Thermodynamics of materials with internal variables in the context of the Green and Naghdi theories

V. Ciancio and R. Quintanilla

Abstract. In this paper we propose some thermodynamic theories of materials with internal variables. In concrete, we extend the arguments of the theories of the thermoelasticity of type I, II and III proposed by Green and Naghdi to the case of materials with internal variables. We state the system of equations in the linear case when the material is centrosymmetric. We end with an existence and uniqueness result for the linear equations in the cases of type I and II.

M.S.C. 2000: 35L05, 49M50, 35E05.
Key words: Thermodynamics, materials with internal variables, linear theory, wave propagation, relaxation phenomena, phenomenological coefficients.

1 Introduction

The usual theory of heat conduction based on the Fourier law allows the phenomena of the "infinite diffusion velocity". This is not well accepted from a physical point of view. The articles of Dreyer and Struchtrup [11] and Caviglia *et al.* [1] provide an extensive survey of work on experiments involving the propagation of heat as a thermal wave. They report instances where the phenomena of second sound has been observed in several kinds of materials. In [10, 9] and in the books of Jou *et al.* [18] and Müller and Ruggeri [23] alternative formulations for the heat conduction are considered. In the recent surveys of Chandrasekharaiah [3, 2] and Hetnarski and Ignaczak [17] the theory proposed by Green and Naghdi [12]-[16] is considered as an alternative way of formulation of the thermomechanical theories. This theory is developed in a rational way to produce a fully consistent. It makes use of a general entropy balance rather than an entropy inequality. The development is quite general and the characterization of material response for the thermal phenomena is based on three types of constitutive functions.

We consider this theory that is a good model to explain the heat conduction for several kinds of solids and fluids. A natural question is to know what happens when the mechanical structure is more complicated than the simple elastic materials. In

Balkan Journal of Geometry and Its Applications, Vol.12, No.1, 2007, pp. 16-31.

fact the works of Green and Naghdi [13] seem to have a similar point of view, but in a more general context. Here, we try to extend this theory to the case of materials with internal variables.

We believe that the mathematical and physical analysis will reveal the usefulness of this theory and it is to this end that the present paper is addressed. In this sense our work tries to follow the work initiated in [24, 25].

Materials with internal variables have deserved a great attention in recent years [19], [20],[22], [21]. This theory allow us to describe several inelastic questions [8], [6], [4]. It is worth recalling the works of Ciancio *et al.* [7, 5], where the authors proposed a thermodynamic theory for thermoelastic and viscoanelastic solids with non-euclidean structure in which an additional dissipation occur no related to heat conduction. In these references the non-euclidean structure is described by a intrinsic material metric in which the metric tensor, G, plays a role of thermodynamic internal variable and whose evolution obeys the usual thermodynamic constrains of positive rate of dissipation and the thermal structure was based on the entropy inequality.

Inspired in this approach we propose a theory of materials with internal variables where the thermal structure is based in an entropy equality following the works of Green and Naghdi.

The structure of this paper is as follows. In the second section we give some preliminaries concerning the basic equations and variables in the theory of Green and Nagdhi. The basic equations for the theories of type I, II and III are developed in the sections three, four and five respectively. In section six we propose the existence, uniqueness and continuous dependence of solutions with respect initial conditions and supply terms for the linear problem of the type I theory. In the last section we make the same, but for the type II theory.

2 Preliminaries

In this section we summarize the main kinetic, basic concepts and balance equation in the form proposed by Green and Naghdi [12]-[16]. We consider a material volume $\mathcal{B}$ with boundary $\partial\mathcal{B}$; and we denote B the corresponding region in the reference configuration.

The equations for mass conservation and balance of linear momentum are

$$\dot{\rho} + \rho \text{ div } \dot{x} = 0, \tag{2.1}$$

$$\rho\dot{x} = \text{ div } T + \rho b. \tag{2.2}$$

In equations (2.1), (2.2), $x(X,t)$ means the motion of the point which in the reference configuration occupies the point X at the moment t, ρ is the mass density in the reference configuration, b is the external body force and T is the stress tensor. The expression div means the divergence operator with respect the material coordinates.

The main thermal variables that we will utilize in the remainder of the paper are:

1. α is the thermal displacement, $\alpha = \alpha(X,t)$.
2. θ is the temperature, $\dot{\alpha} = \theta$.
3. S is the external rate of supply entropy.
4. ξ is the internal rate of production of entropy.

5. $\mathbf{\Phi}$ is the internal flux of entropy.
6. $q = \theta\, \mathbf{\Phi}$ is the heat of flux.
7. η is the entropy.

The balance of the entropy can be postulated as

$$\rho\dot{\eta} = \rho(S + \xi) + \text{ div } \mathbf{\Phi}. \tag{2.3}$$

The balance of the energy can be written in the form

$$\rho\dot{e} = T : \nabla\dot{x} + \text{ div } q + \rho S\theta, \tag{2.4}$$

where e is the internal energy. If we introduce the free energy

$$\Psi = e - \theta\eta, \tag{2.5}$$

we obtain

$$\rho(\dot{\Psi} + \dot{\theta}\eta) - T : \nabla\dot{x} + \rho\theta\xi - \mathbf{\Phi} \cdot \nabla\theta = 0. \tag{2.6}$$

