

Classifications of some special infinity-harmonic maps

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Abstract. ∞ -Harmonic maps are a generalization of ∞ -harmonic functions. They can be viewed as the limiting cases of p -harmonic maps as p goes to infinity. In this paper, we give complete classifications of linear and quadratic ∞ -harmonic maps from and into a sphere, quadratic ∞ -harmonic maps between Euclidean spaces. We describe all linear and quadratic ∞ -harmonic maps between Nil and Euclidean spaces, between Sol and Euclidean spaces. We also study holomorphic ∞ -harmonic maps between complex Euclidean spaces.

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1 Introduction

In this paper, we work in the category of smooth objects so that all manifolds, vector fields, and maps are assumed to be smooth unless there is an otherwise statement.

The infinity-Laplace equation

$$(1.1) \quad \Delta_{\infty} u := \frac{1}{2} \langle \nabla u, \nabla |\nabla u|^2 \rangle = \sum_{i,j=1}^m u_{ij} u_i u_j = 0,$$

where $u : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}$, $u_i = \frac{\partial u}{\partial x^i}$ and $u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$, was first discovered and studied by G. Aronsson in his study of “optimal” Lipschitz extension of functions in the late 1960s ([1], [2]).

To see why this nonlinear and highly degenerate elliptic PDE has been so fascinating, we recall that the well-known minimal surface equation can be written as

$$(1 + |\nabla u|^2) \Delta u + \sum_{i,j=1}^m u_i u_j u_{ij} = 0,$$

from which we see that the ∞ -Laplace equation can be obtained as **harmonic minimal surface equation** meaning the equation for harmonic functions with minimal

graphs. The solutions of the ∞ -Laplace equation are called ∞ -harmonic functions which have the following interpretations:

Lemma 1.1. (see [10]) *Let $u : (M^m, g) \rightarrow \mathbb{R}$ be a function. Then the following conditions are equivalent: (1) u is an ∞ -harmonic function, i.e., $\Delta_\infty u = 0$; (2) u is horizontally homothetic; (3) ∇u is perpendicular to $\nabla|\nabla u|^2$; (4) $\text{Hess}_u(\nabla u, \nabla u) = 0$; (5) $|\nabla u|^2$ is constant along any integral curve of ∇u .*

Also, the ∞ -Laplace equation can be viewed (see [1]) as the formal limit, as $p \rightarrow \infty$, of p -Laplace equation

$$\Delta_p u := |\nabla u|^{p-2} \left(\Delta u + \frac{p-2}{|\nabla u|^2} \Delta_\infty u \right) = 0.$$

Finally, the ∞ -Laplace equation can be viewed as the Euler-Lagrange equation of the L^∞ variational problem of minimizing $E_\infty(u) = \text{ess sup}_\Omega |du|$ among all Lipschitz continuous functions u with given boundary values on $\partial\Omega$ (see e.g., [3]).

Recently, a great deal of research work has been done in the study of the ∞ -Laplace equation after the work of Crandall and Lions (see e.g. [8]) on the theory of viscosity solutions for fully nonlinear problems. Many important results have been achieved and published (see e.g., [3], [8], [9] and the references therein).

On the other hand, the ∞ -Laplace equation has been found to have some very interesting applications in areas such as image processing (see e.g. [6], [12]), mass transfer problems (see e.g. [9]), and the study of shape metamorphism (see e.g. [7]).

The generalization from harmonic functions to harmonic maps between Riemannian manifolds was so fruitful that it has not only opened new fields of study in differential geometry, analysis, and topology but also brought important applications to many branches in mathematics and theoretical physics. It would be interesting to study maps between Riemannian manifolds that generalize ∞ -harmonic functions. This was initiated in [11] where the notion of ∞ -harmonic maps between Riemannian manifolds was introduced as a natural generalization of ∞ -harmonic functions and as the limit case of p -harmonic maps as $p \rightarrow \infty$.

Definition 1.2. A map $\varphi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds is called an ∞ -harmonic map if the gradient of its energy density belongs to the kernel of its tangent map, i.e., φ is a solution of the PDEs

$$(1.2) \quad \tau_\infty(\varphi) := \frac{1}{2} d\varphi(\text{grad } |d\varphi|^2) = 0,$$

where $|d\varphi|^2 = \text{Trace}_g \varphi^* h$ is the energy density of φ .

