta}_{ij}x^i_\alpha(t,0)x^j_\beta(t,0)dv_h.$$

Using the functional $I_F(x_0)=\int_T F^{\alpha\beta}_{ij}(x(t))x^i_\alpha(t)x^j_\beta(t)dv_h$, the previous relation becomes

(2.2)
$$\frac{\partial}{\partial\tau}\bigg|_{\tau=0}E_{(h^\tau,g^\tau)}(\sigma)=I_F(x^0),$$

where x^0 is a minimal submanifold of the metric g^0.

The existence of solutions of this problem for the family (g^τ) implies the existence of an one-parameter group of diffeomorphisms $\Phi^\tau(t,x)=(\psi^\tau(t),\varphi^\tau(x))$, such that $g^\tau=(\varphi^\tau)^*g^0$ and $h^\tau=(\psi^\tau)^*h^0$. Explicitly

(2.3)
$$h^\tau_{\alpha\beta}=\left(h^0_{\mu\nu}\circ\psi^\tau\right)\frac{\partial t'^\mu}{\partial t^\alpha}(t,\tau)\frac{\partial t'^\nu}{\partial t^\beta}(t,\tau),$$

where $\psi^\tau(t)=(\psi^1(t,\tau),\dots,\psi^p(t,\tau))$, $t'=\psi^\tau(t)$,

(2.4)
$$g^\tau_{ij}=\left(g^0_{k\ell}\circ\varphi^\tau\right)\frac{\partial x'^k}{\partial x^i}(x,\tau)\frac{\partial x'^\ell}{\partial x^j}(x,\tau),$$

where $\varphi^\tau(x)=(\varphi^1(x,\tau),\dots,\varphi^n(x,\tau))$, $x'=\varphi^\tau(x)$.

Instead of (2.3), we need

(2.5)
$$h^{\alpha\beta}_\tau=\left(h_0^{\mu\nu}\circ\psi^{-\tau}\right)\frac{\partial t^\alpha}{\partial t'^\mu}(t,\tau)\frac{\partial t^\beta}{\partial t'^\nu}(t,\tau).$$

Theorem 2.1 *The relations* (2.4) *and* (2.5) *imply*

(2.6)
$$f_{ij}=\frac{1}{2}(v_{i;j}+v_{j;i}),$$

where $v^k(x) = \dfrac{\partial}{\partial \tau}(x'^k)(x,\tau)$, $v_i = g^0_{ij} v^j$ *and* $v_{i;j}$ *is the covariant derivative of* (v_i) *and*

$$k^{\alpha\beta} = \frac{1}{2}(u^{\alpha;\beta} + u^{\beta;\alpha}), \tag{2.7}$$

where $\dfrac{\partial}{\partial \tau}\Big|_{\tau=0} (\psi^\alpha)(t,\tau) = u^\alpha$, $u^{\alpha;\beta} = u^\alpha_{;\mu} h_0^{\mu\beta}$ *and* $u^\alpha_{;\mu}$ *is the covariant derivative of* (u^α).

Proof. The relation (2.6) is similar to relation (1.8). Differentiating the relation (2.5) with respect to τ, we have

$$\begin{aligned} 2k^{\alpha\beta} &= \frac{\partial h_0^{\mu\nu}}{\partial t^\gamma}\left(\frac{\partial}{\partial \tau}\Big|_{\tau=0} t^\gamma\right)\frac{\partial t^\alpha}{\partial t'^\mu}(t,0)\frac{\partial t^\beta}{\partial t'^\nu}(t,0) + (h_0^{\mu\nu}\circ\psi^0)\frac{\partial}{\partial \tau}\left(\frac{\partial t^\alpha}{\partial t'^\mu}\right)(t,0)\frac{\partial t^\beta}{\partial t'^\nu}(t,0) \\ &\quad + (h_0^{\mu\nu}\circ\psi^0)\frac{\partial t^\alpha}{\partial t'^\mu}(t,0)\frac{\partial}{\partial \tau}\left(\frac{\partial t^\beta}{\partial t'^\nu}\right)(t,0) = -\frac{\partial h_0^{\mu\nu}}{\partial t^\gamma}\mu^\gamma \delta^\alpha_\mu \delta^\beta_\nu \\ &\quad + h_0^{\mu\nu}\frac{\partial}{\partial t'^\mu}\left(\frac{\partial}{\partial \tau}\Big|_{\tau=0} t^\alpha\right)\delta^\beta_\nu + h_0^{\mu\nu}\delta^\alpha_\mu \frac{\partial}{\partial t'^\nu}\left(\frac{\partial}{\partial \tau}\Big|_{\tau=0} t^\beta\right) \\ &= -\frac{\partial h_0^{\alpha\beta}}{\partial t'^\mu}u^\mu + h_0^{\mu\beta}\frac{\partial u^\beta}{\partial t'^\mu} + h_0^{\nu\alpha}\frac{\partial u^\alpha}{\partial t'^\mu}. \end{aligned}$$