3 Theory of type I

In the theory of type I we postulate the following constitutive equations

$$\Psi = \hat{\Psi}(A),\, T = \hat{T}(A),\, \eta = \hat{\eta}(A),\, \mathbf{\Phi} = \hat{\mathbf{\Phi}}(A),\, \xi = \hat{\xi}(A),\ \dot{G} = \hat{H}(A) \tag{3.1}$$

where

$$A = (\nabla x, \theta, \nabla\theta, G). \tag{3.2}$$

We assume that the response functions are of C^1-class. Substituting the constitutive equations into (2.6), we have

$$\rho(\frac{\partial\Psi}{\partial\theta} + \eta)\dot{\theta} + (\rho\frac{\partial\Psi}{\partial\nabla x} - T) : \nabla\dot{x} + \rho\frac{\partial\Psi}{\partial\nabla\theta} \cdot \nabla\dot{\theta} - \mathbf{\Phi} \cdot \nabla\theta + \rho\xi\theta + \rho\frac{\partial\Psi}{\partial G} : \dot{G} = 0. \tag{3.3}$$

From this equality we see that the constitutive equations which are compatible with the energy equation have the form

$$\frac{\partial\Psi}{\partial\nabla\theta} = 0,\ \ \eta = -\frac{\partial\Psi}{\partial\theta},\ \ T = \rho\frac{\partial\Psi}{\partial\nabla x}, \tag{3.4}$$

$$\rho\theta\xi = \mathbf{\Phi} \cdot \nabla\theta - \rho\frac{\partial\Psi}{\partial G} : \dot{G}. \tag{3.5}$$

In the relation (3.5) we see that the constitutive equation for the internal rate of production of the entropy can be given in terms of the internal flux of the entropy and the constitutive equations for the variation of the internal variables.

The equations which govern the evolution of the thermodynamics of materials with internal variables in the context of the theory of type I are given by the evolutionary equations (2.2), (2.3) and the constitutive equations (3.4), (3.5).

If we assume that

$$\frac{\partial e}{\partial \theta} \neq 0, \tag{3.6}$$

we can consider the set of variables

$$A = (e, \nabla x, \nabla \theta, G). \tag{3.7}$$

If we apply again the energy equation we see that

$$\rho\Big(1 - \theta\frac{\partial \eta}{\partial e}\Big)\dot{e} + (\rho\theta\frac{\partial \eta}{\partial \nabla x} + T) : \nabla \dot{x} + \rho\theta\frac{\partial \eta}{\partial \nabla \theta} \cdot \nabla\dot{\theta} - \mathbf{\Phi} \cdot \nabla\theta + \rho\theta\xi + \rho\theta\frac{\partial \eta}{\partial G} : \dot{G} = 0. \tag{3.8}$$

Thus, we obtain

$$\frac{\partial \eta}{\partial \nabla \theta} = 0, \;\; \theta^{-1} = \frac{\partial \eta}{\partial e}, \;\; T \; = \; -\rho\theta\frac{\partial \eta}{\partial \nabla x} \tag{3.9}$$

and

$$\rho\theta\xi = \mathbf{\Phi} \cdot \nabla\theta - \rho\theta\frac{\partial \eta}{\partial G} : \dot{G}. \tag{3.10}$$

In the remain of this section we assume that at time $t = 0$, we have

$$\theta(X, 0) = T_0, \;\; G(X, 0) = G_0, \tag{3.11}$$

where T_0 and G_0 are constants. We denote by

$$u_i = x_i - \delta_{iA}X_A, \;\; h = G - G_0, \tag{3.12}$$

where δ_{iA} is the Kronecker delta and

$$T = \theta - T_0. \tag{3.13}$$

In the linear theory we assume that

$$u = \epsilon u', \;\; T = \epsilon T', \;\; h = \epsilon h', \tag{3.14}$$

where ϵ is a parameter small enough for the squares and higher powers to be neglected and u', T' and h' are independent of ϵ.

In the case of centrosymmetric materials we assume that

$$\rho\Psi = \frac{1}{2}\nabla u : C\nabla u - \frac{1}{2}aT^2 - T\,\boldsymbol{\beta} : \nabla u + \nabla u : m_1 h - T\,m_2 : h + \frac{1}{2}\,h : m_3 h \tag{3.15}$$

$$\mathbf{\Phi} = K_0\nabla T, \tag{3.16}$$

$$\dot{h} = m_4 \nabla u + T\, m_5 + m_6 h. \tag{3.17}$$

where m_1, m_3, m_4, m_6 are tensors of order four while $\boldsymbol{\beta}, K_0, h, m_2, m_5$ of order two.

Here C and K_0 satisfy the symmetry

$$C = C^t, \quad K_0 = K_0^t. \tag{3.18}$$

The tensor C is called elasticity tensor, a, K_0 and $\boldsymbol{\beta}$ are related with the heat capacity, thermal conductivity and the thermal dilatation and $m_1, ..., m_6$ are related with the internal variables.

From (3.4), (3.5) we have

$$T = C\nabla u - \boldsymbol{\beta} T + m_1 h, \quad \rho\eta = \boldsymbol{\beta} : \nabla u + m_2 : h + aT, \quad \xi = 0. \tag{3.19}$$

Thus, the system of equations that governs our problem is

$$\rho \ddot{u} = \text{ div } \Big(C\nabla u - \boldsymbol{\beta} T + m_1 h \Big) + \rho b, \tag{3.20}$$

$$a\dot{T} = -\boldsymbol{\beta} : \nabla \dot{u} - m_2 : \Big(m_4 \nabla u + m_5 T + m_6 h \Big) + \text{ div } (K_0 \nabla T) + \rho S, \tag{3.21}$$

and the equation (3.17). It is worth noting that equation (3.21) can be written as

$$a\dot{T} = -\boldsymbol{\beta} : \nabla \dot{u} + \text{ div } (K_0 \nabla T) + p_1 : \nabla u + p_2 : h + p_3 T + \rho S, \tag{3.22}$$

where p_1, p_2 and p_3 can be given in terms of the m_i.