A direct computation using local coordinates yields

Corollary 1.3. *In local coordinates, a map $\varphi : (M, g) \rightarrow (N, h)$ with $\varphi(x) = (\varphi^1(x), \dots, \varphi^n(x))$ is ∞ -harmonic if and only if*

$$(1.3) \quad g(\text{grad } \varphi^\alpha, \text{grad } |d\varphi|^2) = 0, \quad \alpha = 1, 2, \dots, n.$$

Clearly, any ∞ -harmonic function is an ∞ -harmonic map by Definition 1.2. It also follows from the definition that any map between Riemannian manifolds with constant energy density, i.e., $|\mathrm{d}\varphi|^2 = \mathrm{Trace}_g \varphi^* h = \text{constant}$ is an ∞ -harmonic map. Thus, the following important and familiar families are all ∞ -harmonic maps: (i) totally geodesic maps, (ii) isometric immersions, (iii) Riemannian submersions, (iv) eigenmaps between spheres, (v) projections of multiply warped products (e.g., the projection of the generalized Kasner spacetimes), (vi) equator maps, and (vii) radial projections. We refer the readers to [11] for details of these and many other examples and other results including methods of constructing ∞ -harmonic maps into Euclidean spaces and spheres, characterizations of ∞ -harmonic immersions and submersions, study of ∞ -harmonic morphisms which can be characterized as horizontally homothetic submersions, and the transformation ∞ -Laplacians under the the conformal change of metrics.

In this paper, we study the classification of ∞ -harmonic maps between certain model spaces. We give complete classifications of linear and quadratic ∞ -harmonic maps from and into a sphere, quadratic ∞ -harmonic maps between Euclidean spaces. We describe all linear and quadratic ∞ -harmonic maps between Nil and Euclidean spaces and between Sol and Euclidean spaces. We also study holomorphic ∞ -harmonic maps complex Euclidean spaces.

2 quadratic ∞ -harmonic maps between Euclidean spaces

As we mentioned in Section 1 that any map with constant energy density is ∞ -harmonic. It follows that any affine map $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ with $\varphi(X) = AX + b$, where A is an $n \times m$ matrix and $b \in \mathbb{R}^n$ is a constant, is an ∞ -harmonic map because of its constant energy density. Note that there are also globally defined ∞ -harmonic maps between Euclidean spaces which are not affine maps. For example, one can check that $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by $\varphi(x, y, z) = (\cos x + \cos y + \cos z, \sin x + \sin y + \sin z)$ is a map with constant energy density $|\mathrm{d}\varphi|^2 = \mathrm{Trace}_g \varphi^* h = 3$ and hence an ∞ -harmonic maps. In this section, we give a complete classification of ∞ -harmonic maps between Euclidean spaces defined by quadratic polynomials. First, we prove the following lemma which will be used frequently in this paper.

Lemma 2.1. *Let A_i , $i = 1, 2, \dots, n$, be symmetric $m \times m$ matrices. Then, $(\sum_{j=1}^n A_j^2)A_i + A_i(\sum_{j=1}^n A_j^2) = 0$ for all $i = 1, 2, \dots, n$ if and only if $A_i = 0$ for $i = 1, 2, \dots, n$.*

Proof. Suppose otherwise, i.e., one of A_i is not zero, without loss of generality, we may assume $A_1 \neq 0$. Then $\mathrm{rank}(A_1) = K$ with $1 \leq K \leq m$. Without loss of generality, we can choose a suitable orthogonal matrix T such that $T^{-1}A_1T$ takes the diagonal form with the first K diagonal entries $\lambda_1, \dots, \lambda_K$ non-zero.

Note that

$$(2.1) \quad T^{-1} \sum_{j=1}^n (A_j^2) T = \sum_{j=1}^n T^{-1} (A_j^2) T = \sum_{j=1}^n (T^{-1} A_j T)^2$$

with each $(T^{-1} A_j T)^2$ being symmetric matrix. It follows that

$$(2.2) \quad 0 = T^{-1} 0 T = \sum_{j=1}^n (T^{-1} A_j T)^2 T^{-1} A_1 T + T^{-1} A_1 T (T^{-1} A_j T)^2.$$

This is impossible because the i -th entry in the main diagonal of the matrix on the right-hand side of Equation (2.2) takes the form

$$(2.3) \quad 2\lambda_i (\lambda_i^2 + \sum_{j \geq 2} |(T^{-1} A_j T)^i|^2),$$

where $(T^{-1} A_j T)^i$ denotes the i -th row vector in $(T^{-1} A_j T)$, and we know that at least one λ_i is not zero. The contradiction proves the Lemma. \square

Theorem 2.2. *Let $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a quadratic map with $\varphi(X) = (X^t A_1 X, \dots, X^t A_n X)$, where $X^t = (x^1, \dots, x^m) \in \mathbb{R}^m$, and $A_i, i = 1, \dots, n$ are symmetric $m \times m$ matrices. Then, φ is an ∞ -harmonic map if and only if φ is a constant map.*

Proof. By a straightforward computation and Corollary 1.3, the ∞ -harmonic map equation reduces to

$$(2.4) \quad X^t A_i \left(\sum_{j=1}^n A_j^2 \right) X = 0, \quad i = 1, 2, \dots, n, \quad \forall X,$$

which is equivalent to $A_i (\sum_{j=1}^n A_j^2) + (\sum_{j=1}^n A_j^2) A_i = 0, i = 1, 2, \dots, n$. It follows from this and Lemma 2.1 that $A_i = 0$ for $i = 1, 2, \dots, n$, and hence $\varphi(X) = 0$, a constant map, from which we obtain the Theorem. \square