On the other hand

$$\begin{aligned} u^{\alpha;\beta} + u^{\beta;\alpha} &= u^\alpha_{;\mu} h_0^{\mu\beta} + u^\beta_{;\mu} h_0^{\alpha\mu} = h_0^{\mu\beta}\left(\frac{\partial u^\alpha}{\partial t'^\mu} + \Gamma^\alpha_{\mu\nu} u^\nu\right) + h^{\alpha\mu}\left(\frac{\partial u^\beta}{\partial t'^\mu} + \Gamma^\beta_{\mu\nu} u^\nu\right) \\ &= h_0^{\mu\nu}\frac{\partial u^\alpha}{\partial t'^\mu} + h_0^{\alpha\mu}\frac{\partial u^\beta}{\partial t'^\mu} + \left(h_0^{\mu\beta}\Gamma^\alpha_{\mu\nu} + h_0^{\alpha\mu}\Gamma^\beta_{\mu\nu}\right)u^\nu \\ &= h_0^{\mu\beta}\frac{\partial u^\alpha}{\partial t'^\mu} + h_0^{\alpha\mu}\frac{\partial u^\beta}{\partial t'^\mu} - \frac{\partial h_0^{\alpha\beta}}{\partial t'^\mu}u^\mu. \end{aligned}$$

The equality (2.7) was proved.

We have the following linearization of the problem $1'$: do integrals (2.2) determine the tensor $\left(F^{\alpha\beta}_{ij}\right)$?

2.4 Multi-ray Transform of a Distinguished Tensor Field

Problem $2'$. Generalizing the problem to tensor fields of any rank, the following question appears: to what extent the integrals

$$I_F(x) = \int_T F^{\alpha_1 \ldots \alpha_m}_{i_1 \ldots i_m}(x(t)) x^{i_1}_{\alpha_1}(t) \cdots x^{i_m}_{\alpha_m}(t) dv_h \tag{2.8}$$

determine a symmetric tensor field F?

The function I_F, determined by the equality (2.8) on the set of submanifolds $\sigma \in \partial M$, is called *multi-ray transform of the tensor* F.

References

[1] V.I. Arnold, *Metodele matematice ale mecanicii clasice*, Editura Ştiinţifică şi Enciclopedică, Bucureşti, 1980.

[2] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, I, II, Interscience Publishers, New York, 1963, 1969.

[3] A. Mishchenko, A. Fomenko, *A Course of Differential Geometry and Topology*, Mir Publishers, Moscow, 1988.

[4] A. Sharafutdinov, *Integral Geometry of Tensor Fields*, VSPBV, Utrecht, 1994.

[5] C. Udrişte, *Linii de câmp*, Editura Tehnică, Bucureşti, 1998.

[6] C. Udrişte, *Geometric Dynamics*, Kluver Academic Publishers, 2000.

[7] C. Udrişte, M. Ferrara, D. Opriş, *Economic Geometric Dynamics*, Geometry Balkan Press, 2004.

[8] C. Udrişte, M. Postolache, *Atlas of Magnetic Geometric Dynamics*, Geometry Balkan Press, 2001.

[9] H. Urakawa, *Calculus of Variations and Harmonic Maps*, Shokabo Publishing Co. Ltd., Tokyo, 1990.

[10] Gh. Vrânceanu, *Geometrie diferenţială globală*, Editura Didactică şi Pedagogică, Bucureşti, 1973.

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