To have a defined problem we need to impose boundary and initial conditions. The Dirichlet homogeneous boundary conditions are

$$u = 0, \quad T = 0, \quad X \in \partial B. \tag{3.23}$$

The initial conditions are

$$u(X,0) = u^0, \quad \dot{u}(X,0) = v^0, \quad T(X,0) = T^0, h(X,0) = h^0. \tag{3.24}$$

Remark. A particular case corresponds when $m_1 = m_4 = 0$. Then $p_1 = 0$ and p_2, p_3 are easily computable.

4 Theory of type II

For the theory of type II we postulate constitutive equations (3.1) where

$$A = (\nabla x, \theta, \nabla \alpha, G). \tag{4.1}$$

In this case we obtain the relation

$$\rho(\frac{\partial \Psi}{\partial \theta} + \eta)\dot{\theta} + (\rho \frac{\partial \Psi}{\partial \nabla x} - T) : \nabla \dot{x} + (\rho \frac{\partial \Psi}{\partial \nabla \alpha} - \boldsymbol{\Phi}) \cdot \nabla \theta + \rho \xi \theta + \rho \frac{\partial \Psi}{\partial G} : \dot{G} = 0. \tag{4.2}$$

From (4.2) we see that the constitutive equations which are compatible with the energy equation take the form

$$\Psi = \Psi(A),\ \eta = -\frac{\partial \Psi}{\partial \theta},\ T = \rho\frac{\partial \Psi}{\partial \nabla x},\ \boldsymbol{\Phi} = \rho\frac{\partial \Psi}{\partial \nabla \alpha},\ \dot{G} = H(A) \tag{4.3}$$

$$\rho\theta\xi = -\rho\frac{\partial \Psi}{\partial G} : \dot{G}. \tag{4.4}$$

Thus, we see that the constitutive equation for the internal rate of production of entropy can be given in terms of the variation of the internal variables. For the theory of type II the system of equations which governs the evolution of the thermodynamics of materials with internal variables is given by the equations (2.2) and (2.3) with the constitutive equations (4.3), (4.4).

When we assume that condition (3.6) holds, we may consider the set of variables

$$A = (\nabla x, e, \nabla\alpha, G). \tag{4.5}$$

The energy equation implies that

(4.6)
$$\rho\Big(1 - \theta\frac{\partial \eta}{\partial e}\Big)\dot{e} + (\rho\theta\frac{\partial \eta}{\partial \nabla x} + T) : \nabla\dot{x} + (\rho\theta\frac{\partial \eta}{\partial \nabla \alpha} - \boldsymbol{\Phi}) \cdot \nabla\theta + \rho\theta\xi + \rho\theta\frac{\partial \eta}{\partial G} : \dot{G} = 0.$$

This relation implies that

$$\theta^{-1} = \frac{\partial \eta}{\partial e},\ T = -\rho\,\theta\,\frac{\partial \eta}{\partial \nabla x},\ \ \boldsymbol{\Phi} = \rho\,\theta\,\frac{\partial \eta}{\partial \nabla \alpha}, \tag{4.7}$$

and

$$\xi = -\frac{\partial \eta}{\partial G} : \dot{G}. \tag{4.8}$$

Now we state the linear equations which govern the problem in the case of the type II theory. In the remain of this section we assume that at time $t = 0$, we have (3.11) and

$$\alpha(X, 0) = \alpha_0, \tag{4.9}$$

where α_0 is constant. We also use the notation of (3.12), (3.13) and

$$\tau = \int_0^t T(s)ds. \tag{4.10}$$

It follows that

$$\alpha = \tau + T_0 t + \alpha_0,\ \ \nabla\alpha = \nabla\tau,\ \ \dot{\tau} = T. \tag{4.11}$$

In the linear theory we assume that u, T, h are given as in (3.14). In the case of centrosymmetric materials we assume that

(4.12)
$$\rho\Psi = \frac{1}{2}\nabla u : C\nabla u - \frac{1}{2}aT^2 + \frac{1}{2}\nabla\tau \cdot K\nabla\tau - \boldsymbol{\beta} : \nabla u T + \nabla u : m_1 h - T m_2 : h + \frac{1}{2}h : m_3 h,$$

and $\dot{h}$ is given in (3.17). Again C satisfies the symmetry (3.18) and K is also symmetric. The meaning of the tensors $C, a, \boldsymbol{\beta}$ and $m_1, ..., m_6$ is given in the previous section. Here we have a new tensor K. This is a tensor which is usual in the type II theory. We have that $T, \rho\eta$ and ξ are given as in (3.19). A relevant difference is that

$$\boldsymbol{\Phi} = K\nabla\tau, \tag{4.13}$$

Thus, the system of equations that governs our problem is

$$\rho\ddot{u} = \text{ div } \Big(C\nabla u - \boldsymbol{\beta}T + m_1 h\Big) + \rho b, \tag{4.14}$$

$$a\ddot{\tau} = -\boldsymbol{\beta} : \nabla\dot{u} - m_2 : \Big(m_4\nabla u + m_5 T + m_6 h\Big) + \text{ div } (K\nabla\tau) + \rho S, \tag{4.15}$$

and the equation (3.17). It is worth noting that equation (4.15) can be written as

$$a\ddot{\tau} = -\boldsymbol{\beta} : \nabla\dot{u} + \text{ div } (K\nabla\tau) + p_1 : \nabla u + p_2 : h + p_3 T + \rho S, \tag{4.16}$$

where p_1, p_2 and p_3 can be given in terms of the m_i.