Theorem 2.3. *Let $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^n, \varphi(X) = (X^t A_1 X, \dots, X^t A_n X) + (AX)^t + b$ be a polynomial map, where A_i is an $m \times m$ symmetric matrix for $i = 1, 2, \dots, n, A$ an $n \times m$ matrix, and $b \in \mathbb{R}^n$. Then, φ is an ∞ -harmonic map if and only if φ is an affine map with $\varphi(X) = (AX)^t + b$.*

Proof. Let $\alpha_i \in \mathbb{R}^m, i = 1, 2, \dots, n$, denote the i -th row vector of the matrix A . A direct computation shows that in this case the map φ is ∞ -harmonic if and only if

$$16 \sum_{j=1}^n X^t A_i A_j^2 X + 8 \sum_{j=1}^n X^t A_i A_j (\alpha_j)^t + 8 \sum_{j=1}^n (\alpha_i A_j^2) X + 4 \sum_{j=1}^n \alpha_i A_j (\alpha_j)^t = 0.$$

By comparing the coefficients of the leading terms of this polynomial identity we have that, if φ is ∞ -harmonic, then

$$(2.5) \quad 16 X^t A_i \sum_{j=1}^n A_j^2 X = 0, \quad i = 1, 2, \dots, n,$$

which is the same as Equation (2.4). Now we can use Lemma 2.1 to conclude that if φ is ∞ -harmonic, then $A_i = 0$ for $i = 1, 2, \dots, n$ and hence $\varphi(X) = (AX)^t + b$ is an affine map. The converse statement clearly true because an affine map has constant energy density. Therefore, we obtain the theorem. \square

Remark 2.4. It would be interesting to know if there are any ∞ -harmonic maps $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ defined by homogeneous polynomials of degree greater than 2.

3 Linear and quadratic ∞ -harmonic maps from and into a sphere

In this section, we first derive an equation for linear ∞ -harmonic maps between conformally flat spaces. We then use it to give a complete classification of linear ∞ -harmonic maps between a Euclidean space and a sphere.

Lemma 3.1. *Let $\varphi : (\mathbb{R}^m, g = F^{-2}\delta_{ij}) \rightarrow (\mathbb{R}^n, h = \lambda^{-2}\delta_{\alpha\beta})$ with*

$$\varphi(X) = AX = (A^1X, \dots, A^nX),$$

where A^i is the i -th row vector of A , be a linear map between conformally flat spaces. Then, φ is ∞ -harmonic if and only if $A = 0$, i.e., $\varphi(X) = AX = 0$ is a constant map, or $\langle A^\alpha, \nabla(\frac{F}{\lambda \circ \varphi}) \rangle = 0$ for $\alpha = 1, 2, \dots, n$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and ∇f denotes the gradient of f taken with respect to the Euclidean metric on \mathbb{R}^m .

Proof. For the linear map $\varphi : (\mathbb{R}^m, g = F^{-2}\delta_{ij}) \rightarrow (\mathbb{R}^n, h = \lambda^{-2}\delta_{\alpha\beta})$ with $\varphi(X) = AX = (A^1X, \dots, A^nX)$ we have $g(\text{grad } \varphi^\alpha, \text{grad } |\text{d}\varphi|^2) = \frac{2F^3|A|^2}{\lambda \circ \varphi} \langle A^\alpha, \nabla(\frac{F}{\lambda \circ \varphi}) \rangle$, from which the Lemma follows. \square

Let (S^n, g_{can}) be the n -dimensional sphere with the standard metric. It is well known that we can identify $(S^n \setminus \{N\}, g_{can})$ with $(\mathbb{R}^n, \lambda^{-2}\delta_{ij})$, where $\lambda = \frac{1+|x|^2}{2}$. Using coordinate $\{x_i\}$ we can write the components of g_U as:

$$\bar{g}_{ij} = \lambda^{-2}\delta_{ij}, \quad \bar{g}^{ij} = \lambda^2\delta_{ij}.$$

As an application of Lemma 3.1 we give the following classification of linear ∞ -harmonic maps between spheres.

Theorem 3.2. *A linear map $\varphi : (\mathbb{R}^m, F^{-2}\delta_{ij}) \cong (S^m \setminus \{N\}, g_{can}) \rightarrow (\mathbb{R}^n, \lambda^{-2}\delta_{ij}) \cong (S^n \setminus \{N\}, g_{can})$ between two spheres with $\varphi(X) = (A^1X, \dots, A^nX)$ is ∞ -harmonic if and only if $A = 0$, i.e., φ is a constant map, or, $A^tA = I_{m \times m}$, i.e., φ is an isometric immersion.*

Proof. To prove the theorem, we applying Lemma 3.1 with $F = \frac{1+|X|^2}{2}$ and $\lambda = \frac{1+|Y|^2}{2}$ we conclude that φ is ∞ -harmonic if and only if $A = 0$, $\varphi(X) = AX = 0$ is a constant map, or $\langle A^\alpha, \nabla(\frac{F}{\lambda \circ \varphi}) \rangle = \frac{1}{2(\lambda \circ \varphi)^2} \langle A^\alpha, (1 + |AX|^2)X - (1 + |X|^2)A^tAX \rangle = 0$, $\alpha = 1, 2, \dots, n$, which is equivalent to

$$(3.1) \quad (1 + |AX|^2)AX - (1 + |X|^2)A^tAX = 0,$$

for any $X \in \mathbb{R}^m$. Since (3.1) is an identity of polynomials we compare the coefficients to have

$$(3.2) \quad \begin{cases} AX - AA^tAX = 0 \\ |AX|^2AX - |X|^2AA^tAX = 0. \end{cases}$$

for any $X \in \mathbb{R}^m$. It is easy to see that Equation (3.2) implies that $A = 0$, or, $A^tA = I_{m \times m}$ and $|\varphi(X)|^2 = |AX|^2 = |X|^2$, from which we obtain the theorem. \square

For linear maps between a Euclidean space and a sphere we have

Theorem 3.3. (1) A linear map $\varphi : \mathbb{R}^m \longrightarrow (\mathbb{R}^n, \lambda^{-2}\delta_{ij}) \equiv (S^n \setminus \{N\}, g_{can})$ from a Euclidean space into a sphere with $\varphi(X) = (A^1X, \dots, A^nX)$ is ∞ -harmonic if and only if $A = 0$, i.e., φ is a constant map.
 (2) A linear map $\varphi : (\mathbb{R}^m, \lambda^{-2}\delta_{ij}) \equiv (S^m \setminus \{N\}, g_{can}) \longrightarrow \mathbb{R}^n$ from a sphere into a Euclidean space with $\varphi(X) = (A^1X, \dots, A^nX)$ is ∞ -harmonic if and only if $A = 0$, i.e., φ is a constant map.

Proof. To prove the first Statement, we applying Lemma 3.1 with $F = 1$ and $\lambda = \frac{1+|Y|^2}{2}$ we conclude that φ is ∞ -harmonic if and only if $A = 0$ and hence $\varphi(X) = AX = 0$ is a constant map, or $\langle A^\alpha, \nabla(\frac{1}{\lambda \circ \varphi}) \rangle = -\frac{1}{(\lambda \circ \varphi)^2} \langle A^\alpha, A^tAX \rangle = 0$, $\alpha = 1, 2, \dots, n$, which is equivalent to

$$(3.3) \quad AA^tAX = 0.$$

for any $X \in \mathbb{R}^m$. By letting $X = (A^i)^t$, $i = 1, \dots, n$ in Equation (3.3) we conclude that φ is ∞ -harmonic if and only if $AA^tAA^t = 0$. Note that $AA^tAA^t = (AA^t)(AA^t)^t = 0$ implies that $\text{Trace}(AA^t) = \sum_{i=1}^n |A^i|^2 = 0$. It follows that $|A| = 0$, i.e., φ is a constant map. This gives the first Statement of the theorem.

For the second Statement, we apply Lemma 3.1 with $\lambda = 1$ and $F = \frac{1+|X|^2}{2}$ to conclude that φ is ∞ -harmonic if and only if $A = 0$, $\varphi(X) = AX = 0$ is a constant map, or

$$(3.4) \quad \langle A^\alpha, \nabla F \rangle = \langle A^\alpha, X \rangle = 0$$

for $\alpha = 1, 2, \dots, n$ and for all $X \in \mathbb{R}^m$. It is easy to see that Equation (3.4) implies that $A^\alpha = 0$ for $\alpha = 1, 2, \dots, n$ and hence $A = 0$, i.e., φ is a constant. This completes the proof of the Theorem. \square

Again, we identify $(S^n \setminus \{N\}, g_{can})$ with $(\mathbb{R}^n, \lambda^{-2}\delta_{ij})$, where $\lambda = \frac{1+|X|^2}{2}$.

Theorem 3.4. (1) A quadratic map $\varphi : \mathbb{R}^m \longrightarrow (\mathbb{R}^n, \lambda^{-2}\delta_{ij}) \equiv (S^n \setminus \{N\}, g_{can})$ into sphere with $\varphi(X) = (X^tA_1X, X^tA_2X, \dots, X^tA_nX)$ is ∞ -harmonic if and only if it is a constant map.
 (2) A quadratic map from a sphere into a Euclidean space $\varphi : (\mathbb{R}^m, \lambda^{-2}\delta_{ij}) \longrightarrow \mathbb{R}^n$ with $\varphi(X) = (X^tA_1X, X^tA_2X, \dots, X^tA_nX)$ is ∞ -harmonic if and only if it is a constant map.