To have a defined problem we need to impose boundary and initial conditions. The Dirichlet homogeneous boundary conditions are (3.23) and

$$\tau = 0, \quad X \in \partial B. \tag{4.17}$$

The initial conditions are (3.23) and

$$\tau(X, 0) = \tau^0. \tag{4.18}$$

Remark. We can do a remark with respect the tensors p_i which is similar to the one at the end of the previous section.

5 Theory of type III

In the case of the theory of type III we have the following set of variables

$$A = (\nabla x, \theta, \nabla\alpha, \nabla\theta, G). \tag{5.1}$$

The energy equation gives

$$\begin{aligned} \rho(\tfrac{\partial\Psi}{\partial\theta} + \eta)\dot{\theta} \quad & +(\rho\tfrac{\partial\Psi}{\partial\nabla x} - T) : \nabla\dot{x} + (\rho\tfrac{\partial\Psi}{\partial\nabla\alpha} - \boldsymbol{\phi}) \cdot \nabla\theta \\ & +\rho\tfrac{\partial\Psi}{\partial\nabla\theta} \cdot \nabla\dot{\theta} + \rho\xi\theta + \rho\tfrac{\partial\Psi}{\partial G} : \dot{G} = 0. \end{aligned} \tag{5.2}$$

In this situation the constitutive equations which are compatible with the energy equation take the form

$$\frac{\partial\Psi}{\partial\nabla\theta} = 0, \, \eta = -\frac{\partial\Psi}{\partial\theta}, \, T = \rho\frac{\partial\Psi}{\partial x}, \dot{G} = H(A) \tag{5.3}$$

$$\rho\,\theta\,\xi = -\rho\frac{\partial\Psi}{\partial G} : \dot{G} - (\rho\frac{\partial\Psi}{\partial\nabla\alpha} - \boldsymbol{\phi})\cdot\nabla\theta. \tag{5.4}$$

When condition (3.6) holds, we can use the set of variables

$$A = (\nabla x, e, \nabla\alpha, \nabla\theta, G). \tag{5.5}$$

A similar argument to those used previously shows that

$$\theta^{-1} = \frac{\partial\eta}{\partial e},\ T = -\rho\theta\frac{\partial\eta}{\partial\nabla x}, \tag{5.6}$$

and

$$\rho\,\theta\,\xi = -\frac{\partial\eta}{\partial G} : \dot{G} - (\rho\frac{\partial\Psi}{\partial\nabla\alpha} - \boldsymbol{\phi})\cdot\nabla\theta. \tag{5.7}$$

Now we state the linear equations which govern the problem in the case of the type III theory. In the remain of this section we assume that at time $t = 0$, we have (3.11) and (3.12). We also use the notation of (3.13) and (4.10). It follows (4.11). Again u, T, h are given as in (3.14). In the case of centrosymmetric materials we have that $\rho\Psi$ is given as in (4.12) and $\dot{h}$ is defined as in (3.17).

We have that $T, \rho\eta$ and ξ are given as in (3.19). A new relevant difference is that

$$\boldsymbol{\phi} = K_0\nabla T + K\nabla\tau \tag{5.8}$$

Here we have tensors K_0, and K. They are usual in the type III theory. Thus, the system of equations that governs our problem is

$$\rho\ddot{u} = \text{ div }\Big(C\nabla u - \boldsymbol{\beta}T + m_1 h\Big) + \rho b, \tag{5.9}$$

$$a\ddot{\tau} = -\boldsymbol{\beta} : \nabla\dot{u} + \text{ div }(K\nabla\tau + K_0\nabla T) + p_1 : \nabla u + p_2 : h + p_3 T + \rho S, \tag{5.10}$$

and the equation (3.17).

6 Existence in type I

In this section we obtain an existence theorem of solutions of the problem determined by system (3.20), (3.22), (3.17), initial conditions (3.24) the boundary conditions (3.23). To this end we assume that

(i) $\rho > 0$ and $a > 0$.

(ii) There exists a positive constant C_0 such that

$$\nabla u : C\nabla u \geq C_0\nabla u : \nabla u. \tag{6.1}$$

(iii) There exists a positive constant K_0 such that

$$\nabla T\cdot K_0\nabla T \geq K_0\nabla T\cdot\nabla T. \tag{6.2}$$

We now transform the boundary-initial-value problem into an abstract problem on a suitable Hilbert space. We denote

$$\mathcal{Z} = \{(u, v, T, h); u \in W_0^{1,2}, v \in L^2, T \in L^2, h \in \left[L^2\right]^2\},$$

where $W_0^{1,2}$ and L^2 are the usual Sobolev spaces and $W_0^{1,2}$ and L^2 means $\left[W_0^{1,2}\right]^3$ and $\left[L^2\right]^3$ respectively.