Proof. For Statement (1), by a straightforward computation and (1.3) we conclude that the quadratic map φ is ∞ -harmonic iff

$$(3.5) \quad 0 = 16\sigma(X^t \sum_{j=1}^n A_j^2 X) \langle X^t A_\alpha, \nabla \sigma \rangle + 16\sigma^2 X^t A_\alpha \sum_{j=1}^n A_j^2 X,$$

where $\sigma = \frac{1}{\lambda \circ \varphi}$. Note that (3.5) is equivalent to

$$-32(X^t \sum_{j=1}^n A_j^2 X)(X^t A_\alpha A_1 X y_1 + \dots + X^t A_\alpha A_n X y_n) + \frac{16}{\sigma} X^t A_\alpha \sum_{j=1}^n A_j^2 X = 0, \text{ or,}$$

$$(3.6) \quad 0 = -P(X) + 8X^t A_\alpha \sum_{j=1}^n A_j^2 X$$

where $P(X)$ denotes a polynomial in X of degree greater than 2. Noting that the equation is an identity of polynomials we conclude that if φ is ∞ -harmonic, then

$$(3.7) \quad X^t A_\alpha \sum_{j=1}^3 A_j^2 X = 0, \quad \alpha = 1, 2, \dots, n,$$

which is exactly the Equation (2.4) and the same arguments used in the proof of Theorem 2.2 apply to give the required results.

To prove the second statement, let $\nabla f = (f_1, \dots, f_m)$ denotes the Euclidean gradient of function f . Then, a straightforward computation gives:

$$(3.8) \quad \nabla \varphi^\alpha = 2X^t A_\alpha, \quad |\mathrm{d}\varphi|^2 = \lambda^2 \delta^{\alpha\beta} \varphi_\alpha^i \varphi_\beta^j \delta_{ij} = 4\lambda^2 \sum_{j=1}^n X^t A_j^2 X,$$

and

$$(3.9) \quad \nabla |\mathrm{d}\varphi|^2 = 8\lambda(X^t \sum_{j=1}^n A_j^2 X)X + 8\lambda^2 X^t \sum_{j=1}^n A_j^2,$$

where we have used the fact that $\nabla \lambda = X$. The ∞ -harmonic map equation becomes

$$(3.10) \quad 16\lambda^3(X^t \sum_{j=1}^n A_j^2 X)X^t A_\alpha X + 16\lambda^4 X^t A_\alpha \sum_{j=1}^n A_j^2 X = 0$$

for all $X \in \mathbb{R}^m$ and for $\alpha = 1, 2, \dots, n$. Note that Equation (3.10) is an identity of polynomials since λ is also a polynomial. Comparing the coefficients of the polynomials we conclude that φ is ∞ -harmonic implies that $X^t A_\alpha \sum_{j=1}^n A_j^2 X = 0$ for $\alpha = 1, 2, \dots, n$, which is exactly the Equation (2.4) and the same arguments used in the proof of Theorem 2.2 apply to give the required results. \square

Remark 3.5. It is well known that any eigenmap between spheres is of constant energy density, so any eigenmap is ∞ -harmonic. It would be interesting to know if there is any ∞ -harmonic maps between spheres which is not an eigenmap.

4 Linear ∞ -harmonic maps from and into Nil space

In this section we will give a complete classification of linear ∞ -harmonic maps between Euclidean spaces and Nil space.

Theorem 4.1. *Let (\mathbb{R}^3, g_{Nil}) denote Nil space, where the metric with respect to the standard coordinates (x, y, z) in \mathbb{R}^3 is given by $g_{Nil} = dx^2 + dy^2 + (dz - xdy)^2$. Then*

- (1) *A linear function $f : (\mathbb{R}^3, g_{Nil}) \rightarrow \mathbb{R}$, $f(x, y, z) = Ax + By + Cz$ is an ∞ -harmonic function if and only if $A = 0$ or $C = 0$.*
- (2) *A linear map $\varphi : (\mathbb{R}^3, g_{Nil}) \rightarrow \mathbb{R}^n$ ($n \geq 2$) is ∞ -harmonic if and only if φ is a composition of the projection $\pi_1 : (\mathbb{R}^3, g_{Nil}) \rightarrow \mathbb{R}^2$, $\pi_1(x, y, z) = (x, y)$ followed by a linear map $\mathbb{R}^2 \rightarrow \mathbb{R}^n$, or, φ is a composition of the projection $\pi_2 : (\mathbb{R}^3, g_{Nil}) \rightarrow \mathbb{R}^2$, $\pi_2(x, y, z) = (y, z)$ followed by a linear map $\mathbb{R}^2 \rightarrow \mathbb{R}^n$.*

Proof. For Statement (1), we note that it has been proved in [10] that a linear function $f : (\mathbb{R}^3, g_{Nil}) \rightarrow \mathbb{R}$, $f(x, y, z) = Ax + By + Cz$ is an 1-harmonic if and only if it is horizontally homothetic which is equivalent to f being ∞ -harmonic. It was further shown that this is equivalent to $A = 0$ or $C = 0$. To prove Statement (2), one can easily compute the following components of Nil metric:

$$(4.1) \quad g_{11} = 1, \quad g_{12} = g_{13} = 0, \quad g_{22} = 1 + x^2, \quad g_{23} = -x, \quad g_{33} = 1;$$