Let us consider the operators

$$A(u) = \rho^{-1} \text{ div } (C\nabla u),$$

$$BT = -\rho^{-1} \text{ div } \boldsymbol{\beta}T, \quad Dh = \rho^{-1} \text{ div } m_1 h,$$
$$L(u) = a^{-1}(p_1 : \nabla u), \quad M(v) = -a^{-1}(\boldsymbol{\beta} : \nabla v),$$
$$QT = a^{-1} \text{ div } (K_0 \nabla T) + a^{-1}(p_3 T), Nh = a^{-1}(p_2 : h),$$
$$J_1(u) = m_4 \nabla u, J_2 T = m_5 T, J_3 h = m_6 h,$$

and

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 \\ A & 0 & C & D \\ L & M & Q & N \\ J_1 & 0 & J_2 & J_3 \end{pmatrix}, \tag{6.3}$$

where I is the identity operator.

We note that

$$(W_0^{1,2} \cap W^{2,2}) \times W_0^{1,2} \times W_0^{1,2} \times \left[W^{1,2}\right]^2,$$

is a subset of the domain of $\mathcal{A}$ that is dense in $\mathcal{Z}$. Our boundary-initial-value problem can be transformed into the abstract equation

$$\frac{d\omega}{dt} = \mathcal{A}\omega(t) + \mathcal{F}(t), \quad \omega(0) = \omega_0, \tag{6.4}$$

where

$$\omega = (u, v, T, h), \quad \omega_0 = (u^0, v^0, T^0, h^0),$$

and

$$\mathcal{F}(t) = (0, \rho^{-1}b, \rho a^{-1}S, 0).$$

We introduce in $\mathcal{Z}$ the following inner product

$$< (u, v, T, h), (u^*, v^*, T^*, h^*) > \tag{6.5}$$

$$= \frac{1}{2} \int_B \Big(\rho v.v^* + aTT^* + \lambda h : h^* + \nabla u : C\nabla u^* + \big(\nabla u : m_1 h^* + \nabla u^* : m_1 h\big)\Big) dV,$$

where λ is a sufficiently large positive constant in order to guarantee that this bilinear form defines an inner product in the Hilbert space $\mathcal{Z}$. It is equivalent to the usual one in this space

It is worth noting the inclusion of the term $\nabla u : m_1 h^* + \nabla u^* : m_1 h$ in the definition of the inner product. The point is to control the derivative of the term $v.v$. Then, we need to include the parameter (sufficiently large) λ to assure the positivity of the energy.

Now, we use the theory of semigroups of linear operators to obtain the existence of solutions for the equation (6.4).

Lemma 6.1. *The operator $\mathcal{A}$ satisfies that there exists a positive constant μ_1 such that*

$$< \mathcal{A}\omega, \omega > \leq \mu_1 ||\omega||^2. \tag{6.6}$$

for any ω in the domain of $\mathcal{A}$.

Proof. Let ω be in the domain of $\mathcal{A}$. Using the divergence theorem and the boundary conditions, we find that

$$< \mathcal{A}\omega, \omega > = \int_B \Big(\nabla u : m_1(m_4 \nabla u + m_5 T + m_6 h) + T p_1 : \nabla u + T p_2 : h + p_3 T^2$$

$$+ \lambda (m_4 \nabla u + m_5 T + m_6 h) : h - \nabla T . K_0 \nabla T \Big) dV.$$

It is clear that we can select a positive constant ν such that

$$< \mathcal{A}\omega, \omega > \leq \nu \int_B \Big(\nabla u : \nabla u + T^2 + h : h) dV.$$

Then, we can obtain μ_1 satisfying estimate (6.6).

Lemma 6.2. *The operator $\mathcal{A}$ satisfies that there exists a positive constant μ_2 such that*

$$Rang\,(\mu_2 \mathcal{I} - \mathcal{A}) = \mathcal{Z}. \tag{6.7}$$

Proof. Let $\varpi^* = (u^*, v^*, T^*, h^*) \in (W^{2,2})^2 \times W^{2,2} \times \Big[W^{1,2} \Big]^2$. We must prove that the equation

$$\mu_2 \varpi - \mathcal{A}\varpi = \varpi^*, \tag{6.8}$$

has a solution $\varpi = (u, v, T, h) \in \mathcal{D}(\mathcal{A})$ for μ_2 sufficiently large. From the definition of $\mathcal{A}$, we obtain the system:

$$\mu_2 u - v = u^*, \tag{6.9}$$

$$\mu_2 v - Au - CT - Dh = v^*, \tag{6.10}$$

$$\mu_2 T - Lu - Mv - QT - Nh = T^*, \tag{6.11}$$

$$\mu_2 h - J_1 u - J_2 T - J_3 h = h^*. \tag{6.12}$$

Substituting (6.9), in (6.10), (6.11), we get

$$\mu_2^2 u - Au - CT - Dh = \mu_2 u^* + v^*, \tag{6.13}$$

$$\mu_2 T - Lu - \mu_2 Mu - QT - Nh = T^* - Mu^*, \tag{6.14}$$

$$\mu_2 h - J_1 u - J_2 T - J_3 h = h^*. \tag{6.15}$$

To study the system (6.13)-(6.15) we introduce the following bilinear form on $W_0^{1,2} \times W_0^{1,2} \times \Big[L^2\Big]^2$

(6.16)

$$\mathcal{B}_{\mu_2}\Big[(u,T,h),(\tilde{u},\tilde{T},\tilde{h})\Big] =< \begin{pmatrix} \mu_2^2 - A & -C & -D \\ -\mu_2 M - L & \mu_2 - Q & -N \\ -J_1 & -J_2 & \mu_2 - J_3 \end{pmatrix} \begin{pmatrix} u \\ T \\ h \end{pmatrix}, (\tilde{u},\tilde{T},\tilde{h}) > .$$

After several uses of the divergence theorem and the arithmetic-geometric mean inequality we can prove that $\mathcal{B}_{\mu_2}$ (for μ_2 sufficiently large) is bounded and coercive. The right hand side of (6.13)-(6.15) lies in $W^{-1} \times W^{-1} \times L^2$. Hence Lax- Milgram theorem implies the existence of a solution of the system (6.13)-(6.15). Thus, we can solve the equation (6.8).

The previous lemmas and the use of the Lumer-Phillips corollary to the Hille-Yosida theorem allow us to obtain the theorems:

Theorem 6.1. *The operator $\mathcal{A}$ generates a quasi-contractive semigroup in $\mathcal{Z}$.*

Theorem 6.2. . *Let us assume that the conditions (i)-(iii) are satisfied and the supply terms $b \in C^1([0,T], L^2(B)) \cap C^0([0,T], W_0^{1,2}(B))$, $S \in C^1([0,T], L^2(B)) \cap C^0([0,T], W_0^{1,2}(B))$. Then, for any (u_0, v_0, T_0, h_0) in $\mathcal{D}$, there exists a unique solution to the evolution equations; namely, there exists a unique $(u(t), v(t), T(t), h(t)) \in C^1([0,T], \mathcal{Z}) \cap C^0([0,T], \mathcal{D})$.*

Since the solutions are defined by means of a quasi-contractive semigroup, we may obtain a continuous dependence of the solutions upon initial data and body loads. In particular we have the following estimate for the solutions

$$||(u(t), v(t), T(t), h(t))|| \leq \exp(\delta t)\Big(||(u_0, v_0, T_0, h_0)|| + \int_0^t \Big(\int_B (b.b + S^2)dv\Big)^{1/2}\Big). \tag{6.17}$$

Thus, under assumptions (i)-(iii) the problem of the linear thermodynamic theory materials with internal variables in the context of the type I theory is well posed.

Remark. When $m_1 = 0$ we can take in (7.5) the parameter $\lambda = 1$. When we assume that $m_4 = 0$, we obtain that

$$< \mathcal{A}\omega, \omega > \leq \int_B \Big(Tp_2 : h + p_3 T^2 + Tm_5 : h + m_6 h : h - \nabla T . K_0 \nabla T\Big) dV.$$

Thus if we assume that the expression inside the integral is less or equal than zero the semigroup is dissipative and when the supply terms are absent we obtain stability of solutions.

7 Existence in type II

In this section we obtain an existence theorem of solutions of the problem determined by system (4.14), (4.16), (3.17), initial conditions (3.24), (4.18) the boundary conditions (3.23), (4.17). To this end we assume that conditions (i), (ii) of the section 6 hold. We also assume that

(iii') There exists a positive constant K_0 such that

$$\nabla\tau \cdot K\nabla\tau \geq K_0 \nabla\tau \cdot \nabla\tau. \tag{7.1}$$

Again we transform the boundary-initial-value problem into an abstract problem on a suitable Hilbert space. We denote

$$\mathcal{Z} = \{u, v, \tau, T, h); u \in W_0^{1,2}, \tau \in W_0^{1,2}, v \in L^2, T, h \in \left[L^2\right]^2\}.$$

Let us consider the operators

$$A(u) = \rho^{-1} \text{ div } (C\nabla u),$$
$$BT = -\rho^{-1} \text{ div } \boldsymbol{\beta} T, \quad Dh = \rho^{-1} \text{ div } m_1 h,$$
$$L(u) = a^{-1}(p_1 : \nabla u), \quad M(v) = -a^{-1}(\boldsymbol{\beta} : \nabla v),$$
$$Q\tau = a^{-1} \text{ div } (K\nabla\tau), RT = a^{-1}(p_3 T), Nh = a^{-1}(p_2 : h),$$
$$J_1(u) = m_4 \nabla u, J_2 T = m_5 T, J_3 h = m_6 h,$$

and

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ A & 0 & 0 & C & D \\ 0 & 0 & 0 & I & 0 \\ L & M & Q & R & N \\ J_1 & 0 & 0 & J_2 & J_3 \end{pmatrix}, \tag{7.2}$$

where I and I are the identity operator in the respective spaces.

We note that

$$(W_0^{1,2} \cap W^{2,2}) \times W_0^{1,2} \times (W_0^{1,2} \cap W^{2,2}) \times W_0^{1,2} \times \left[W^{1,2}\right]^2,$$

is a subset of the domain of $\mathcal{A}$ that is dense in $\mathcal{Z}$. Our boundary-initial-value problem can be transformed into the abstract equation (6.4) where

$$\omega = (u, v, \tau, T, h), \quad \omega_0 = (u^0, v^0, \tau^0, T^0, h^0),$$

and

$$\mathcal{F}(t) = (0, \rho^{-1}b, 0, \rho a^{-1}S, 0).$$

We introduce in $\mathcal{Z}$ the following inner product

$$< (u, v, \tau, T, h), (u^*, v^*, \tau^*, T^*, h^*) > \tag{7.3}$$
$$= \frac{1}{2}\int_B \Big(\rho v.v^* + aTT^* + \lambda h : h^* + \nabla u : C\nabla u^* + \nabla\tau.K\nabla\tau^* + \big(\nabla u : m_1 h^* + \nabla u^* : m_1 h\big)\Big) dV,$$

where λ is a sufficiently large positive constant in order to guarantee that this bilinear form defines an inner product in the Hilbert space $\mathcal{Z}$. It is equivalent to the usual one in this space

Lemma 7.1. *The operator $\mathcal{A}$ satisfies that there exists a positive constant μ_1 such that*

$$< \mathcal{A}\omega, \omega > \leq \mu_1 ||\omega||^2. \tag{7.4}$$

for any ω in the domain of $\mathcal{A}$.