Let $\varphi : (\mathbb{R}^3, g_{Nil}) \rightarrow \mathbb{R}^n$ ($n \geq 2$) be a linear map $\varphi(X) = AX$ with A being an $n \times 3$ matrix and $X^t = (x, y, z)$. By a straightforward computation and Corollary 1.3 we see that φ is ∞ -harmonic iff

$$(4.2) \quad g(\nabla \varphi^i, \nabla |\mathrm{d}\varphi|^2) = 2a_{i1} \left(\sum_{i=1}^n a_{i3}^2 x + \sum_{i=1}^n a_{i2} a_{i3} \right) = 0, \quad i = 1, 2, \dots, n,$$

where and in the rest of the paper a_{ij} denotes the entry at the i th row and j th column of the matrix A . Solving Equation (4.2) we have $a_{i1} = 0$, for $i = 1, 2, \dots, n$, or $a_{i3} = 0$, for $i = 1, 2, \dots, n$, from which we conclude that the linear map $\varphi : (\mathbb{R}^3, g_{Nil}) \rightarrow \mathbb{R}^n$ ($n \geq 2$) with $\varphi(X) = AX$ is ∞ -harmonic if and only if the first or the third column of A vanishes. Thus, we obtain the theorem. \square

Remark 4.2. (i) We remark that in both cases, the maximum possible rank of the linear ∞ -harmonic map φ is 2.

(ii) We can check that in first case the linear ∞ -harmonic map φ has constant energy density given by a quadratic polynomial whilst in the second case the linear ∞ -harmonic map φ has non-constant energy density.

(iii) It follows from our Theorem that we can choose to have submersion $\varphi : (\mathbb{R}^3, g_{Nil}) \rightarrow \mathbb{R}^2$ with $\varphi(x, y, z) = (a_{12}y + a_{13}z, a_{22}y + a_{23}z)$ so that φ has non-constant energy density. Clearly, φ cannot be a Riemannian submersion because the energy density is not constant.

Theorem 4.3. *A linear map $\varphi : \mathbb{R}^m \rightarrow (\mathbb{R}^3, g_{Nil})$ into Nil space is ∞ -harmonic if and only if φ is a composition of a linear map $\mathbb{R}^m \rightarrow \mathbb{R}^2$ followed by the inclusion map $i_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $i_1(y, z) = (0, y, z)$, or, φ is a composition of a linear map $\mathbb{R}^m \rightarrow \mathbb{R}^2$ followed by the inclusion map $i_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $i_2(x, z) = (x, 0, z)$.*

Proof. Let $\varphi : \mathbb{R}^m \longrightarrow (\mathbb{R}^3, g_{Nil})$ be a linear map with $\varphi(X) = AX$, where A is a $3 \times m$ matrix and $X^t = (x_1, \dots, x_m)$. In this case, the ∞ -harmonic map equation becomes

$$2 \sum_{k=1}^m a_{ik} a_{1k} \left(\sum_{j=1}^m a_{2j}^2 x_j - \sum_{j=1}^m a_{2j} a_{3j} \right) = 0, \quad i = 1, 2, 3,$$

Solving this system we have $a_{1k} = 0$, for $k = 1, 2, \dots, m$ or $a_{2k} = 0$, for $k = 1, 2, \dots, m$, from which we obtain the theorem. \square

Remark 4.4. We remark that in both cases, the maximum possible rank of the linear ∞ -harmonic map φ is 2 and the linear ∞ -harmonic map φ has constant energy density.

5 Linear ∞ -harmonic maps from and into Sol space

In this section we give a complete classification of linear ∞ -harmonic maps between Euclidean spaces and Sol space.

Theorem 5.1. *Let (\mathbb{R}^3, g_{Sol}) denote Sol space, where the metric with respect to the standard coordinates (x, y, z) in \mathbb{R}^3 is given by $g_{Sol} = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2$. Then*

- (1) *A linear function $f : (\mathbb{R}^3, g_{Sol}) \longrightarrow \mathbb{R}$, $f(x, y, z) = Ax + By + Cz$ is an ∞ -harmonic function if and only if $C = 0$ or $A = B = 0$.*
- (2) *A linear map $\varphi : (\mathbb{R}^3, g_{Sol}) \longrightarrow \mathbb{R}^n$ ($n \geq 2$) is ∞ -harmonic if and only if φ is a composition of the projection $\pi_1 : (\mathbb{R}^3, g_{Sol}) \longrightarrow \mathbb{R}^2$, $\pi_1(x, y, z) = (x, y)$ followed by a linear map $\mathbb{R}^2 \longrightarrow \mathbb{R}^n$, or, φ is a composition of the projection $\pi_2 : (\mathbb{R}^3, g_{Sol}) \longrightarrow \mathbb{R}^2$, $\pi_2(x, y, z) = (z)$ followed by a linear map $\mathbb{R} \longrightarrow \mathbb{R}^n$.*