Proof. Let ω be in the domain of $\mathcal{A}$. Using the divergence theorem and the boundary conditions, we find that

$$< \mathcal{A}\omega, \omega >= \int_B \Big(\nabla u : m_1(m_4 \nabla u + m_5 T + m_6 h) + T p_1 : \nabla u + T p_2 : h + p_3 T^2$$
$$+\lambda (m_4 \nabla u + m_5 T + m_6 h) : h\Big) dV.$$

It is clear that we can select a positive constant ν such that

$$< \mathcal{A}\omega, \omega > \leq \nu \int_B \Big(\nabla u : \nabla u + T^2 + h : h) dV.$$

Then, we can obtain μ_1 satisfying estimate (7.5).

Lemma 7.2. *The operator $\mathcal{A}$ satisfies that there exists a positive constant μ_2 such that*

$$Rang\,(\mu_2 \mathcal{I} - \mathcal{A}) = \mathcal{Z}. \tag{7.5}$$

Proof. Let $\varpi^* = (u^*, v^*, \tau^*, T^*, h^*) \in (W^{2,2})^2 \times (W^{2,2})^2 \times \Big[W^{1,2}\Big]^2$. We must prove that the equation

$$\mu_2 \varpi - \mathcal{A}\varpi = \varpi^*, \tag{7.6}$$

has a solution $\varpi = (u, v, \tau, T, h) \in \mathcal{D}(\mathcal{A})$ for μ_2 sufficiently large. From the definition of $\mathcal{A}$, we obtain the system:

$$\mu_2 u - v = u^*, \tag{7.7}$$

$$\mu_2 v - Au - CT - Dh = v^*, \tag{7.8}$$

$$\mu_2 \tau - T = \tau^*, \tag{7.9}$$

$$\mu_2 T - Lu - Mv - Q\tau - RT - Nh = T^*, \tag{7.10}$$

$$\mu_2 h - J_1 u - J_2 T - J_3 h = h^*. \tag{7.11}$$

Substituting (7.7), (7.9) in (7.8), (7.10), we get

$$\mu_2^2 u - Au - C\tau - Dh = \mu_2 u^* + v^* + C\tau^*, \tag{7.12}$$

$$\mu_2^2\tau - Lu - \mu_2 Mu - Q\tau - \mu_2 R\tau - Nh = \mu_2\tau^* - R\tau^* - Mu^*, \tag{7.13}$$

$$\mu_2 h - J_1 u - \mu_2 J_2\tau - J_3 h = h^* - J_2\tau^*. \tag{7.14}$$

To study the system (7.12)-(7.14) we introduce the following bilinear form on $W_0^{1,2} \times W_0^{1,2} \times \Big[L^2\Big]^2$

(7.15)

$$\mathcal{B}_{\mu_2}\Big[(u,\tau,h),(\tilde{u},\tilde{\tau},\tilde{h})\Big] =< \begin{pmatrix} \mu_2^2 - A & -\mu_2 C & -D \\ -\mu_2 M - L & \mu_2^2 - Q - \mu_2 R & -N \\ -J_1 & -\mu_2 J_2 & \mu_2 - J_3 \end{pmatrix} \begin{pmatrix} u \\ \tau \\ h \end{pmatrix}, (\tilde{u},\tilde{\tau},\tilde{h}) > .$$

After several uses of the divergence theorem an the arithmetic-geometric mean inequality we can prove that $\mathcal{B}_{\mu_2}$ (for μ_2 sufficiently large) is bounded and coercive. The right hand side of (7.12)-(7.14) lies in $W^{-1} \times W^{-1} \times \Big[L^2\Big]^2$. Hence Lax- Milgram theorem implies the existence of a solution of the system (7.12)-(7.14). Thus, we can solve the equation (7.6).

The previous lemmas and the use of the Lumer-Phillips corollary to the Hille-Yosida theorem allow us to obtain the theorems:

Theorem 7.1. *The operator $\mathcal{A}$ generates a quasi-contractive semigroup in $\mathcal{Z}$.*

Theorem 7.2. *. Let us assume that the conditions (i)-(ii) and (iii') are satisfied and the supply terms $b \in C^1([0,T], L^2(B)) \cap C^0([0,T], W_0^{1,2}(B))$, $S \in C^1([0,T], L^2(B)) \cap C^0([0,T], W_0^{1,2}(B))$. Then, for any $(u_0, v_0, \tau_0, T_0, h_0)$ in $\mathcal{D}$, there exists a unique solution to the evolution equations; namely, there exists a unique $(u(t), v(t), \tau(t), T(t), h(t)) \in C^1([0,T], \mathcal{Z}) \cap C^0([0,T], \mathcal{D})$.*

Since the solutions are defined by means of a quasi-contractive semigroup, we may obtain a continuous dependence of the solutions upon initial data and body loads. In particular we have the following estimate for the solutions

$$||(u(t), v(t), \tau(t), T(t), h(t))|| \leq \exp(\delta t)\Big(||(u_0, v_0, \tau_0, T_0, h_0)|| + \int_0^t \Big(\int_B (b.b + S^2) dv\Big)^{1/2}\Big). \tag{7.16}$$

Thus, under assumptions (i)-(ii) and (iii') the problem of the linear thermodynamic theory materials with internal variables in the context of the type II theory is well posed.

Remark. When $m_1 = 0$ we can take in (7.5) the parameter $\lambda = 1$. When we assume that $m_4 = 0$, we obtain that

$$< \mathcal{A}\omega, \omega >= \int_B \Big(Tp_2 : h + p_3 T^2 + Tm_5 : h + m_6 h : h\Big) dV.$$

Thus if we assume that the expression inside the integral is less or equal than zero the semigroup is dissipative and when the supply terms are absent we obtain stability of solutions.

Remark. An existence theorem could be obtained in a similar way in the type III theory.

Acknowledgement. The work of the second author (R.Q.) is supported by the project "Stability Aspects in Thermomechanics" (BFM 2003-00309). It was started when the second author (R.Q.) visited the Department of Mathematics of the University of Messina. R.Q. thanks the hospitality and kindness of this Department.

References

[1] G. Caviglia, A. Morro and B. Straughan, *Thermoelasticity at cryogenic temperatures*, Int. J. Nonlin. Mech., 27 (1992), 251-263.

[2] D.S. Chandrasekharaiah, *Hyperbolic thermoelasticity: A review of recent literature*, Appl. Mech. Rev. 51 (1998), 705-729.

[3] D.S. Chandrasekharaiah, *Thermoelasticity with second sound: A review*, Appl. Mech. Rev. 39 (1986), 355-376.

[4] V. Ciancio, V.A. Cimmelli, *Thermomechanics of interestitial working at liquid boundaries*, Technishe Mechanik. 22 (2002), 187-192.

[5] V. Ciancio, V.A. Cimmelli and K.C. Valanis, *A thermodynamic theory of thermoelastic and viscoanelastic solids with non-euclidean struture*, J. Non-Equilib. Thermodyn., 26 (2001), 153-166.

[6] V. Ciancio, M. Dolfin, M. Francaviglia, *Localization of deformations in finite elastoplasticity*, Technische Mechanik 22, 2 (2002), 111-117.

[7] V. Ciancio, M. Francaviglia, *Non-euclidean structures as internal variable in non-equilibrium thermomechanics*, Balkan Journal of Geometry and its Applications (BJGA), 8, 1 (2003), 33-43.

[8] V. Ciancio, L. Restuccia, *Non linear dissipative waves in viscoanelastic media*, Physica 132 A (1985), 606-616.

[9] V. Ciancio, J. Verhás, *A thermodynamic theory for heat radiation through the atmosphere*, J. Non-Equilib. Thermodynamics 16 (1991), 57-65.

[10] V. Ciancio, J. Verhás, *A thermodynamic theory for radiating heat transfer*, J. Non-Equilib. Thermodynamics, 15 (1990), 33-43.

[11] W. Dreyer, H. Struchtrup, *Heat pulse experiments revisited*, Continuum Mech. Thermodyn, 5 (1993), 3-50.

[12] A.E. Green, P.M. Naghdi, *A re-examination of the basic postulates of themomechanics*, Proc. Royal Society London A 432 (1991),171-194.

[13] A.E. Green, P.M. Naghdi, *A unified procedure for contruction of theories of deformable media.I. Classical continuum physics,II. Generalized continua, III. Mixtures of interacting continua*, Proc. Royal Society London A 448 (1995), 335-356, 357-377, 379-388.

[14] A.E. Green, P.M. Naghdi, *On undamped heat waves in an elastic solid*, J. Thermal Stresses 15 (1992), 253-264.

[15] A.E. Green, P.M. Naghdi, *Thermoelasticity without energy dissipation*, J. Elasticity 31 (1993), 189-208.

[16] A.E. Green, P.M. Naghdi, *A new thermoviscous theory for fluids*, Journal of Non-Newtonian Fluid Mech. 56 (1995a), 289-306.

[17] R.B. Hetnarski, J. Ignaczak, *Generalized thermoelasticity*, J. Thermal Stresses 22 (1999), 451-470.

[18] D. Jou, J. Casas-Vazquez, G. Lebon, *Extended Irreversible Thermodynamics*, Springer-Verlag, Berlin, 1996.

[19] G.A. Kluitenberg, V. Ciancio, *On linear dynamical equations of state for inelastic media*, Physica 93 A (1978), 273-286.

[20] G.A. Maugin, *Internal variables and dissipative structures*, J. Non-Equilib. Thermodyn. 195 (1990), 173-192.

[21] G.A. Maugin, *The Thermomechanics of Nonlinear Irreversible Behaviors. An Introduction*, World Scientific Publ., Singapore and River Edge, 1999.

[22] G.A. Maugin, W. Muschik, *Thermodynamics with internal variables - Part I - General concepts*, J. Non-Equilib. Thermodyn. 19 (1994), 217-249.

[23] I. Müller, T. Ruggeri, *Rational and Extended Thermodynamics*, Springer-Verlag, New-York, 1998.

[24] R. Quintanilla, *Existence in thermoelasticity without energy dissipation*, Jour. Thermal Stresses, 25 (2002), 195-202.

[25] R. Quintanilla and B. Straughan, *Growth and uniqueness in thermoelasticity*, Proc. Royal Society London A 456 (1999), 1419-1429.

Authors' addresses:

V.Ciancio
Department of Mathematics, University of Messina
c.da Papardo, s.ta Sperone, 31, 98166 Messina, Italy.
e-mail: ciancio@unime.it

R.Quintanilla
Universidad Politécnica de Catalunya, Departamento Matemática Applicada 2,
08222 Terrassa, Barcelona, Spain.