Proof. The Statement (1) is proved in [10]. To prove Statement (2), one can easily compute the following components of Sol metric:

$$(5.1) \quad g_{11} = e^{2z}, \quad g_{22} = e^{-2z}, \quad g_{33} = 1, \quad \text{all other, } g_{ij} = 0;$$

Let $\varphi : (\mathbb{R}^3, g_{Sol}) \longrightarrow \mathbb{R}^n$ ($n \geq 2$) be a linear map $\varphi(X) = AX$ with A being an $n \times 3$ matrix and $X^t = (x, y, z)$. Then, a straightforward computation shows that φ is ∞ -harmonic iff

$$(5.2) \quad g(\nabla \varphi^i, \nabla |\mathrm{d}\varphi|^2) = -2a_{i3} \left(\sum_{i=1}^n a_{i1}^2 e^{-2z} - \sum_{i=1}^n a_{i2} e^{2z} \right) = 0, \quad i = 1, 2, \dots, n.$$

Solving Equation (5.2) we obtain the theorem. \square

Remark 5.2. It follows from Theorem 5.1 that the maximum rank of the linear ∞ -harmonic maps from Sol space into Euclidean space is 2. In the case when the linear ∞ -harmonic map with the third column of A vanishing it has non-constant energy density.

Theorem 5.3. *A linear map $\varphi : \mathbb{R}^m \longrightarrow (\mathbb{R}^3, g_{Sol})$ is ∞ -harmonic if and only if it is a composition of a linear map $\mathbb{R}^m \longrightarrow \mathbb{R}^2$ followed by the inclusion map $i_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $i_1(x, y) = (x, y, 0)$, or, φ is a composition of a linear map $\mathbb{R}^m \longrightarrow \mathbb{R}$ followed by an inclusion map $i_2 : \mathbb{R} \longrightarrow \mathbb{R}^3$, $i_2(z) = (0, 0, z)$.*

Proof. For the linear map $\varphi : \mathbb{R}^m \longrightarrow (\mathbb{R}^3, g_{Sol})$ $\varphi(X) = AX$, where A is a $3 \times m$ matrix and $X^t = (x_1, \dots, x_m)$, the ∞ -harmonic map equation reads

$$(5.3) \quad 2 \sum_{k=1}^m a_{ik} a_{3k} \left(\sum_{j=1}^m a_{1j}^2 e^{2z} - \sum_{j=1}^m a_{2j}^2 e^{-2z} \right) = 0, \quad i = 1, 2, 3,$$

The solutions of this system give the statements in the Theorem. \square

Remark 5.4. It follows from theorem 5.3 that the maximum rank of the linear ∞ -harmonic maps from Euclidean space into Sol space is 2, and any linear ∞ -harmonic map into Sol space has constant energy density.

6 Quadratic ∞ -harmonic maps into Sol and Nil spaces

Theorem 6.1. *Let (\mathbb{R}^3, g_{Sol}) denote Sol space, where the metric with respect to the standard coordinates (x, y, z) in \mathbb{R}^3 is given by $g_{Sol} = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2$. Then, a quadratic map $\varphi : \mathbb{R}^m \longrightarrow (\mathbb{R}^3, g_{Sol})$ with $\varphi(X) = (X^t A_1 X, X^t A_2 X, X^t A_3 X)$ is an ∞ -harmonic map if and only if it is a constant map.*

Proof. In this case the map φ is ∞ -harmonic iff

$$(6.1) \quad 16(X^t A_i A_1^2 X + X^t A_1^2 X X^t A_i A_3 X) e^{2z} + 16(X^t A_i A_2^2 X - X^t A_2^2 X X^t A_i A_3 X) e^{-2z} + 16X^t A_i A_3^2 X = 0, \quad \text{for all } X \in \mathbb{R}^m, \text{ and } i = 1, 2, 3.$$

Note that Equation (6.1) is an identity of functions which are analytic. We can substitute the Taylor expansions for $e^{2X^t A_3 X}$ and $e^{-2X^t A_3 X}$ into (6.1) and compare the coefficients of the second degree terms to get

$$16(X^t A_i \sum_{j=1}^3 A_j^2 X) = 0, \quad i = 1, 2, 3.$$

From this we obtain $A_i (\sum_{j=1}^3 A_j^2) + (\sum_{j=1}^3 A_j^2) A_i = 0$, and Lemma 2.1 applies to complete the proof of the Theorem. \square

In a similar way we can prove the following

Theorem 6.2. *A quadratic map $\varphi : \mathbb{R}^m \longrightarrow (\mathbb{R}^3, g_{Nil})$, $\varphi(X) = (X^t A_1 X, X^t A_2 X, X^t A_3 X)$ into Nil space is ∞ -harmonic if and only if it is a constant map.*

Example 6.3. We remark that there are many polynomial ∞ -harmonic maps $\varphi : (\mathbb{R}^3, g_{Nil}) \longrightarrow \mathbb{R}^m$, for instance,

$$\varphi : (\mathbb{R}^3, g_{Nil}) \longrightarrow \mathbb{R}^2$$

with $\varphi(x, y, z) = (z - xy/2, 2z - xy)$ is an ∞ -harmonic map which has nonconstant energy density $|\mathrm{d}\varphi|^2 = 5(1 + (x^2 + y^2)/4)$ (see [11] for details).

7 Holomorphic ∞ -harmonic maps

In this section, we study ∞ -harmonicity of holomorphic maps $\mathbb{C}^m \rightarrow \mathbb{C}^n$. Let $(z_1, \dots, z_m) \in \mathbb{C}^m$ and $(w_1, \dots, w_n) \in \mathbb{C}^n$ with $z_j = x_j - iy_j$ $j = 1, \dots, m$ and $w_\alpha = u_\alpha - iv_\alpha$ $\alpha = 1, \dots, n$. Then, a map $\varphi : \mathbb{C}^m \rightarrow \mathbb{C}^n$, $\varphi(z_1, \dots, z_m) = (\varphi^1, \dots, \varphi^n)$ is associated to a map $\varphi : \mathbb{R}^{2m} \rightarrow \mathbb{R}^{2n}$ with $\varphi(x_1, \dots, x_m, y_1, \dots, y_m) = (u_1, \dots, u_n, v_1, \dots, v_n)$. We write the map as $\varphi(X + iY) = \phi(X, Y) + i\psi(X, Y)$, where $X = (x_1, \dots, x_m), Y = (y_1, \dots, y_m) \in \mathbb{R}^m$ and the maps $\phi(X, Y) = (u^1(X, Y), \dots, u^n(X, Y))$ and $\psi(X, Y) = (v^1(X, Y), \dots, v^n(X, Y))$ are called the real and imaginary parts of φ . We have

Theorem 7.1. *A holomorphic map $\varphi : \mathbb{C}^m \rightarrow \mathbb{C}^n$ with $\varphi(X, +iY) = \phi(X, Y) + i\psi(X, Y)$ is ∞ -harmonic if and only if its real and imaginary parts $\phi(X, Y)$ and $\psi(X, Y)$ are ∞ -harmonic.*

Proof. It is well known that $\varphi : \mathbb{C}^m \rightarrow \mathbb{C}^n$ is holomorphic if and only if

$$(7.1) \quad \frac{\partial u^\alpha}{\partial x_j} = \frac{\partial v^\alpha}{\partial y_j}, \quad \frac{\partial u^\alpha}{\partial y_j} = -\frac{\partial v^\alpha}{\partial x_j}; \quad j = 1, 2, \dots, m, \quad \alpha = 1, 2, \dots, n.$$

We can easily check that

$$\begin{aligned} |\nabla u^\alpha|^2 &= |\nabla v^\alpha|^2, \quad |\mathrm{d}\varphi|^2 = 2 \sum_{\alpha=1}^n |\nabla u^\alpha|^2 = 2 \sum_{\alpha=1}^n |\nabla v^\alpha|^2, \\ \nabla |\mathrm{d}\varphi|^2 &= 2\nabla |\nabla \phi|^2 = 2\nabla |\nabla \psi|^2. \end{aligned}$$

Substitute these into the ∞ -harmonic map Equation (1.3) we obtain that φ is ∞ -harmonic if and only if

$$(7.2) \quad 2g(\nabla \phi^\alpha, \nabla |\nabla \phi|^2) = 0, \quad 2g(\nabla \psi^\alpha, \nabla |\nabla \psi|^2) = 0, \quad \alpha = 1, 2, \dots, n,$$

which gives the Theorem. \square

Theorem 7.2. *Let $\varphi : \mathbb{C}^m \rightarrow \mathbb{C}$ be a nonconstant holomorphic map. Then, φ is an ∞ -harmonic map if and only if φ is a composition of an orthogonal projection $\mathbb{C}^m \rightarrow \mathbb{C}$ followed by a homothety $\mathbb{C} \rightarrow \mathbb{C}$, i.e., $\varphi(z_1, \dots, z_m) = \lambda z_i + z_0$, where $\lambda \in \mathbb{R}, z_0 \in \mathbb{C}$ are constants.*

Proof. Notice that a holomorphic map $\varphi : \mathbb{C}^m \rightarrow \mathbb{C}$ is automatically a horizontally weakly conformal harmonic map (see e.g., [4]). It follows from the relationship among tension, p -tension, and ∞ -tension fields of a map

$$(7.3) \quad \tau_p(\varphi) = |\mathrm{d}\varphi|^{p-2} \tau_2(\varphi) + (p-2)|\mathrm{d}\varphi|^{p-4} \tau_\infty(\varphi)$$

that if φ is also an ∞ -harmonic map, then it must be p -harmonic for any p . In this case φ is a p -harmonic morphism (being horizontally weakly conformal and p -harmonic map) for any p . By a theorem in [5], φ must be a horizontally homothetic. Now the classification of horizontally homothetic maps between Euclidean spaces [10] applies to give the required results. \square

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